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ON NH-EMBEDDED AND SS-QUASINORMAL SUBGROUPS OF FINITE GROUPS

Weicheng Zheng, Liang Cui, Wei Meng and Jiakuan Lu

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Abstract. Let G be a finite group. A subgroup H is called S-semipermutable in G if $HG_p = G_pH$ for any $G_p \in Syl_p(G)$ with $(|H|, p) = 1$, where p is a prime number divisible $|G|$. Furthermore, H is said to be NH-embedded in G if there exists a normal subgroup T of G such that HT is a Hall subgroup of G and $H \cap T \leq H_{\overline{s}G}$, where $H_{\overline{s}G}$ is the largest S-semipermutable subgroup of G contained in H, and H is said to be SS-quasinormal in G provided there is a supplement B of H to G such that H permutes with every Sylow subgroup of B. In this paper, we obtain some criteria for p-nilpotency and Supersolvability of a finite group and extend some known results concerning NH-embedded and SS-quasinormal subgroups.

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1. Introduction

Throughout, all groups considered in this paper will be finite. Let G be a group, H and K of are subgroups of G, they are said to be permutable if $HK = KH$, i.e. HK is also a subgroup of G. H is a subgroup of G, it is said to be quasinormal in G if H permutes with all subgroups of G. Kegel [8] introduced the concept of S-quasinormal (or S-permutable), subgroup H of G said to be S-quasinormal if H permutes with all Sylow subgroup of G. Recall that a supplement of H to G is a subgroup B such that $G = HB$. As a generalization of S-quasinormal subgroup, Li [9] introduced the following definition:

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Definition 1.1. [9] A subgroup H of G is said to be SS-quasinormal in G provided there is a supplement B of H to G such that H permutes with every Sylow subgroup of B.

Li [9] investigated the p-nilpotency and supersolvability of finite groups by some SS-quasinormal subgroups of prime power order.

Recall that a subgroup H is called S-semipermutable in G if $HG_p = G_pH$ for any $G_p \in Syl_p(G)$ with $(|H|, p) = 1$, p is a prime number divisible G(see [2]). Recently, Gao and Li [5] introduce the following concept:

Definition 1.2. [5] A subgroup H of a group G is said to be NH-embedded in G if there exists a normal subgroup T of G such that HT is a Hall subgroup of G and $H \cap T \leq H_{\overline{s}G}$, where $H_{\overline{s}G}$ is the largest S-semipermutable subgroup of G contained in H.

Gao and Li [5] showed that the finite group whose maximal subgroups of Sylow subgroups are NH-embedded in G are supersolvable.

By the definition of NH-embedded and SS-quasinormal subgroups, it is obvious that all Hall subgroups, normal subgroups and S-semipermutable subgroups are NH-embedded subgroups. But the converse does not hold. Moreover, an NH-embedded subgroup need not be SS-quasinormal. Conversely, it easy to see that an SS-quasinormal subgroup need not be NH-embedded too.

In the light of above results, it is seem interesting to study the structure of finite groups assuming that maximal subgroups of Sylow subgroups are SS-quasinormal or NH-embedded in G. In this paper, we obtain some criteria for p-nilpotency and supersolvability of a finite group. The main results are as follows.

Theorem 1.3. *Let G be a finite group and G_p a Sylow p-subgroup of G, where p is the smallest prime dividing $|G|$. Assume that every maximal subgroup of G_p is either NH-embedded or SS-quasinormal in G. Then G is p-nilpotent.*

Theorem 1.4. *Let G be a finite group. Suppose that every maximal subgroup of every Sylow subgroup of G is either NH-embedded or SS-quasinormal in G. Then G is supersolvable.*

Theorem 1.5. *Let G be a finite group and H a normal subgroup of G. Suppose that G/H is supersolvable and every maximal subgroup of every Sylow subgroup of H is either NH-embedded or SS-quasinormal in G. Then G is supersolvable.*

All unexplained notations and terminologies are standard and can be found in [4,6].

2. Preliminaries

In this section, we collect some results which will be used in the proof of main results.

Lemma 2.1. [9] *Suppose that H is SS-quasinormal in a group G, $K \leq G$, and N is a normal subgroup of G. We have*

(1) *if $H \leq K$, then H is SS-quasinormal in K;*
(2) *HN/N is SS-quasinormal in G/N;*
(3) *if $N \leq K$ and K/N is SS-quasinormal in G, then K is SS-quasinormal in G;*
(4) *if K is quasinormal in G, then HK is SS-quasinormal in G.*

Lemma 2.2. [9] *Let H be a p-subgroup of G. Then the following statements are equivalent:*

(1) *H is S-quasinormal in G;*
(2) *$H \leq O_p(G)$, and H is SS-quasinormal in G.*

Lemma 2.3. [3] *If H is an S-quasinormal subgroup of the group G, then H/H_G is nilpotent, where H_G is the core of H in G.*

Lemma 2.4. [11] *If H is S-quasinormal in a group G and H is a p-group for some prime p, then $O^p(G) \leq N_G(H)$.*

Lemma 2.5. [1] *Let H be a nilpotent subgroup of a group G. Then the following statements are equivalent:*

(1) *H is an S-quasinormal subgroup of G;*
(2) *the Sylow subgroups of H are S-quasinormal in G.*

Lemma 2.6. [1] *Let P be a Sylow p-subgroup of a group G, and let P_0 be a maximal subgroup of P. Then the following statements are equivalent:*

(1) *P_0 is normal in G;*
(2) *P_0 is S-quasinormal in G.*

Lemma 2.7. [5] *Let G be a group and $H \leq G$. Suppose that H is NH-embedded in G. Let N be a normal subgroup of G. Then*

(1) *If $H \leq K \leq G$ and K is subnormal in G, then H is NH-embedded in K.*
(2) *Suppose that H is a p-group for some $p \in \pi(G)$. If $N \leq H$, then H/N is NH-embedded in G/N.*
(3) *Suppose that H is a p-group for some $p \in \pi(G)$ and N is a p'-subgroup of G. Then HN/N is NH-embedded in G/N.*

Lemma 2.8. [12] *Let G be a group and $H \leq K \leq G$.*

(1) *If H is S-semipermutable in G, then H is S-semipermutable in K;*
(2) *Suppose that N is normal in G and H is a p-group. If H is S-semipermutable in G, then HN/N is S-semipermutable in G/N;*
(3) *If H is S-semipermutable in G and $H \leq O_p(G)$, then H is S-quasinormal in G.*

Lemma 2.9. [10] *Let G be a group and H an S-semipermutable subgroup of G. Suppose that H is a p-subgroup of G for some prime $p \in \pi(G)$ and N is normal in G. Then $H \cap N$ is also an S-semipermutable subgoup of G.*

Lemma 2.10. [7] *Let H be an S-semipermutable π-subgroup of G. Then H^G contains a nilpotent π-complement, and all π-complements in H^G are conjugate. Also, if π consists of a single prime, then H^G is solvable.*

Lemma 2.11. [4] *Let U, V and W be subgroups of a group G. Then the following statements are equivalent:*

(1) *$U \cap VW = (U \cap V)(U \cap W)$;*
(2) *$UV \cap UW = U(V \cap W)$.*

3. Proof of Theorem

Proof of Theorem 1.3 Assume that the theorem is false and let G be a counterexample of minimal order. Let G_p be a Sylow p-subgroup of G and $\mathcal{M}(G_p) = \{P_1, P_2, \cdots, P_m\}$ denote the set of all maximal subgroups of G_p. By Theorem hypothesis, every member P_i of $\mathcal{M}(G_p)$ is either NH-embedded or SS-quasinormal in G. Without loss of generality, suppose that every member of the subset $\mathcal{M}_1(G_p) = \{P_1, \cdots, P_k\}$ of $\mathcal{M}(G_p)$ is NH-embedded in G, and every member of the subset $\mathcal{M}_2(G_p) = \{P_{k+1}, \cdots, P_m\}$ of $\mathcal{M}(G_p)$ is SS-quasinormal in G for some $1 \leq k \leq m$. The proof of theorem will be divided into five steps as follows.

Step (1). G has a unique minimal normal subgroup N and G/N is p-nilpotent.

Let N is a minimal normal subgroup of G. Then G_pN/N is a Sylow p-subgroup of G/N. For any $M/N \in \mathcal{M}(G_pN/N)$, let $P = M \cap G_p$, then

$$M = M \cap G_pN = (M \cap G_p)N = PN$$

and

$$P \cap N = (M \cap G_p) \cap N = PN \cap G_p \cap N = G_p \cap N.$$

As $|G_p : P| = |G_p : (G_p \cap M)| = |G_pM : M| = p$, we know that $P \in \mathcal{M}(G_p)$. By Theorem hypothesis, P is either NH-embedded or SS-quasinormal in G.

Suppose that P is SS-quasinormal in G, then $M/N = PN/N$ is also SS-quasinormal in G/N by Lemma 2.1(2).

Now, we assume that P is NH-embedded in G, then there is a normal subgroup T of G such that PT is a Hall subgroup of G and $P \cap T \leq P_{\overline{s}G}$. It is easy seen that TN/N is normal in G/N and $PN/N \cdot TN/N = PTN/N$ is a Hall subgroup of G/N. As $P \cap N = G_p \cap N$ is a Sylow p-subgroup of N, we have

$$|N \cap PT|_p = |N|_p = |N \cap P|_p = |(N \cap P)(N \cap T)|_p$$

and

$$|N \cap PT|_{p'} = \frac{|PT|_{p'}|N|_{p'}}{|NPT|_{p'}} = \frac{|T|_{p'}|N|_{p'}}{|NT|_{p'}} = |N \cap P|_{p'} = |(N \cap P)(N \cap T)|_{p'}.$$

This implies that $N \cap PT = (N \cap P)(N \cap T)$ and hence $PN \cap TN = (P \cap T)N$ by Lemma 2.11. Therefore,

$$PN/N \cap TN/N = (PN \cap TN)/N = (P \cap T)N/N \leq P_{\overline{s}G}N/N.$$

As $P_{\overline{s}G}$ is S-semipermuted in G, we get that $P_{\overline{s}G}N/N$ is also S-semipermuted in G/N by Lemma 2.8(2). This leads to $P_{\overline{s}G}N/N \leq (PN/N)_{\overline{s}G/N}$. So $M/N = PN/N$ is NH-embedded in G/N.

By above arguments, we know that G/N satisfies the hypothesis of theorem. By the choice of G, we know that G/N is p-nilpotent. Moreover, as the class of all p-nilpotent groups is saturated formation, we obtain that N is the unique minimal normal subgroup of G.

Step (2). $O_{p'}(G) = 1$.

Suppose that $O_{p'}(G) > 1$, then $N \leq O_{p'}(G)$ by (1). As G/N is p-nilpotent, we know that $G/O_{p'}(G)$ is p-nilpotent and hence G is also p-nilpotent, a contradiction. Therefore, $O_{p'}(G) = 1$.

Step (3). $N \leq P_i$ for any $P_i \in \mathcal{M}_1(G_p)$.

For any $H \in \mathcal{M}_1(G_p)$, H is NH-embedded in G, then there is a normal subgroup T of G such that HT is a Hall subgroup of G and $H \cap T \leq H_{\overline{s}G}$. If $T = 1$, then H is a Hall subgroup of G and hence $H = 1$. This implies that $|G_p| = p$, as P is a maximal subgroup of G_p. By Burnside theorem, G is p-nilpotent, a contradiction. Hence $T \neq 1$ and $N \leq T$. If $H \cap N = 1$, then $|N|_p \leq p$. So N is p-nilpotent by Burnside theorem. Let U be a normal Hall p'-subgroup of N, then U is normal in G. By minimality of N, we know that $U = 1$ and hence N is a subgroup of order p. Consequently, the nilpotency of G/N implies that G is p-nilpotent, a contradiction. Therefore, we have $H \cap N \neq 1$. Since $H \cap N \leq H \cap T \leq H_{\overline{s}G} \leq H$, we get that $H_{\overline{s}G} \cap N \leq H \cap N$ and hence $H \cap N = H_{\overline{s}G} \cap N$. By Lemma 2.9, $H \cap N$ is S-semipermuted in G. In the other hand, observe that $N \leq H_{\overline{s}G}^G$, we know that N is solvable by Lemma 2.10. This implies that N is a p-group and hence $N \leq O_p(G)$. In particular, $H \cap N \leq O_p(G)$. Applying Lemma 2.8(3), we get that $H \cap N$ is S-quasinormal in G. Thus, $O^p(G) \leq N_G(H \cap N)$ by Lemma 2.4. Noting that $H \cap N$ is normal in G_p which implies that $H \cap N$ is normal in G. Thus, $H \cap N = N$. This leads to $N \leq H$.

Step (4). For every $P_j \in \mathcal{M}_2(G_p)$, there exists a normal subgroup M_j of G such $N \leq M_j$.

For any $H \in \mathcal{M}_2(G_p)$, we know that H is SS-quasinormal in G. So there exists a subgroup B of G such that $G = HB$, and $HB_p = B_pH$ for every Sylow subgroup B_p of B. So $|B : H \cap B|_p = |G : H|_p = p$ from $G = HB$. Thus, $B_p \nsubseteq H$ and $B_pH = HB_p$ is a Sylow p-subgroup of G. In view of $H \in \mathcal{M}_2(G_p)$ and by comparison of orders, $|H \cap B|_p = H \cap B$. Therefore,

$$H \cap B = \bigcap_{b \in B} (B_p^b \cap H) \leq \bigcap_{b \in B} B_p^b = O_p(B).$$

From $|O_p(B) : B \cap H| = p$ or 1, we obtain $|B/O_p(B)|_p = 1$ or p. As p is the smallest prime dividing $|G|$, by Burnside theorem, $B/Op(B)$ is p-nilpotent. So B is p-solvable. And there is a Hall $p^{'}$-subgroup of B from ([6], IV, 1.7). Let K be a Hall $p^{'}$-subgroup of B, $\pi(K) = (q_2, \dots, q_s)$, $Q_i \in Syl_{q_i}(K)$. According to the definition of SS-quasinormal, H and $< Q_2, \dots, Q_s >= K$ are permutations. So HK is a subgroup of G. Obviously, K is a Hall $p^{'}$-group of G, and HK is a subgroup of index p in G. As p is the smallest prime dividing $|G|$, $HK \trianglelefteq G$. If $HK = 1$, then G is elementary commutative p-group, contradiction. So, $N \leq HK = M_j$.

Step (5). Final contradiction.

Set $V = (\bigcap_{i=1}^{k} P_i) \bigcap (\bigcap_{j=k+1}^{m} M_j)$. By above arguments, we know that $N \leq V$. Moreover, we have

$$N = N \cap G_p \leq V \cap G_p = ((\bigcap_{i=1}^{k} P_i) \bigcap (\bigcap_{j=k+1}^{m} M_j)) \bigcap G_p = \bigcap_{i=1}^{m} P_i = \Phi(G_p).$$

By ([6]. III. 3.3), we know that $\Phi(G_p) \leq \Phi(G)$ and hence $N \leq \Phi(G)$. Since G/N is p-nilpotent, we get that $G/\Phi(G)$ is p-nilpotent. As the class of all p-nilpotent is a statured formation, G will be p-nilpotent. This is finally contradiction. The proof of theorem is complete. □

Proof of Theorem 1.4 Assume that the theorem is false and let G be a counterexample of minimal order. Let p be the smallest prime dividing $|G|$. Then G is p-nilpotent by Theorem 1.3. Let U be a Hall normal p'-subgroup of G. It is easy seen that U satisfies the theorem hypothesis by Lemma 2.1(1) and Lemma 2.7(1). By induction, U is supersolvable and hence G possesses Sylow tower property of supersolvable type. Let q be the largest prime dividing $|G|$ and Q is a Sylow q-subgroup of G. Then Q is normal in G. By Lemmas 2.1(2) and 2.7(3), we know that G/Q satisfies the theorem hypothesis and hence G/Q is supersolvable by the choice of G.

Let N be a minimal normal subgroup of G. Similar to the proof of Theorem 1.3, G/N satisfies the theorem hypothesis and hence G/N is supersolvable by minimality of G. As the class of all supersolvable group is a statured formation, N will be a unique minimal normal subgroup of G. Therefore, we have $N \leq Q$.

We claim that $N \leq H$ for any $H \in \mathcal{M}(Q)$. By theorem hypothesis, we know that H is either NH-embedded or SS-quasinormal in G. If H is SS-quasinormal in G. As Q is normal in G, we have $H \leq Q = O_q(G)$. Applying Lemma 2.2, H is S-quasinormal in G. Moreover, H is normal in G by Lemma 2.6. So we have $N \leq H$.

Now, assume that H is NH-embedded in G. Then there exists a normal subgroup T of G such that HT is a Hall subgroup of G and $H \cap T \leq H_{\overline{s}G}$. If $T = 1$, then $H = HT$ is a Hall subgroup of G. This implies that $H = 1$ and hence $|Q| = q$. As G/Q is supersolvable, we get that G would be supersolvable, a contradiction. Therefore, $T \neq 1$ and hence $N \leq T$. Consequently, G/T is supersolvable. If $H \cap T = 1$, then $|Q \cap T| = q$. This forces that $N = Q \cap T$ is a subgroup of order q. Therefore, G is supersolvable, a contradiction. Consequently, $H \cap T \neq 1$. Observe that $H \cap T \leq H_{\overline{s}G} \leq H$, we have $H \cap T = H_{\overline{s}G} \cap T$ is S-semipermuted in G by Lemma 2.9. In the other hand, $H \cap T \leq Q = O_q(G)$ which implies that $H \cap T$ is S-quasinormal in G by Lemma 2.8(3). Applying Lemma 2.4, $O^q(G) \leq N_G(H \cap T)$. Furthermore, noting that $H \cap T$ is normal in Q. We can get that $H \cap T$ is normal in G. Therefore, $N \leq H \cap T \leq H$. The claim as desired.

By above arguments, we get that $N \leq \bigcap_{H \in \mathcal{M}(Q)} H = \Phi(Q)$. By ([6], III, 3.3), $\Phi(Q) \leq \Phi(G)$ and hence $N \leq \Phi(G)$. So $G/\Phi(G)$ is supersolvable and hence G is supersolvable. This is a finally contradiction. The proof of theorem is complete. □

Proof of Theorem 1.5 Assume that the result is false and let G be a counterexample of minimal order. Applying Lemma 2.1(1) and Lemma 2.7(1), we know that every maximal subgroup of Sylow subgroups of H is either NH-embedded or SS-quasinormal in H. By Theorem 1.4, H is supersolvable. Let q be the largest prime dividing $|H|$ and $Q = O_q(G)$ char $Syl_q(H)$. Then Q is normal in H and so it is in G. Obviously, $(G/Q)/(H/Q) \cong G/H$ is supersolvable. By Lemmas 2.1(2) and 2.7(3), we know that every maximal subgroup of Sylow subgroups of H/Q is either NH-embedded or SS-quasinormal in G/Q. Therefore, G/Q satisfies the theorem hypothesis and hence G/Q is supersolvable.

Let N be a minimal normal subgroup of G contained in Q. Similar to the proof of Theorem 1.3, we know that G/N satisfies the theorem hypothesis and hence G/N

is supersolvable. As the class of all supersolvable group is a statured formation, N will be a unique minimal normal subgroup of G contained in Q.

In the following, similar to the proof of Theorem 1.4, we can get that $N \leq \Phi(Q)$ and hence $G/\Phi(Q)$ is supersolvable. By ([6], III, 3.3), $\Phi(Q) \leq \Phi(G)$. So $G/\Phi(G)$ is supersolvable. As the class of all supersolvable groups is a statured formation, we obtain that G is supersolvable. This is a final contradiction. The proof of Theorem is complete. □

4. Some applications

As an immediate consequence of Theorem 1.3, we can get the corollaries as follows.

Corollary 4.1. ([5, Theorem 3.1]) *Let p be the smallest prime dividing $|G|$ and G_p a Sylow p-subgroup of G. Suppose that every maximal subgroup of G_p is NH-embedded in G. Then G is p-nilpotent.*

Corollary 4.2. ([1, Theorem 3.1]) *Let p be the smallest prime dividing $|G|$ and G_p a Sylow p-subgroup of G. Suppose that every maximal subgroup of G_p is SS-quasinormal in G. Then G is p-nilpotent.*

Theorem 1.5 immediately implies the following corollaries.

Corollary 4.3. ([9, Theorem 1]) *Let G be a finite group. If the maximal subgroups of the Sylow subgroups of G are SS-quasinormal in G, then G is supersolvable.*

Corollary 4.4. ([5, Theorem 3.4]) *Let G be a group with a normal subgroup E such that G/E is supersolvable. If every maximal subgroup of every Sylow subgroup of E is NH-embedded in G, then G is supersolvable.*

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References

[1] M. Asaad and A. Heliel, *On S-quasinormally embedded subgroups of finite groups,* J. Pure Appl. Algebra, 165(2) (2001), 129-135.

[2] Z. Chen, *On a theorem of Srinivasan,* J. Southwest Normal Univ. Nat. Sci. (Chinese), 12(1) (1987), 1-4.

[3] W. E. Deskins, *On quasinormal subgroups of finite groups,* Math. Z., 82 (1963), 125-132.

[4] K. Doerk and T. Hawkes, Finite Soluble Groups, Walter de Gruyter, Berlin, 1992.

[5] Y. Gao and X. Li, *On NH-embedded subgroups of finite groups,* J. Algebra Appl., 21(10) (2022), 2250200 (11 pp).

[6] B. Huppert, Endliche Gruppen, Springer-Verlag, Berlin, 1967.

[7] I. M. Isaacs, *Semipermutable* π*-subgroups,* Arch. Math. (Basel), 102(1) (2014), 1-6.

[8] O. H. Kegel, *Sylow-Gruppen und Subnormalteiler endlicher Gruppen,* Math. Z., 78 (1962), 205-221.

[9] S. Li, Z. Shen, J. Liu and X. Liu, *The influence of SS-quasinormality of some subgroups on the structure of finite groups,* J. Algebra, 319(10) (2008), 4275-4287.

[10] Y. Li, S. Qiao, N. Su and Y. Wang, *On weakly s-semipermutable subgroups of finite groups,* J. Algebra, 371 (2012), 250-261.

[11] P. Schmid, *Subgroups permutable with all Sylow subgroups,* J. Algebra, 207(1) (1998), 285-293.

[12] Q. Zhang and L. Wang, *The influence of s-semipermutable subgroups on the structure of finite groups,* Acta Math. Sinica (Chinese Ser.), 48 (2005), 81-88.

Weicheng Zheng and **Liang Cui**
School of Mathematics and Computing Science
Guilin University of Electronic Technology
Guilin, Guangxi, 541002, P.R. China
e-mails: weicheng10051123@163.com (W. Zheng)
cl15135518972@163.com (L. Cui)

Wei Meng (Corresponding Author)
a. School of Mathematics and Computing Science
Guilin University of Electronic Technology
Guilin, Guangxi, 541002, P.R. China
b. Center for Applied Mathematics of Guangxi (GUET)
Guilin, Guangxi, 541002, P.R. China
e-mail: mlwhappyhappy@163.com

Jiakuan Lu
School of Mathematics and Statistics
Guangxi Normal University
Guilin, Guangxi, 541004, P.R. China
e-mail: jklu@gxnu.edu.cn