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SHARP COEFFICIENT ESTIMATES FOR ϑ-SPIRALLIKE FUNCTIONS INVOLVING GENERALIZED q-INTEGRAL OPERATOR

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Abstract. The aim of this article is to identify a new subfamily of spirallike functions and then to demonstrate necessary and sufficient conditions, sharp coefficients estimates for functions in this subfamily.

1. Introduction

Stand by $\mathbb{A}$ the family of functions $f(\zeta) = \zeta + \sum_{k=2}^{\infty} a_k \zeta^k$ analytic in the open unit disk $\mathcal{D} = \{\zeta \in \mathbb{C} : |\zeta| < 1\}$ with the normalization condition $f(0) = 0 = f'(0) - 1$. A function $f \in \mathbb{A}$ is named univalent in $\mathcal{D}$ provided that it does not take the same value twice. Stand by $\mathbb{S}$ the subfamily of $\mathbb{A}$ involving univalent functions. For analytic functions f_1 and f_2 in $\mathcal{D}$, we ensure that f_1 is subordinate to f_2, expressed by $f_1 \prec f_2$, for a Schwarz function

$$\boldsymbol{\Lambda}(\zeta) = \sum_{k=1}^{\infty} \boldsymbol{\kappa}_k \zeta^k \quad \left(\boldsymbol{\Lambda}(0) = 0, |\boldsymbol{\Lambda}(\zeta)| < 1\right),$$

analytic in $\mathcal{D}$ such that $f_1(\zeta) = f_2(\boldsymbol{\Lambda}(\zeta))$ $(\zeta \in \mathcal{D})$.

Now, we shall deal with a subfamily of $\mathbb{S}$ which is of special interest in its own right, namely the spirallike functions.

For $-\infty < t < \infty$ and $\vartheta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, the logarithmic ϑ-spiral curve is expressed by $w = w_0 \exp(-e^{-i\vartheta} t)$, where w_0 is a nonzero complex number. We must mention here that 0-spirals are radial half-lines. For an analytic function, we can call it ϑ-spirallike provided that its range is ϑ-spirallike. Stand by $\mathcal{S}_\vartheta$ the family of ϑ-spirallike functions. Analytically, $f \in \mathbb{A}$ belongs to the family $\mathcal{S}_\vartheta$ iff

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$\Re\left(e^{i\vartheta}\frac{\zeta f'(\zeta)}{f(\zeta)}\right) > 0$ [17]. Libera [10] used this approach to ϑ-spirallike functions of order σ

$$\Re\left(e^{i\vartheta}\frac{\zeta f'(\zeta)}{f(\zeta)}\right) > \sigma\cos\vartheta$$

and asserted by $\mathcal{S}_\vartheta(\sigma)$. Clearly, $\mathcal{S}_\vartheta(\sigma) \subset \mathcal{S}_\vartheta$. Further, the general coefficient bounds for functions in $\mathcal{S}_\vartheta(\sigma)$ was proved:

$$|a_k| \leq \prod_{j=0}^{k-2}\left(\frac{\left|2\left(1-\sigma\right)e^{-i\vartheta}\cos\vartheta + j\right|}{j+1}\right) \quad \left(k \in \mathbb{N}\backslash\left\{1\right\}, \quad \mathbb{N} = \left\{1, 2, \cdots\right\}\right).$$

This result is sharp. Finding sharp results for functions belonging to the different families of analytic functions is of special interest because of the geometric properties of such functions [12], [14], [20], [21].

The age of quantum calculus (q-calculus) is as old as calculus and because of its applications to wider disciplines from physical sciences to social sciences, it was revived during the last three decades. The first study on the q-calculus dates back to 1908 [8]. On the other hand, q-calculus is connection with function theory. The study of q-calculus in Geometric Function Theory was partially provided by Srivastava [18]. This application is still among the most popular subject of many mathematicians today [1], [2], [3], [5], [7], [15], [19].

In the course of the paper, suppose $0 < q < 1$ and the definitions deal with the complex-valued function f.

The q-derivative of f expressed by [8]:

$$D_q f(\zeta) = \begin{cases} \dfrac{f(\zeta) - f(q\zeta)}{(1-q)\zeta}, & \zeta \neq 0 \\ \\ f'(0), & \zeta = 0 \end{cases}. \tag{1}$$

If f is differentiable at ζ, then $\lim_{q\to 1^-} D_q f(\zeta) = f'(\zeta)$.

The q-integral of f expressed by [9]:

$$\int_0^\zeta f(u)d_q u = \zeta(1-q)\sum_{k=0}^{\infty} q^k f(\zeta q^k),$$

provided the series converges.

Next, the q-gamma function is expressed by

$$\Gamma_q(u) = (1-q)^{1-u}\prod_{k=0}^{\infty}\frac{1-q^{k+1}}{1-q^{k+u}} \ (u > 0),$$

which has the following properties

$$\Gamma_q(u+1) = [u]_q\Gamma_q(u), \quad \Gamma_q(u+1) = [u]_q!, \tag{2}$$

where $u \in \mathbb{N}$ and

$$[u]_q! = \begin{cases} [u]_q[u-1]_q \dots [2]_q[1]_q, & u \geq 1 \\ 1, & u = 0. \end{cases}$$

If we set $q \to 1^-$, we find $\Gamma_q(u) \to \Gamma(u)$ [8].

The q-beta function

$$B_q(u,s) = \int_0^1 \zeta^{u-1}(1-q\zeta)_q^{s-1} d_q\zeta, \quad (u,s>0) \tag{3}$$

is the q-analogue of Euler's formula [9] with

$$B_q(u,s) = \frac{\Gamma_q(u)\Gamma_q(s)}{\Gamma_q(u+s)}, \tag{4}$$

Next, the q-binomial coefficients are expressed by [6]

$$\binom{k}{n}_q = \frac{[k]_q!}{[n]_q![k-n]_q!}. \tag{5}$$

In a recent study [11], the generalized q-integral operator $\chi^{\alpha}_{\beta,q}f : \mathbb{A} \to \mathbb{A}$ is expressed by

$$\chi^{\alpha}_{\beta,q}f(\zeta) = \binom{\alpha+\beta}{\beta}_q \frac{[\alpha]_q}{\zeta^{\beta}} \int_0^{\zeta} \left(1-\frac{qu}{\zeta}\right)_q^{\alpha-1} u^{\beta-1} f(u) d_q u \quad (\alpha>0, \beta>-1). \tag{6}$$

From (2), (3), (4) and (5), they arrive

$$\chi^{\alpha}_{\beta,q}f(\zeta) = \zeta + \sum_{k=2}^{\infty} \frac{\Gamma_q(\beta+n)\Gamma_q(\alpha+\beta+1)}{\Gamma_q(\alpha+\beta+n)\Gamma_q(\beta+1)} a_k \zeta^k. \tag{7}$$

For some special values, we find the following integral operators previously known.

(i) If $\alpha = 1$, the q-Bernardi integral operator $J_{\beta,q}f$ is obtained [13]

$$J_{\beta,q}f(\zeta) = \frac{[1+\beta]_q}{\zeta^{\beta}} \int_0^{\zeta} u^{\beta-1} f(u) d_q u = \sum_{k=1}^{\infty} \frac{[1+\beta]_q}{[n+\beta]_q} a_k \zeta^k.$$

(ii) If $\alpha = 1, q \to 1^-$, the Bernardi integral operator is obtained [4]

$$J_{\beta}f(\zeta) = \frac{1+\beta}{\zeta^{\beta}} \int_0^{\zeta} u^{\beta-1} f(u) du = \sum_{k=1}^{\infty} \frac{1+\beta}{n+\beta} a_k \zeta^k.$$

(iii) If $\alpha = 1, \beta = 0, q \to 1^-$, the Alexander integral operator is obtained [16]

$$J_0 f(\zeta) = \int_0^{\zeta} \frac{f(u)}{u} du = \zeta + \sum_{k=2}^{\infty} \frac{1}{n} a_k \zeta^k.$$

2. Main Results

Firstly, we introduce the new subfamily $SC^{\alpha}_{\beta,q}(\sigma,\nu)$ of $\vartheta-$spirallike functions inserting the function $\chi^{\alpha}_{\beta,q}f$.

Definition 1. *A function $f\in\mathbb{A}$ is in $SC^{\alpha}_{\beta,q}(\sigma,\nu)$ if*

$$\Re\left(e^{i\vartheta}\frac{\zeta\left(\chi^{\alpha}_{\beta,q}f(\zeta)\right)'}{\nu\zeta\left(\chi^{\alpha}_{\beta,q}f(\zeta)\right)'+(1-\nu)\chi^{\alpha}_{\beta,q}f(\zeta)}\right)>\sigma\cos\vartheta,$$

where $|\vartheta|<\frac{\pi}{2},\ 0\le\sigma<1,\alpha>0,\ \beta>-1,\ 0\le\nu\le1.$

Note that

1) Letting $q\to1^{-}$ and $\alpha=1$ in Definition 1, we arrive the class $SC^{\alpha}_{\beta,q}(\sigma,\nu):=SC_{\beta}(\sigma,\nu)$ involving Bernardi integral operator given in (ii).

2) Letting $q\to1^{-}$, $\alpha=1$ and $\beta=0$ in Definition 1, we arrive the class $SC^{\alpha}_{\beta,q}(\sigma,\nu):=SC(\sigma,\nu)$ involving Alexander integral operator given in (iii).

This paper deals with the new class $SC^{\alpha}_{\beta,q}(\sigma,\nu)$ of $\vartheta-$spirallike functions involving a generalized $q-$integral operator and its several properties.

Next, we get coefficient conditions and sharp bounds for functions in $SC^{\alpha}_{\beta,q}(\sigma,\nu)$.

Theorem 1. *Assume $\chi^{\alpha}_{\beta,q}f(\zeta)\neq0$ for $\zeta\in\mathcal{D}\setminus\{0\}$. Then, f is in $SC^{\alpha}_{\beta,q}(\sigma,\nu)$ if and only if*

$$\sum_{k=2}^{\infty}\Big[(k-1)(1+e^{2i\vartheta})(1-\sigma\nu+i(1-\nu)tan\vartheta)$$
$$+2(1-\sigma)e^{2i\vartheta}-(k-1)(1-e^{2i\vartheta})(1-\sigma)\nu\Big]\times\frac{\Gamma_q(\beta+k)\Gamma_q(\alpha+\beta+1)}{\Gamma_q(\alpha+\beta+k)\Gamma_q(\beta+1)}a_k\zeta^k\neq0.$$

Proof. Let us put

$$\Delta(\zeta)=\chi^{\alpha}_{\beta,q}f(\zeta)=\zeta+\sum_{k=2}^{\infty}X_k\zeta^k\quad(\zeta\in\mathcal{D}),$$

where $X_n=\frac{\Gamma_q(\beta+k)\Gamma_q(\alpha+\beta+1)}{\Gamma_q(\alpha+\beta+k)\Gamma_q(\beta+1)}a_k$ with $X_1=1$. Now, consider the function

$$\Sigma(\zeta)=\frac{\left(\frac{\zeta\Delta'(\zeta)}{\nu\zeta\Delta'(\zeta)+(1-\nu)\Delta(\zeta)}\right)e^{i\vartheta}sec\vartheta-itan\vartheta-\sigma}{1-\sigma}.$$

is an analytic, $\Sigma(0)=1$ and $\Re\Sigma(\zeta)>0$, then $f\in SC^{\alpha}_{\beta,q}(\sigma,\nu)$ iff

$$\Sigma(\zeta)\neq\frac{1-e^{2i\vartheta}}{1+e^{2i\vartheta}}$$

or, equivalently

$$\frac{e^{i\vartheta} sec\vartheta \zeta \Delta'(\zeta) - (\sigma + itan\vartheta)(\nu\zeta\Delta'(\zeta) + (1-\nu)\,\Delta(\zeta))}{(1-\sigma)(\nu\zeta\Delta'(\zeta) + (1-\nu)\,\Delta(\zeta))} \neq \frac{1-e^{2i\vartheta}}{1+e^{2i\vartheta}}.$$

Now, from the series expansion of $\Delta(\zeta)$, we arrive

$$\frac{\sum_{k=1}^{\infty}\left[(k-1)(1-\sigma\nu + i(1-\nu)tan\vartheta) + (1-\sigma)\right] X_k\zeta^k}{(1-\sigma)\sum_{k=1}^{\infty}(1+(k-1)\nu)X_k\zeta^k} \neq \frac{1-e^{2i\vartheta}}{1+e^{2i\vartheta}},$$

which yields for $\zeta \neq 0$

$$\sum_{k=2}^{\infty}\big[(k-1)(1+e^{2i\vartheta})(1-\sigma\nu + i(1-\nu)tan\vartheta) + 2(1-\sigma)e^{2i\vartheta} - (k-1)(1-e^{2i\vartheta})(1-\sigma)\nu\big]X_k\zeta^k \neq 0.$$

□

Theorem 2. *Let* $\chi^{\alpha}_{\beta,q}f(\zeta) \neq 0$ *for* $\zeta \in \mathcal{D}\setminus\{0\}$*. If* f *is in* $SC^{\alpha}_{\beta,q}(\sigma,\nu)$*, then*

$$|a_k| \leq \frac{\Gamma_q(\alpha+\beta+k)\Gamma_q(\beta+1)}{\Gamma_q(\beta+k)\Gamma_q(\alpha+\beta+1)(k-1)!\,(1-\nu)^{k-1}} \times \prod_{j=0}^{k-2}\left|j(1-\nu) + 2(1-\sigma)e^{i\vartheta}cos\vartheta(1+\nu j)\right|, \tag{8}$$

where $k \in \mathbb{N}\setminus\{1\}$ *with* $a_1 = 1$*. This result is sharp.*

Proof. Since $f \in SC^{\alpha}_{\beta,q}(\sigma,\nu)$, we can use a Schwarz function $\Lambda(\zeta)$ such that

$$\left(\frac{\zeta\left(\chi^{\alpha}_{\beta,q}f(\zeta)\right)'}{\nu\zeta\left(\chi^{\alpha}_{\beta,q}f(\zeta)\right)' + (1-\nu)\chi^{\alpha}_{\beta,q}f(\zeta)}\right)e^{i\vartheta}sec\vartheta - itan\vartheta = \frac{1+(1-2\sigma)\Lambda(\zeta)}{1-\Lambda(\zeta)}.$$

If we put the function $\Delta(\zeta)$, we find

$$\sum_{k=1}^{\infty}\left[ke^{i\vartheta}sec\vartheta - (1+itan\vartheta)(1+(k-1)\nu)\right]X_k\zeta^k = \left(\sum_{k=1}^{\infty}\left[ke^{i\vartheta}sec\vartheta + (1-2\sigma-itan\vartheta)(1+(k-1)\nu)\right]X_k\zeta^k\right)\Lambda(\zeta).$$

Now, for $k \in \mathbb{N}$, we can write

$$\sum_{k=1}^{m}\left[ke^{i\vartheta}sec\vartheta - (1+itan\vartheta)(1+(k-1)\nu)\right]X_k\zeta^k + \sum_{k=m+1}^{\infty}b_k\zeta^k = \left(\sum_{k=1}^{m-1}\left[ke^{i\vartheta}sec\vartheta + (1-2\sigma-itan\vartheta)(1+(k-1)\nu)\right]X_k\zeta^k\right)\Lambda(\zeta). \tag{9}$$

For $m = 2, 3, \cdots$, the LHS of (9) is convergent in $\mathcal{D}$. Since $|\Lambda(\zeta)| < 1$, it is easy to get by appealing to Parseval's Theorem that

$$\sum_{k=1}^{m-1} \left|ke^{i\vartheta} sec\vartheta + (1 - 2\sigma - itan\vartheta)(1 + (k-1)\nu)\right|^2 |X_k|^2$$

$$\geq \sum_{k=2}^{m} \left|ne^{i\vartheta} sec\vartheta - (1 + itan\vartheta)(1 + (k-1)\nu)\right|^2 |X_k|^2$$

or

$$\sum_{k=1}^{m-1} 4(1-\sigma)(1+(k-1)\nu)\left(k - \sigma(1+(k-1)\nu\right)|X_k|^2 \geq \frac{(m-1)^2(1-\nu)^2}{cos^2\vartheta}|X_m|^2, \tag{10}$$

where $X_1 = 1$. Now, we claim that

$$|X_k| \leq \frac{1}{(k-1)!\,(1-\nu)^{k-1}} \prod_{j=0}^{k-2} \left|j(1-\nu) + 2(1-\sigma)e^{i\vartheta}cos\vartheta(1+\nu j)\right|. \tag{11}$$

For $k = 2$, we find from (10)

$$|X_2| \leq \frac{2(1-\sigma)cos\vartheta}{1-\nu},$$

which is equivalent to (11). The equation (11) is found for larger k from (10) by the principle of the mathematical induction.

Fix $k, k \geq 3$ and let the equation (8) holds for $n = 2, 3, ..., k-1$. From (10), we arrive

$$|X_k|^2 \leq \frac{4(1-\sigma)cos^2\vartheta}{(k-1)^2\,(1-\nu)^2}\left\{1 - \sigma + \sum_{n=2}^{k-1} X(n,j,\sigma)\right\}, \tag{12}$$

where

$$X(n,j,\sigma) = \frac{(1+(n-1)\nu)(n-\sigma(n-1)\nu)}{((n-1)!\,(1-\nu)^{n-1})^2} \prod_{j=0}^{n-2}\left|j(1-\nu) + 2(1-\sigma)e^{i\vartheta}cos\vartheta(1+\nu j)\right|^2.$$

Now, we will indicate that the square of RSH of (11) is equal to RSH of (12), that is,

$$\prod_{j=0}^{k-2} \frac{\left|j(1-\nu) + 2(1-\sigma)e^{i\vartheta}cos\vartheta(1+\nu j)\right|^2}{((k-1)!\,(1-\nu)^{k-1})^2}$$

$$= \frac{4(1-\sigma)cos^2\vartheta}{(k-1)^2\,(1-\nu)^2}\left\{1 - \sigma + \sum_{n=2}^{k-1} X(n,j,\sigma)\right\} \tag{13}$$

for $k = 3, 4, \cdots$. After further calculations, we indicate that (13) is true for $k = 3$ and prove the claim. Assume the equation (13) is valid for all n, $3 < n \leq (k-1)$.

From (9) and (12), we find

$$\begin{aligned}
|X_k|^2 &\le \frac{4(1-\sigma)cos^2\vartheta}{(k-1)^2\,(1-\nu)^2}\left\{1-\sigma+\sum_{n=2}^{k-2}X(n,j,\sigma)+X(k-1,j,\sigma)\right\}\\
&\le \frac{4(1-\sigma)cos^2\vartheta}{(k-1)^2\,(1-\nu)^2}\times\Bigg\{1-\sigma+\sum_{n=2}^{k-2}\frac{(1+(n-1)\nu)(n-\sigma(n-1)\nu)}{((n-1)!\,(1-\nu)^{n-1})^2}\\
&\qquad\times\prod_{j=0}^{n-2}\left|j(1-\nu)+2(1-\sigma)e^{i\vartheta}cos\vartheta(1+\nu j)\right|^2\\
&\qquad+\frac{(1+(k-2)\nu)(k-1-\sigma(k-2)\nu)}{((k-2)!\,(1-\nu)^{k-2})^2}\\
&\qquad\times\prod_{j=0}^{k-3}\left|j(1-\nu)+2(1-\sigma)e^{i\vartheta}cos\vartheta(1+\nu j)\right|^2\Bigg\}\\
&=\frac{\prod_{j=0}^{k-3}\left|j(1-\nu)+2(1-\sigma)e^{i\vartheta}cos\vartheta(1+\nu j)\right|^2}{((k-2)!\,(1-\nu)^{k-2})^2}\\
&\quad\times\left\{\frac{(k-2)^2}{(k-1)^2}+\frac{4(1-\sigma)cos^2\vartheta(1+(k-2)\nu)(k-1-\sigma(k-2)\nu)}{(k-1)^2\,(1-\nu)^2}\right\}\\
&=\frac{\prod_{j=0}^{k-3}\left|j(1-\nu)+2(1-\sigma)e^{i\vartheta}cos\vartheta(1+\nu j)\right|^2}{((k-1)!\,(1-\nu)^{k-1})^2}\\
&\quad\times\left\{(k-2)^2\,(1-\nu)^2+4(1-\sigma)cos^2\vartheta(1+(k-2)\nu)(k-1-\sigma(k-2)\nu)\right\}
\end{aligned}$$

yields

$$|X_k|\le\frac{1}{((k-1)!\,(1-\nu)^{k-1})^2}\prod_{j=0}^{k-2}\left|j(1-\nu)+2(1-\sigma)e^{i\vartheta}cos\vartheta(1+\nu j)\right|^2.$$

Since

$$X_k=\frac{\Gamma_q(\beta+k)\Gamma_q(\alpha+\beta+1)}{\Gamma_q(\alpha+\beta+k)\Gamma_q(\beta+1)}a_k\quad(X_1=1),$$

we obtain the desired result.

To prove the estimate is sharp, we need following equality

$$\chi^{\alpha}_{\beta,q}f(\zeta)=\frac{\zeta}{(1+K\zeta)^{\frac{2(\sigma-1)e^{-i\vartheta}cos\vartheta}{K}}}$$

where $K=(1-\nu)-2\nu(1-\sigma)e^{-i\vartheta}cos\vartheta$. □

3. Conclusions

It is obvious that the link between q-calculus and Geometric Function Theory presents original and interesting results. Hence, in the present work, we use a generalized q-integral operator to establish a new subfamily $SC^{\alpha}_{\beta,q}(\sigma,\nu)$ of ϑ-spiralike functions. We also derive sharp upper bounds for Taylor Maclaurin coefficients of functions in this family.

Letting $\alpha = 1$, we have coefficients bounds for functions defined by q-Bernardi integral operator.

Corollary 1. *Let $J_{\beta,q}f(\zeta) \neq 0$ for $\zeta \in \mathcal{D} \setminus \{0\}$. If f is in $SC_{\beta,q}(\sigma,\nu)$, then*

$$|a_k| \leq \frac{[\beta+k]_q}{[\beta+1]_q (k-1)! (1-\nu)^{k-1}} \prod_{j=0}^{k-2} \left| j(1-\nu) + 2(1-\sigma)e^{i\vartheta} cos\vartheta (1+\nu j) \right|,$$

where $k \in \mathbb{N} \setminus \{1\}$ with $a_1 = 1$. This result is sharp.

Letting $\alpha = 1$ and $q \to 1^-$, we obtain following coefficients bounds for functions given by Bernardi integral operator.

Corollary 2. *Let $J_{\beta}f(\zeta) \neq 0$ for $\zeta \in \mathcal{D} \setminus \{0\}$. If f is in $SC_{\beta}(\sigma,\nu)$, then*

$$|a_k| \leq \frac{(\beta+k)}{(\beta+1)(k-1)! (1-\nu)^{k-1}} \prod_{j=0}^{k-2} \left| j(1-\nu) + 2(1-\sigma)e^{i\vartheta} cos\vartheta (1+\nu j) \right|,$$

where $k \in \mathbb{N} \setminus \{1\}$ with $a_1 = 1$. This result is sharp.

If $\alpha = 1$, $\beta = 0$ and $q \to 1^-$, we have following result for functions given in terms of Alexander integral operator.

Corollary 3. *Let $J_0 f(\zeta) \neq 0$ for $\zeta \in \mathcal{D} \setminus \{0\}$. If f is in $SC(\sigma,\nu)$, then*

$$|a_k| \leq \frac{k}{(k-1)! (1-\nu)^{k-1}} \prod_{j=0}^{k-2} \left| j(1-\nu) + 2(1-\sigma)e^{i\vartheta} cos\vartheta (1+\nu j) \right|,$$

where $k \in \mathbb{N} \setminus \{1\}$ with $a_1 = 1$. This result is sharp.

Our consequences are also applicable for various subfamilies of analytic functions.

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