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# BOUNDS FOR THE MAXIMUM EIGENVALUES OF THE FIBONACCI-FRANK AND LUCAS-FRANK MATRICES

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Abstract. Frank matrix is one of the popular test matrices for eigenvalue routines because it has well-conditioned and poorly conditioned eigenvalues. In this paper, we investigate the bounds for the maximum eigenvalues of the special cases of the generalized Frank matrices which are called Fibonacci-Frank and Lucas-Frank matrices. Then, we obtain the Euclidean norms and the upper bounds for the spectral norms of these matrices.

## 1. Introduction

The Fibonacci and Lucas number sequences which are the most famous integer sequences, are defined by the recurrence relations ($n \geq 1$) [8]

$$f_{n+1} = f_n + f_{n-1} \quad \text{with} \quad f_0 = 0, f_1 = 1 \tag{1}$$

and

$$l_{n+1} = l_n + l_{n-1} \quad \text{with} \quad l_0 = 2, l_1 = 1. \tag{2}$$

The Binet formulas for the Fibonacci and Lucas number sequences are

$$f_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad l_n = \alpha^n + \beta^n, \tag{3}$$

respectively, where $\alpha = \dfrac{1+\sqrt{5}}{2}$ and $\beta = \dfrac{1-\sqrt{5}}{2}$ [8]. Also, there are some summation formulas for these number sequences, for example [8, 22]

$$\sum_{i=1}^{n} f_i = f_{n+2} - 1, \quad \sum_{i=1}^{n} f_i^2 = f_n f_{n+1} \tag{4}$$

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and

$$\sum_{i=1}^{n} l_i = l_{n+2} - 3, \quad \sum_{i=1}^{n} l_i^2 = l_n l_{n+1} - 2. \tag{5}$$

Matrix theory plays an important role in mathematics, engineering and many other sciences because the matrices are very useful tool to solve multidimensional equation systems. A matrix may be assigned numerical items in various ways, for example the determinant, trace, eigenvalues, singular values, spectral radius, matrix norm, etc. Norms for matrices are used to measure the "sizes" of the matrices, have an importance in the matrix theory. Due to the various applications of the Fibonacci and Lucas number sequences, there have been many studies on the norms of the special matrices with entries of the Fibonacci and Lucas numbers [1,2,7,15,17–20]. The Euclidean (Frobenius) and spectral norm of an $m \times n$ matrix $A$ are defined as

$$\|A\|_F = \left[\sum_{i=1}^{m}\sum_{j=1}^{n} \mid a_{ij} \mid^2\right]^{\frac{1}{2}} \quad \text{and} \quad \|A\|_2 = \sqrt{\max_{1 \le i \le n} \lambda_i \left(A^H A\right)}, \tag{6}$$

respectively, where $A^H$ is the conjugate transpose of the matrix $A$ and $\lambda_i \left(A^H A\right)$'s are the eigenvalues of $A^H A$ [6]. The maximum row length norm $r_1(A)$ and the maximum column length norm $c_1(A)$ of any matrix $A$ are defined by

$$r_1(A) = \max_i \sqrt{\sum_j \mid a_{ij} \mid^2} \quad \text{and} \quad c_1(A) = \max_j \sqrt{\sum_i \mid a_{ij} \mid^2}, \tag{7}$$

respectively [6]. Moreover, for any $m \times n$ matrices $A = [a_{ij}]$, $B = [b_{ij}]$ and $C = [c_{ij}]$, if $A = B \circ C$, then

$$\|A\|_2 \le r_1(B)\, c_1(C), \tag{8}$$

where $B \circ C$ is the Hadamard product of the matrices $B$ and $C$, which is defined by $B \circ C = [b_{ij} c_{ij}]$ [6].

Frank [3] defined the matrix of order $n$

$$F_n = \begin{bmatrix} n & n-1 & 0 & 0 & \dots & 0 & 0 \\ n-1 & n-1 & n-2 & 0 & \dots & 0 & 0 \\ n-2 & n-2 & n-2 & n-3 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 2 & 2 & 2 & 2 & \dots & 2 & 1 \\ 1 & 1 & 1 & 1 & \dots & 1 & 1 \end{bmatrix}, \tag{9}$$

which is called Frank matrix. The elements of the Frank matrix $F_n = [g_{ij}]$ are characterized by the formula

$$g_{ij} = \begin{cases} n + 1 - \max(i, j), & i > j - 2 \\ 0, & \text{orherwise.} \end{cases} \tag{10}$$

Hake [5] investigated the determinant, inverse, $LU$-decomposition and characteristic polynomials of the matrix $F_n$. Because of its well conditioned and poorly conditioned eigenvalues, the Frank matrix is one of the popular test matrices for eigenvalue routines. As a consequence of Sturm's Theorem, all eigenvalues of the matrix $F_n$ are real and positive [5]. Varah [23] gave a generalization of the Frank matrix and computed its eigensystem. The generalized Frank matrix $F_{a_n}$ is defined as

$$F_{a_n} = \begin{bmatrix} a_n & a_{n-1} & 0 & 0 & \dots & 0 & 0 \\ a_{n-1} & a_{n-1} & a_{n-2} & 0 & \dots & 0 & 0 \\ a_{n-2} & a_{n-2} & a_{n-2} & a_{n-3} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_2 & a_2 & a_2 & a_2 & \dots & a_2 & a_1 \\ a_1 & a_1 & a_1 & a_1 & \dots & a_1 & a_1 \end{bmatrix}, \tag{11}$$

where $a = (a_1, a_2, a_3, \dots, a_n)$ is a finite sequence with any $a_i$ real numbers [12]. The elements of the generalized Frank matrix $F_{a_n} = \left[(f_a)_{ij}\right]$ are characterized by

$$(f_a)_{ij} = \begin{cases} a_{n+1-\max(i,j)}, & i > j-2 \\ 0, & \text{otherwise.} \end{cases} \tag{12}$$

It is clear that for $a_i = i, (i = 1, 2, \dots n)$ the generalized Frank matrix turns into the well-known Frank matrix. The authors investigated some properties of the matrix $F_{a_n}$ and presented that the set of all $n \times n$ generalized Frank matrices is an $n$-dimensional vector space. They obtained the characteristic polynomials of the matrix $F_{a_n}$ as

$$P_n(\lambda) = (\lambda - a_n + a_{n-1}) P_{n-1}(\lambda) - a_{n-1}\lambda P_{n-2}(\lambda), \tag{13}$$

with the initial conditions

$$P_1(\lambda) = \lambda - a_1 \quad \text{and} \quad P_2(\lambda) = \lambda^2 - (a_1 + a_2)\lambda + a_1 a_2 - a_1^2.$$

The Sturm's Theorem gives the exact number of zeros in an interval for any polynomial without multiple zeros, is used for computing the eigenvalues of symmetric or tridiagonal matrices [4,9,16,21,24]. According to the Sturm's Theorem, if the sequence $P_0(x), P_1(x), \dots, P_n(x)$ has the Sturm sequence properties on $(a, b)$ and $\alpha, \beta$ $(\alpha < \beta)$ are any numbers in $(a, b)$, then $P_n(x)$ has exactly $c(\beta) - c(\alpha)$ different zeros in the interval $(\alpha, \beta)$, where $c(\alpha)$ denotes the number of changes in sign of consecutive members of the sequence $P_0(\alpha), P_1(\alpha), \dots, P_n(\alpha)$ [4]. Mersin and Bahşi [11] showed that the characteristic polynomial of the generalized Frank matrix $F_{a_n}$ is the form of the Strum sequence for the positive and strictly increasing (or negative and strictly decreasing) sequence $\{a_n\}$. They obtained all eigenvalues of the matrix $F_{a_n}$ are different and positive, also the eigenvalues of the matrices $F_{a_i}$ and $F_{a_{i-1}}$ are interlaced for $1 \leq i \leq n$, by considering the Sturm's Theorem.

Moreover, as a conclusion of the Sturm's Theorem, for the positive and strictly increasing sequence $\{a_n\}$, the inequalities

$$\lambda_n < a_1 \quad \text{and} \quad a_n < \lambda_1 \tag{14}$$

are hold, where $\lambda_n$ and $\lambda_1$ are the minimum and the maximum eigenvalues of $F_{a_n}$ for $n \geq 2$, respectively [11].

As the special forms of the generalized Frank matrices, Fibonacci-Frank matrix $F_{f_n}$ and Lucas-Frank matrix $F_{l_n}$ are defined as

$$F_{f_n} = \begin{bmatrix} f_n & f_{n-1} & 0 & 0 & \dots & 0 & 0 \\ f_{n-1} & f_{n-1} & f_{n-2} & 0 & \dots & 0 & 0 \\ f_{n-2} & f_{n-2} & f_{n-2} & f_{n-3} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ f_2 & f_2 & f_2 & f_2 & \dots & f_2 & f_1 \\ f_1 & f_1 & f_1 & f_1 & \dots & f_1 & f_1 \end{bmatrix} \tag{15}$$

and

$$F_{l_n} = \begin{bmatrix} l_n & l_{n-1} & 0 & 0 & \dots & 0 & 0 \\ l_{n-1} & l_{n-1} & l_{n-2} & 0 & \dots & 0 & 0 \\ l_{n-2} & l_{n-2} & l_{n-2} & l_{n-3} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ l_2 & l_2 & l_2 & l_2 & \dots & l_2 & l_1 \\ l_1 & l_1 & l_1 & l_1 & \dots & l_1 & l_1 \end{bmatrix}, \tag{16}$$

where $f_n$ and $l_n$ are the ordinary Fibonacci and Lucas numbers [10]. The elements of the matrices $F_{f_n} = [f_{ij}]$ and $F_{l_n} = [l_{ij}]$ are

$$f_{ij} = \begin{cases} f_{n+1-\max(i,j)}, & i > j-2 \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad l_{ij} = \begin{cases} l_{n+1-\max(i,j)}, & i > j-2 \\ 0, & \text{otherwise.} \end{cases} \tag{17}$$

Since the determinant of the matrix in equation (15) is zero, Fibonacci-Frank matrix $F_{f_n}$ is used as

$$F_{f_n} = \begin{bmatrix} f_{n+1} & f_n & 0 & 0 & \dots & 0 & 0 \\ f_n & f_n & f_{n-1} & 0 & \dots & 0 & 0 \\ f_{n-1} & f_{n-1} & f_{n-1} & f_{n-2} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ f_3 & f_3 & f_3 & f_3 & \dots & f_3 & f_2 \\ f_2 & f_2 & f_2 & f_2 & \dots & f_2 & f_2 \end{bmatrix} \tag{18}$$

in [10].

We note that $F_{f_n}$ will represent the matrix given by (18), throughout the paper. The determinants, inverses, $LU$-decompositions and characteristic polynomials of the matrices $F_{f_n}$ and $F_{l_n}$ are examined in [10]. The characteristic polynomial of the Fibonacci-Frank matrix $F_{f_n}$ is obtained from equation (13) as

$$P_n(\lambda) = (f_{n-1} - \lambda) P_{n-1}(\lambda) - f_n \lambda P_{n-2}(\lambda), \tag{19}$$

with the initial conditions $P_0(\lambda) = 1$, $P_1(\lambda) = 1 - \lambda$, $P_2(\lambda) = \lambda^2 - 3\lambda + 1$, and the characteristic polynomial of the Lucas-Frank matrix $F_{l_n}$ is

$$Q_n(\mu) = (l_{n-2} - \mu) Q_{n-1}(\mu) - l_{n-1} \mu Q_{n-2}(\mu), \tag{20}$$

with $Q_0(\mu) = 1$, $Q_1(\mu) = 1 - \mu$, $Q_2(\mu) = \mu^2 - 4\mu + 2$. The characteristic polynomials in equations (19) and (20) have the properties of the Sturm sequences [11].

In this paper, firstly we obtain the number of the eigenvalues of the matrices $F_{f_n}$ and $F_{l_n}$ in the interval $(0, 1)$. We examine the bounds for the maximum eigenvalues of the matrices $F_{f_n}$ and $F_{l_n}$. Then, we present the Euclidean norm and the upper bounds for the spectral norms of these matrices. Additionally, we give an example to illustrate our results.

## 2. Main Results

**Lemma 1.** *Let the characteristic polynomials of the Fibonacci-Frank matrix $F_{f_n}$ and Lucas-Frank matrix $F_{l_n}$ be $P_n(\lambda)$ and $Q_n(\mu)$, respectively. Then, for the value of $\lambda = \mu = 1$, we have*

(i) $P_n(1) < 2P_{n-1}(1)$, *for* $n \geq 6$,
(ii) $Q_n(1) > 3Q_{n-1}(1)$, *for* $n \geq 3$.

*Proof.* (i) We use the induction method on $n$. Since $P_6(1) = 0 < 2P_5(1) = 12$, the result is true for $n = 6$. Suppose that the result is true for $n = k > 6$. Then,

$$P_k(1) < 2P_{k-1}(1). \tag{21}$$

Hence, we have

$$0 > P_7(1) > P_8(1) > P_9(1) > \ldots > P_{k-1}(1). \tag{22}$$

For $n = k + 1$,

$$\begin{aligned} P_{k+1}(1) - 2P_k(1) &= (f_k - 1) P_k(1) - f_{k+1} P_{k-1}(1) - 2P_k(1) \\ &= f_k P_k(1) - 3P_k(1) - (f_k + f_{k-1}) P_{k-1}(1) \\ &= f_k (P_k(1) - P_{k-1}(1)) - f_{k-1} P_{k-1}(1) - 3P_k(1) \\ &< f_k P_{k-1}(1) - f_{k-1} P_{k-1}(1) - 3P_k(1) \\ &= f_{k-2} P_{k-1}(1) - 3P_k(1) \\ &< f_{k-2} P_{k-1}(1) - 6P_{k-1}(1) \\ &< (f_{k-2} - 6) P_{k-1}(1) \\ &< 0. \end{aligned}$$

Hence,

$$P_{k+1}(1) < 2P_k(1). \tag{23}$$

(ii) The proof is similar to the proof of (i).

□

**Theorem 1.** *The number of the eigenvalues of the Fibonacci-Frank matrix $F_{f_n}$ in the interval $(0,1)$ is three for $n \geq 7$.*

*Proof.* Considering the Sturm's Theorem, we must show that $c(1) - c(0) = 3$, where $c(x)$ denotes the number of changes in sign of consecutive members of the sequence in equation (19), for $n \geq 7$.

TABLE 1. The number of sign changes of $P_{i\leq 7}(\lambda)$ for $\lambda = 0$, $\lambda = 1$

| Characteristic polynomials $P_i(\lambda)$ for $i \leq 7$ | Sign of $P_i(\lambda)$ for $\lambda = 0$ | Sign of $P_i(\lambda)$ for $\lambda = 1$ |
|---|---|---|
| $P_0(\lambda) = 1$ | + | + |
| $P_1(\lambda) = 1 - \lambda$ | + | 0 |
| $P_2(\lambda) = \lambda^2 - 3\lambda + 1$ | + | − |
| $P_3(\lambda) = -\lambda^3 + 6\lambda^2 - 6\lambda + 1$ | + | 0 |
| $P_4(\lambda) = \lambda^4 - 11\lambda^3 + 27\lambda^2 - 16\lambda + 2$ | + | + |
| $P_5(\lambda) = -\lambda^5 + 19\lambda^4 - 90\lambda^3 + 127\lambda^2 - 55\lambda + 6$ | + | + |
| $P_6(\lambda) = \lambda^6 - 32\lambda^5 + 273\lambda^4 - 793\lambda^3 + 818\lambda^2 - 297\lambda + 30$ | + | 0 |
| $P_7(\lambda) = -\lambda^7 + 53\lambda^6 - 776\lambda^5 + 4147\lambda^4 - 8813\lambda^3 + 7756\lambda^2 - 2484\lambda + 240$ | + | − |
| Number of sign changes | $c_7(0) = 0$ | $c_7(1) = 3$ |

From Table 1, we have $c_7(0) = 0$ and $c_7(1) = 3$. Then, $P_7(\lambda)$ has $c_7(1) - c_7(0) = 3$ eigenvalues in the interval $(0,1)$. The eigenvalues of the matrix $F_{f_7}$ are $\lambda_1 = 33.108$, $\lambda_2 = 11.495$, $\lambda_3 = 4.834$, $\lambda_4 = 2.083$, $\lambda_5 = 0.882$, $\lambda_6 = 0.433$, $\lambda_7 = 0.164$. Then, the eigenvalues in the interval $(0,1)$ are $\lambda_5$, $\lambda_6$ and $\lambda_7$, so it is clear that our result is correct for $n = 7$. As it seen in Table 1, $P_7(1) < 0$. From Lemma 1, we have $P_n(1) < 2P_{n-1}(1)$, then $P_n(1) < 0$ for $n > 7$. That is, there is no sign change of $P_n(1)$ for $n > 7$. Hence, $c_n(1) = 3$ is true for $n > 7$. To complete the proof we must show that $c_n(0) = 0$ is true for $n > 7$. From the initial condition of the recurrence relation in equation (19), we have $P_1(0) = 1$. Considering the recurrence relation in equation (19), we have

$$\begin{aligned} P_n(0) &= f_{n-1}P_{n-1}(0) \\ &= f_{n-1}f_{n-2}P_{n-2}(0) \\ &\vdots \\ &= f_{n-1}f_{n-2}f_{n-3}\cdots f_1P_1(0) \\ &> 0. \end{aligned}$$

That is, there is no sign change of $P_n(0)$ for any positive integer $n$. Thus, $c_n(0) = 0$ for $n \geq 1$. Hence, we have $c_n(1) - c_n(0) = 3$ for $n \geq 7$. That is, the number of the eigenvalues of the Fibonacci-Frank matrix $F_{f_n}$ in the interval $(0,1)$ is three for $n \geq 7$, as desired. □

**Theorem 2.** *The number of the eigenvalues of the Lucas-Frank matrix $F_{l_n}$ in the interval $(0,1)$ is two for $n \geq 4$.*

*Proof.* The proof is similar to the proof of Theorem 1. □

**Lemma 2.** *The equalities for the Fibonacci-Frank matrix $F_{f_n}$*

(i) $trF_{f_n} = f_{n+3} - 2,$

(ii) $trF_{f_n}^2 = 3\left(f_n f_{n+1} - 1\right) + f_{n+1}^2,$

(iii) $\sum_{i=1}^{n}\left(\lambda_i - \frac{trF_{f_n}}{n}\right)^2 = 3\left(f_n f_{n+1} - 1\right) + f_{n+1}^2 - \frac{1}{n}\left(f_{n+3} - 2\right)^2$

*are valid and also the equalities for the Lucas-Frank matrix $F_{l_n}$*

(i') $trF_{l_n} = l_{n+2} - 3,$

(ii') $trF_{l_n}^2 = 3\left(l_n l_{n+1} - 2\right) - 2l_n^2,$

(iii') $\sum_{i=1}^{n}\left(\mu_i - \frac{trF_{l_n}}{n}\right)^2 = 3\left(l_n l_{n+1} - 2\right) - 2l_n^2 - \frac{1}{n}\left(l_{n+2} - 3\right)^2$

*are hold, where $\lambda_i$'s and $\mu_i$'s $(i = 1, 2, \ldots, n)$ are the eigenvalues of the matrices $F_{f_n}$ and $F_{l_n}$, respectively.*

*Proof.* (i) For the Fibonacci-Frank matrix $F_{f_n} = [f_{ij}]$, we have

$$trF_{f_n} = \sum_{i=2}^{n+1} f_i = \sum_{i=1}^{n} f_i + f_{n+1} - f_1 = f_{n+2} + f_{n+1} - 2 = f_{n+3} - 2.$$

(ii) For the matrix $F_{f_n}^2 = \left[f_{ij}^{(2)}\right]$, we have

$$trF_{f_n}^2 = \sum_{i=1}^{n} f_{ii}^{(2)} = \sum_{i=1}^{n}\left(\sum_{k=1}^{n} f_{ik} f_{ki}\right).$$

From the following equalities

$$f_{ik} = \begin{cases} f_{n+2-\max(i,k)}, & i > k-2 \\ 0, & \text{otherwise} \end{cases}$$

and

$$f_{ki} = \begin{cases} f_{n+2-\max(k,i)}, & k > i-2 \\ 0, & \text{otherwise,} \end{cases}$$

we can say that if $\mid i-k \mid < 2$, then $f_{ik} f_{ki} \neq 0$, otherwise $f_{ik} f_{ki} = 0$. $\mid i-k \mid < 2$ yields $i = k$, $i = k-1$ and $i = k+1$ for $1 < i < n$. Since,

${f_{11}}^{(2)} = f_{n+1}^2 + f_n^2$ and ${f_{nn}}^{(2)} = f_2^2 + f_2^2$, we have

$$\begin{aligned}
trF_{f_n}^2 &= \sum_{i=1}^{n} {f_{ii}}^{(2)} = f_{n+1}^2 + f_n^2 + \sum_{i=2}^{n-1}\left(\sum_{k=i-1}^{i+1} f_{ik}f_{ki}\right) + f_2^2 + f_2^2 \\
&= \sum_{i=2}^{n-1}\left(2f_{n+2-i}^2 + f_{n+1-i}^2\right) + f_{n+1}^2 + f_n^2 + 2f_2^2 \\
&= 2\sum_{i=2}^{n-1} f_{n+2-i}^2 + \sum_{i=2}^{n-1} f_{n+1-i}^2 + f_{n+1}^2 + f_n^2 + 2f_2^2 \\
&= 2\left(f_n^2 + f_{n-1}^2 + \ldots + f_3^2\right) + \left(f_{n-1}^2 + f_{n-2}^2 + \ldots + f_3^2 + f_2^2\right) \\
&\qquad + f_n^2 + f_{n+1}^2 + 2f_2^2 \\
&= 2\sum_{i=2}^{n} f_i^2 + \sum_{i=2}^{n} f_i^2 + f_{n+1}^2 \\
&= 3\left(f_n f_{n+1} - 1\right) + f_{n+1}^2.
\end{aligned}$$

(iii) By using (i) and (ii), we get

$$\begin{aligned}
\sum_{i=1}^{n}\left(\lambda_i - \frac{trF_{f_n}}{n}\right)^2 &= \sum_{i=1}^{n}(\lambda_i)^2 - 2\frac{trF_{f_n}}{n}\sum_{i=1}^{n}\lambda_i + \sum_{i=1}^{n}\left(\frac{trF_{f_n}}{n}\right)^2 \\
&= \sum_{i=1}^{n}(\lambda_i)^2 - 2\frac{(trF_{f_n})^2}{n} + n\left(\frac{trF_{f_n}}{n}\right)^2 \\
&= 3\left(f_n f_{n+1} - 1\right) + f_{n+1}^2 - \frac{1}{n}\left(f_{n+3} - 2\right)^2.
\end{aligned}$$

The proofs of (i'), (ii') and (iii') are similar to the proofs of (i), (ii) and (iii), respectively. □

**Theorem 3.** *There are the following inequalities for the Fibonacci-Frank matrix* $F_{f_n}$ *and Lucas-Frank matrix* $F_{l_n}$ *whose eigenvalues are ordered as* $\lambda_1 > \lambda_2 > \cdots > \lambda_n$ *and* $\mu_1 > \mu_2 > \cdots > \mu_n$*, respectively*

(i) $$f_{n+1} \le \lambda_1 \le \sqrt{\left(1 - \frac{1}{n}\right)\left(3\left(f_n f_{n+1} - 1\right) + f_{n+1}^2 - \frac{1}{n}\left(f_{n+3} - 2\right)^2\right)} + \frac{1}{n}\left(f_{n+3} - 2\right),$$

(ii) $$l_n \le \mu_1 \le \sqrt{\left(1 - \frac{1}{n}\right)\left(3\left(l_n l_{n+1} - 2\right) - 2{l_n}^2 - \frac{1}{n}\left(l_{n+2} - 3\right)^2\right)} + \frac{1}{n}\left(l_{n+2} - 3\right).$$

*Proof.* (i) The equation

$$\lambda_1 - \frac{trF_{f_n}}{n} = -\sum_{i=2}^{n}\left(\lambda_i - \frac{trF_{f_n}}{n}\right) \tag{24}$$

is clearly holds for the Fibonacci-Frank matrix $F_{f_n}$. Then, we can write the inequality

$$|\lambda_1 - \frac{trF_{f_n}}{n}| \leq \sum_{i=2}^{n} |\lambda_i - \frac{trF_{f_n}}{n}|. \tag{25}$$

By means of [13, 14], we have for the sequence of positive real numbers $q = (q_i)$ and the sequences of non-negative real numbers with similar monocity $a = (a_i)$ and $b = (b_i)$, $(i = 1, 2, \ldots, m)$

$$\sum_{i=1}^{m} q_i \sum_{i=1}^{m} q_i a_i b_i \geq \sum_{i=1}^{m} q_i a_i \sum_{i=1}^{m} q_i b_i. \tag{26}$$

Moreover, if $a = (a_i)$ and $b = (b_i)$ has opposite monocity, then the sense of the inequality in (26) reverses [13, 14].

If equation (26) is applied to the right hand side of the inequality (25) by using as $a_i = \dfrac{1}{\mid \lambda_i - \frac{trF_{f_n}}{n} \mid}$ and $b_i = q_i = \mid \lambda_i - \dfrac{trF_{f_n}}{n} \mid$, then we get

$$\mid \lambda_1 - \frac{trF_{f_n}}{n} \mid \leq \sum_{i=2}^{n} \mid \lambda_i - \frac{trF_{f_n}}{n} \mid \leq \sqrt{(n-1) \sum_{i=2}^{n} \mid \lambda_i - \frac{trF_{f_n}}{n} \mid^2}.$$

Hence, we have

$$\begin{aligned}
\left(\lambda_1 - \frac{trF_{f_n}}{n}\right)^2 &\leq (n-1)\left(\sum_{i=1}^{n}\left(\lambda_i - \frac{trF_{f_n}}{n}\right)^2 - \left(\lambda_1 - \frac{trF_{f_n}}{n}\right)^2\right), \\
\frac{n}{n-1}\left(\lambda_1 - \frac{trF_{f_n}}{n}\right)^2 &\leq \sum_{i=1}^{n}\left(\lambda_i - \frac{trF_{f_n}}{n}\right)^2, \\
\lambda_1 - \frac{trF_{f_n}}{n} &\leq \sqrt{\left(1 - \frac{1}{n}\right)\sum_{i=1}^{n}\left(\lambda_i - \frac{trF_{f_n}}{n}\right)^2}.
\end{aligned}$$

Then, by using Lemma 2 (iii),

$$\lambda_1 \leq \sqrt{\left(1 - \frac{1}{n}\right)\left(3\left(f_n f_{n+1} - 1\right) + f_{n+1}^2 - \frac{1}{n}\left(f_{n+3} - 2\right)^2\right)} + \frac{1}{n}\left(f_{n+3} - 2\right)$$

as desired. Considering equation (14), we have $f_{n+1} < \lambda_1$. This completes the proof.

(ii) The proof is similar to the proof of (i).

□

**Theorem 4.** *The Euclidean (Frobenius) norms of the Fibonacci-Frank matrix $F_{f_n}$ and Lucas-Frank matrix $F_{l_n}$ are*

(i) $\|F_{f_n}\|_F = \sqrt{\frac{1}{5}(3l_{2n+1}+l_{2n}) + f_{n+1}^2 - n - \frac{1}{2}((-1)^n+5)}$,

(ii) $\|F_{l_n}\|_F = \sqrt{2l_{2n-1}+l_{2n}+l_n^2 - 2n - \frac{1}{2}(5(-1)^n+11)}$,

*where $f_n$ and $l_n$ are the ordinary Fibonacci and Lucas numbers, respectively.*

*Proof.* (i) By using the Binet formulas for the Fibonacci numbers, we have

$$\begin{aligned}
\|F_{f_n}\|_F^2 &= \sum_{i=1}^{n-1}(n-i+2) f_{i+1}^2 + f_{n+1}^2 \\
&= \sum_{i=1}^{n-1}(n+2) f_{i+1}^2 - \sum_{i=1}^{n-1} i f_{i+1}^2 + f_{n+1}^2 \\
&= \frac{(n+2)}{5}\sum_{i=1}^{n-1}\left(\alpha^{i+1}-\beta^{i+1}\right)^2 - \frac{i}{5}\sum_{i=1}^{n-1}\left(\alpha^{i+1}-\beta^{i+1}\right)^2 + f_{n+1}^2 \\
&= \frac{n+2}{5}\sum_{i=1}^{n-1}\left(\alpha^2\left(\alpha^2\right)^i + \beta^2\left(\beta^2\right)^i - 2(-1)^{i+1}\right) \\
&\quad -\frac{i}{5}\sum_{i=1}^{n-1}\left(\alpha^2\left(\alpha^2\right)^i + \beta^2\left(\beta^2\right)^i - 2(-1)^{i+1}\right) + f_{n+1}^2.
\end{aligned}$$

Using the well known equalities

$$\sum_{i=1}^{n-1}\alpha^i = \frac{\alpha^n-\alpha}{\alpha-1} \quad \text{and} \quad \sum_{i=1}^{n-1} i\alpha^i = \frac{\alpha - n\alpha^n + (n-1)\alpha^{n+1}}{(\alpha-1)^2}, \tag{27}$$

we have

$$\begin{aligned}
\|F_{f_n}\|_F^2 = \frac{n+2}{5}&\left(\alpha^2\frac{\left(\alpha^2\right)^n-\alpha^2}{\alpha^2-1} + \beta^2\frac{\left(\beta^2\right)^n-\beta^2}{\beta^2-1} + 2\sum_{i=1}^{n-1}(-1)^i\right) \\
&-\frac{1}{5}\left(\alpha^2\left(\frac{\alpha^2-n\left(\alpha^2\right)^n+(n-1)\left(\alpha^2\right)^{n+1}}{\left(\alpha^2-1\right)^2}\right)\right. \\
&\left.+\beta^2\left(\frac{\beta^2-n\left(\beta^2\right)^n+(n-1)\left(\beta^2\right)^{n+1}}{\left(\beta^2-1\right)^2}\right) + 2\sum_{i=1}^{n-1} i(-1)^i\right) + f_{n+1}^2
\end{aligned}$$

$$
\begin{aligned}
&= \frac{n+2}{5}\left(\left(\alpha^{2n+1}+\beta^{2n+1}\right)-\left(\alpha^{3}+\beta^{3}\right)-\left((-1)^{n}+1\right)\right)\\
&\quad -\frac{1}{5}\left(\left(\alpha^{2}+\beta^{2}\right)-n\left(\alpha^{2n}+\beta^{2n}\right)+(n-1)\left(\alpha^{2n+2}+\beta^{2n+2}\right)\right.\\
&\qquad\qquad \left.-\frac{1+(2n-1)(-1)^{n}}{2}\right)+f_{n+1}^{2}\\
&= \frac{n+2}{5}\left(l_{2n+1}-l_{3}\right)-\frac{1}{5}\left(l_{2}-nl_{2n}+(n-1)\,l_{2n+2}\right)-\frac{(-1)^{n}}{2}\\
&\qquad\qquad -\frac{2n+3}{10}+f_{n+1}^{2}\\
&= \frac{1}{5}\left(3l_{2n+1}+l_{2n}\right)+f_{n+1}^{2}-n-\frac{1}{2}\left((-1)^{n}+5\right).
\end{aligned}
$$

Thus, desired result is obtained.

(ii) The proof is similar to the proof of (i).

□

**Theorem 5.** *There are the following upper bounds for the spectral norms of the Fibonacci-Frank matrix $F_{f_n}$ and Lucas-Frank matrix $F_{l_n}$*

(i) $\|F_{f_n}\|_2 \leq \sqrt{\left(f_{n+1}^2+1\right)\left(f_n^2+n-1\right)}$,

(ii) $\|F_{l_n}\|_2 \leq \sqrt{\left(l_n^2+1\right)\left(l_{n-1}^2+n-1\right)}$,

*where $f_n$ and $l_n$ are the ordinary Fibonacci and Lucas numbers, respectively.*

*Proof.* (i) By using the Hadamard product, the matrix $F_{f_n}$ can be written as

$$
F_{f_n} = \underbrace{\begin{bmatrix}
f_{n+1} & 1 & 0 & 0 & \dots & 0 & 0\\
f_n & f_n & 1 & 0 & \dots & 0 & 0\\
f_{n-1} & f_{n-1} & f_{n-1} & 1 & \dots & 0 & 0\\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\
f_3 & f_3 & f_3 & f_3 & \dots & f_3 & 1\\
f_2 & f_2 & f_2 & f_2 & \dots & f_2 & f_2
\end{bmatrix}}_{A} \circ \underbrace{\begin{bmatrix}
1 & f_n & 0 & 0 & \dots & 0 & 0\\
1 & 1 & f_{n-1} & 0 & \dots & 0 & 0\\
1 & 1 & 1 & f_{n-2} & \dots & 0 & 0\\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\
1 & 1 & 1 & 1 & \dots & 1 & f_2\\
1 & 1 & 1 & 1 & \dots & 1 & 1
\end{bmatrix}}_{B}.
$$

Considering the inequality $2f_n^2 < f_{n+1}^2$, which can be proven by the mathematical induction method, we have the maximum row length norm of the matrix $A = [a_{ij}]$ and maximum column length norm of the matrix $B = [b_{ij}]$ as

$$
r_1(A) = \max_i \sqrt{\sum_j |\, a_{ij}\,|^2} = \sqrt{f_{n+1}^2+1}, \tag{28}
$$

$$c_1(B) = \max_j \sqrt{\sum_i |\, b_{ij} \,|^2} = \sqrt{f_n^2 + n - 1}, \tag{29}$$

respectively. Then, by using equation (8), we have

$$\|F_{f_n}\|_2 \leq r_1(A)\, c_1(B) = \sqrt{\left(f_{n+1}^2 + 1\right)\left(f_n^2 + n - 1\right)}. \tag{30}$$

(ii) The proof is similar to the proof of (i).

□

Now, we give the following example to illustrate our results:

**Example 1.** *Consider the Fibonacci-Frank and Lucas-Frank matrices for $n = 8$. Then, the matrices are*

$$F_{f_8} = \begin{bmatrix} 34 & 21 & 0 & 0 & 0 & 0 & 0 & 0 \\ 21 & 21 & 13 & 0 & 0 & 0 & 0 & 0 \\ 13 & 13 & 13 & 8 & 0 & 0 & 0 & 0 \\ 8 & 8 & 8 & 8 & 5 & 0 & 0 & 0 \\ 5 & 5 & 5 & 5 & 5 & 3 & 0 & 0 \\ 3 & 3 & 3 & 3 & 3 & 3 & 2 & 0 \\ 2 & 2 & 2 & 2 & 2 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad F_{l_8} = \begin{bmatrix} 47 & 29 & 0 & 0 & 0 & 0 & 0 & 0 \\ 29 & 29 & 18 & 0 & 0 & 0 & 0 & 0 \\ 18 & 18 & 18 & 11 & 0 & 0 & 0 & 0 \\ 11 & 11 & 11 & 11 & 7 & 0 & 0 & 0 \\ 7 & 7 & 7 & 7 & 7 & 4 & 0 & 0 \\ 4 & 4 & 4 & 4 & 4 & 4 & 3 & 0 \\ 3 & 3 & 3 & 3 & 3 & 3 & 3 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

*The characteristic polynomials of the matrices $F_{f_8}$ and $F_{l_8}$ are*

$$\begin{aligned} P_8(\lambda) &= \lambda^8 - 87\lambda^7 + 2137\lambda^6 - 19968\lambda^5 + 79377\lambda^4 - 139303\lambda^3 + 106949\lambda^2 \\ &\qquad - 33162\lambda + 3120, \\ Q_8(\mu) &= \mu^8 - 120\mu^7 + 4054\mu^6 - 51792\mu^5 + 278231\mu^4 - 647740\mu^3 \\ &\qquad + 652566\mu^2 - 268188\mu + 33264 \end{aligned}$$

*and the eigenvalues of the matrices $F_{f_8}$ and $F_{l_8}$ are*

$$\begin{aligned} \lambda_1 &= 53.563, & \mu_1 &= 74.018, \\ \lambda_2 &= 18.591, & \mu_2 &= 25.689, \\ \lambda_3 &= 7.889, & \mu_3 &= 10.920, \\ \lambda_4 &= 3.753, & \mu_4 &= 5.232, \\ \lambda_5 &= 1.756, & \mu_5 &= 2.300, \\ \lambda_6 &= 0.851, & \mu_6 &= 1.009, \\ \lambda_7 &= 0.432, & \mu_7 &= 0.618, \\ \lambda_8 &= 0.164, & \mu_8 &= 0.214. \end{aligned}$$

*It is clear that $\lambda_6$, $\lambda_7$ and $\lambda_8$ are in the interval $(0, 1)$, then the matrix $F_{f_8}$ has three eigenvalues in the interval $(0, 1)$. Similarly, since $\mu_7$ and $\mu_8$ are in the interval $(0, 1)$, the matrix $F_{l_8}$ has two eigenvalues in the interval $(0, 1)$.*

*There are the following bounds for the maximum eigenvalues of the matrices* $F_{f_8}$ *and* $F_{l_8}$ *from Theorem 3*

$$f_9 \leq \lambda_1 \leq \sqrt{\left(1-\frac{1}{8}\right)\left(3\left(f_8 f_9-1\right)+f_9^2-\frac{1}{8}\left(f_{11}-2\right)^2\right)}+\frac{1}{8}\left(f_{11}-2\right),$$

$$34 \leq \lambda_1 = 53.563 \leq 56.210$$

*and*

$$l_8 \leq \mu_1 \leq \sqrt{\left(1-\frac{1}{8}\right)\left(3\left(l_8 l_9-2\right)-2{l_8}^2-\frac{1}{8}\left(l_{10}-3\right)^2\right)}+\frac{1}{8}\left(l_{10}-3\right),$$

$$47 \leq \mu_1 = 74.018 \leq 77.694.$$

*Considering Theorem 4, the Euclidean norms of the matrices* $F_{f_8}$ *and* $F_{l_8}$ *are*

$$\|F_{f_8}\|_F = \sqrt{\frac{1}{5}\left(3l_{17}+l_{16}\right)+f_9^2-8-\frac{1}{2}\left(\left(-1\right)^8+5\right)} = 61.065$$

*and*

$$\|F_{l_8}\|_F = \sqrt{2l_{15}+l_{16}+l_8^2-16-\frac{1}{2}\left(5\left(-1\right)^8+11\right)} = 84.380.$$

*By using Theorem 5 we have the following upper bounds for the spectral norms of the matrices* $F_{f_8}$ *and* $F_{l_8}$

$$\|F_{f_8}\|_2 = 56.911 \leq \sqrt{\left(f_9^2+1\right)\left(f_8^2+7\right)} = 719.955$$

*and*

$$\|F_{l_8}\|_2 = 78.643 \leq \sqrt{\left(l_8^2+1\right)\left(l_7^2+7\right)} = 1368.970.$$

In Example 1, we gave our results for Fibonacci-Frank matrix $F_{f_n}$ and Lucas-Frank matrix $F_{l_n}$ for n=8. The bounds we have obtained for the maximum eigenvalues of these matrices for increasing values of $n$ obtained from Theorem 3 are given in the following tables:

TABLE 2. The bounds for the maximum eigenvalues of the matrix $F_{f_n}$ according to the increasing values of $n$.

| | |
|---|---|
| $n = 2$ | $2 \leq \lambda_1 = 2.618 \leq 2.618$ |
| $n = 3$ | $3 \leq \lambda_1 = 4.791 \leq 4.828$ |
| $n = 4$ | $5 \leq \lambda_1 = 7.796 \leq 8.095$ |
| $n = 5$ | $8 \leq \lambda_1 = 12.654 \leq 13.130$ |
| $n = 6$ | $13 \leq \lambda_1 = 20.455 \leq 21.337$ |
| $n = 7$ | $21 \leq \lambda_1 = 33.108 \leq 34.654$ |
| $n = 8$ | $34 \leq \lambda_1 = 53.563 \leq 56.210$ |
| $n = 9$ | $55 \leq \lambda_1 = 86.672 \leq 91.153$ |
| $n = 10$ | $89 \leq \lambda_1 = 140.235 \leq 147.760$ |
| $n = 20$ | $10946 \leq \lambda_1 = 17247.848 \leq 18340.237$ |
| $n = 30$ | $1346269 \leq \lambda_1 = 2121345.008 \leq 2262287.634$ |
| $n = 40$ | $165580141 \leq \lambda_1 = 260908188.115 \leq 278631218.037$ |
| $n = 50$ | $20365011074 \leq \lambda_1 = 32089585793.157 \leq 34297378604.5$ |

TABLE 3. The bounds for the maximum eigenvalues of the matrix $F_{l_n}$ according to the increasing values of $n$.

| | |
|---|---|
| $n = 2$ | $3 \leq \mu_1 = 3.414 \leq 3.414$ |
| $n = 3$ | $4 \leq \mu_1 = 6.702 \leq 6.722$ |
| $n = 4$ | $7 \leq \mu_1 = 10.761 \leq 11.034$ |
| $n = 5$ | $11 \leq \mu_1 = 17.512 \leq 18.186$ |
| $n = 6$ | $18 \leq \mu_1 = 28.258 \leq 29.494$ |
| $n = 7$ | $29 \leq \mu_1 = 45.762 \leq 47.918$ |
| $n = 8$ | $47 \leq \mu_1 = 74.018 \leq 77.694$ |
| $n = 9$ | $76 \leq \mu_1 = 119.780 \leq 125.988$ |
| $n = 10$ | $123 \leq \mu_1 = 193.798 \leq 204.213$ |
| $n = 20$ | $15127 \leq \mu_1 = 23835.939 \leq 25345.591$ |
| $n = 30$ | $1860498 \leq \mu_1 = 2931626.699 \leq 3126404.622$ |
| $n = 40$ | $228826127 \leq \mu_1 = 360566248.032 \leq 385058873.003$ |
| $n = 50$ | $28143753123 \leq \mu_1 = 44346716881.237 \leq 47397811506.4$ |

According to Table 2 and Table 3, the bounds are quite close to the exact values of the maximum eigenvalues of the matrices $F_{f_n}$ and $F_{l_n}$. Also, the upper bounds are closer to the maximum eigenvalues than the lower bounds. Additionally, we give the following figures to better illustrate this result.

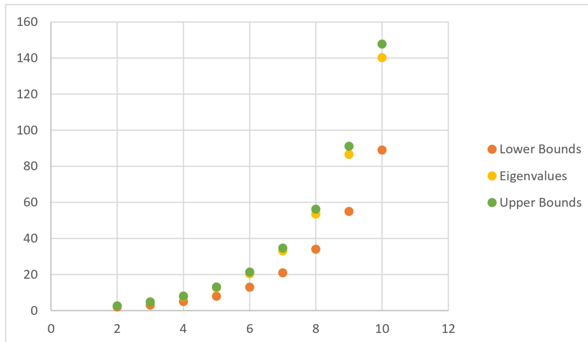


FIGURE 1. The graph of the maximum eigenvalues of the matrix $F_{f_n}$ and their lower and upper bounds for $n = 2, 3, 4, \ldots, 10$.

In Figure 1, the horizontal axis contains the values of $n$ from 2 to 10, and the vertical axis contains the maximum eigenvalues of the matrix $F_{f_n}$ corresponding to these values of $n$, as well as the values of its lower and upper bounds.

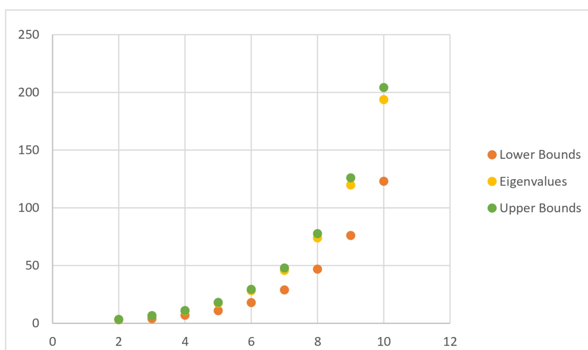


FIGURE 2. The graph of the maximum eigenvalues of the matrix $F_{l_n}$ and their lower and upper bounds for $n = 2, 3, 4, \ldots, 10$.

Similarly, in Figure 2, the values of $n$ between 2 and 10 are on the horizontal axis, and the maximum eigenvalue of the matrix $F_{l_n}$ corresponding to these $n$ values and their lower and upper bounds are on the horizontal axis.

As indicated by the graphs in Figures 1 and 2, the lower and upper bounds are very close to the maximum eigenvalues of the matrices $F_{f_n}$ and $F_{l_n}$ for small values of $n$. As the value of n increases, the distance between these bounds and the maximum eigenvalues widens. In this case, it is observed that the upper bounds remain closer to the maximum eigenvalues than the lower bounds.

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