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Quantum Difference Problem with Point Interaction

Yelda Aygar[1*], Güher Gülçehre Özbey[2] and Güler Başak Öznur[3]

[1*]*Department of Mathematics, Ankara University, 06100 Tandoğan, Ankara, Türkiye*
[2]*Middle Technical University, Üniversiteler Mahallesi, Dumlupınar Bulvarı, No:1, Ankara,06800, Türkiye*
[3]*Department of Mathematics, Gazi University, Teknik Okullar, Ankara, 06500, Türkiye*
[*]*Corresponding author*

Abstract

The main aim of this study is to examine the spectral analysis of q-difference equation with point interaction. We first find Jost solution and Jost function of this problem. Next, we establish the resolvent operator, continuous spectrum and discrete spectrum of the problem. At last, we demonstrate that the quantum boundary value problem with point interaction has finite number of eigenvalues and spectral singularities with finite multiplicities under certain conditions.

Keywords: *Jost solution; point interaction;q-difference equation; spectral analysis.*
2010 Mathematics Subject Classification: *[2010] 39A70, 47A10, 47A75.*

1. Introduction

Spectral analysis of discrete equations with spectral singularities have been investigated intensively by many mathematicians [1],[13], [14], [15], [16], [17], [18], [19], [25], [28], [33]. In recent years quantum calculus has received a lot of attention because it has wide applications in several mathematical areas such as orthogonal polynomials, number theory, combinatorics, fractal geometry, basic hypergeometric functions, mathematical physics [28], [37], [39]. Quantum calculus does not use the concept of classical limit. Hence the functions that are not differentiable in the classical sense can be quantum differentiable and it also provides some physical applications which consist some definitions and theorems of q-calculus, we refer the readers to [20], [21], [23], [26], [27]. Note that spectral analysis of quantum difference equations has been investigated by some researchers [2], [3], [4], [5], [6], [8], [9], [10], [38]. On the other hand, spectral analysis of q-difference equations with spectral singularities and point interactions has not been investigated yet. There are also only a few papers about the scattering analysis of q-difference equations with point interactions [7], [8], [9], [10], [11] which are related to scattering analysis. As is well known, difference equations with point interactions serve as basic models to study the dynamics of processes that are related to sudden changes in their states. These changes are so short as to be negligible when compared to the whole duration. In order to explain these processes mathematically, some conditions are applied to the discontinuous points. These points are called impulsive conditions or some other names such as jump conditions, interface conditions, point interactions and transmission conditions. The theory of difference equations with point interactions depends on the theory of differential equations with point interactions. They have been extensively studied in the past several years; see [12], [29], [32], [35], [36] and the references cited therein. In this paper, we shall investigate the spectral analysis of a discrete quantum Sturm-Liouville problem with point interactions in terms of Jost solution, Jost function, resolvent operator and continuous spectrum. We will present a condition that guarantees the finiteness of eigenvalues and spectral singularities of the problem with finite multiplicities. The set up of this paper is as follows: The next section features some introduction and notations on the q-difference problem with point interactions under consideration. In Section 3, we present some auxiliary results about the Jost solution and Jost function of the problem. Section 4 contains the resolvent operator and continuous spectrum which are needed in the proof of our main results in Section 5 with the results given in Section 3 together. The main results are related to the qualitative properties of eigenvalues and spectral singularities. In Section 6, we make some conclusions.

2. Preliminaries

Notation 2.1. *Let $q > 1$, $\mathbb{N}$ and $\mathbb{N}_0$ denote the set of the natural numbers and the set of nonnegative integers, respectively. Moreover, use the notations*

$$q^{\mathbb{N}} := \{q^n : n \in \mathbb{N}\} \quad q^{\mathbb{N}_0} := \{q^n : n \in \mathbb{N}_0\}.$$

Email addresses: yaygar@ankara.edu.tr (Yelda Aygar), gulcehre@metu.edu.tr (Güher Gülçehre Özbey), basakoznur@gazi.edu.tr (Güler Başak Öznur)

A q-difference equation is an equation that contains q-derivatives of a function defined on $q^{\mathbb{N}_0}$. The $q-$derivative of a function $f: q^{\mathbb{N}_0} \to \mathbb{C}$ is defined by

$$f^{\triangle}(t) := \frac{f(qt) - f(t)}{\mu(t)},$$

where $\mu(t) = (q-1)t$ is the graininess function [20].

Let $\ell_2\left(q^{\mathbb{N}_0}\right)$ be the Hilbert space of all complex-valued functions with the inner product

$$< f, g >_q := \sum_{t \in q^{\mathbb{N}_0}} \mu(t) f(t) \overline{g(t)}, \quad f, g : q^{\mathbb{N}_0} \to \mathbb{C}$$

and the norm

$$\|f\|_q := \left(\sum_{t \in q^{\mathbb{N}_0}} \mu(t) f(t)^2 \right)^{\frac{1}{2}}, \quad f : q^{\mathbb{N}_0} \to \mathbb{C}$$

for all $t \in q^{\mathbb{N}_0}$. Consider the following q-difference problem with point interactions (QP). For this problem, we will consider a second order q-difference equation

$$qa(t)y(qt,z) + b(t)y(t,z) + a\left(\frac{t}{q}\right)y\left(\frac{t}{q},z\right) = \lambda y(t,z), \quad t \in q^{\mathbb{N}} \backslash \left\{q^{m_0-1}, q^{m_0}, q^{m_0+1}\right\} \tag{2.1}$$

with the boundary condition

$$y(1,z) = 0 \tag{2.2}$$

and the point interaction

$$\begin{bmatrix} y\left(q^{m_0+1},z\right) \\ \triangle y\left(q^{m_0+1},z\right) \end{bmatrix} = \begin{bmatrix} y\left(q^{m_0-1},z\right) \\ \nabla y\left(q^{m_0-1},z\right) \end{bmatrix}, \quad B = \begin{bmatrix} \delta_1 & \delta_2 \\ \delta_3 & \delta_4 \end{bmatrix}, \tag{2.3}$$

where $\lambda = 2\sqrt{q}\cos z$ is a spectral parameter, $\det B \neq 0$ for $i = 1,2,3,4$ and δ_i are complex numbers. $\triangle$ denotes the backward difference operator, ∇ denotes the forward difference operator and defined by

$$\triangle y(t,z) = y(t,z) - y\left(\frac{t}{q},z\right)$$

and

$$\nabla y(t,z) = y(qt,z) - y(t,z),$$

respectively. Throughout this study, we will assume that the complex sequences $\{a(t)\}_{t \in q^{\mathbb{N}_0}}$ and $\{b(t)\}_{t \in q^{\mathbb{N}}}$ satisfy the following condition

$$\sum_{t \in q^{\mathbb{N}}} \frac{\ln t}{\ln q} \{|1 - a(t)| + |b(t)|\} < \infty \tag{2.4}$$

and $a(t) \neq 0$ for all $t \in q^{\mathbb{N}_0}$. Furthermore, we will denote by T the q-difference operator generated in $\ell_2\left(q^{\mathbb{N}}\right)$ by the q-difference expression

$$(ly)(t) := qa(t)y(qt,z) + b(t)y(t,z) + a\left(\frac{t}{q}\right)y\left(\frac{t}{q},z\right), \quad t \in q^{\mathbb{N}_0} \backslash \left\{q^{m_0-1}, q^{m_0}, q^{m_0+1}\right\}$$

with the boundary condition (2.2) and the point interactions (2.3).
Let us define two semi-strips

$$\Pi_+ := \left\{ z \in \mathbb{C} : \operatorname{Im} z > 0, \ -\frac{\pi}{2} \leq \operatorname{Re} z \leq \frac{3\pi}{2} \right\}$$

and

$$\Pi := \Pi_+ \cup \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right].$$

We will show the fundamental solutions of (2.1) for $z \in \Pi$ and $t \in \left\{1, q, q^2, \ldots, q^{m_0-1}\right\}$ by $R(t,z)$ and $Q(t,z)$ satisfying the initial conditions

$$R(1,z) = 0 \quad , \quad R(q,z) = 1$$

and

$$Q(1,z) = \frac{1}{a(1)} \quad , \quad Q(q,z) = 0,$$

respectively. The Wronskian of two solutions $y=\{y(t,\lambda)\}_{t\in q^{\mathbb{N}_0}}$ and $u=\{u(t,\lambda)\}_{t\in q^{\mathbb{N}_0}}$ of (2.1) is defined by

$$W[y,u](t)=\mu(t)a(t)\{y(t,\lambda)u(qt,\lambda)-y(qt,\lambda)u(t,\lambda)\},\quad t\in q^{\mathbb{N}_0}.$$

It follows from that $W[R(t,z),Q(t,z)]=-1$ for all $t\in q^{\mathbb{N}_0}$. It is easily seen that $R(t,z)$ and $Q(t,z)$ are entire functions of z. On the other hand, (2.1) admits another solution

$$e(t,z)=\rho(t)\frac{e^{i\frac{\ln t}{\ln q}z}}{\sqrt{\mu(t)}}\left(1+\sum_{r\in q^{\mathbb{N}}}A(t,r)e^{i\frac{\ln r}{\ln q}z}\right),\quad t\in q^{\mathbb{N}_0}$$

satisfying the condition

$$\lim_{t\to\infty}e(t,z)e^{-i\frac{\ln t}{\ln q}z}\sqrt{\mu(t)}=1,\quad z\in\Pi,$$

where $\rho(t)$ and $A(t,r)$ are expressed in terms of $\{a(t)\}$ and $\{b(t)\}$ [2]. Moreover, under the condition (2.4), $A(t,r)$ satisfies the following inequality

$$|A(t,r)|\le C\sum_{s\in\left[tq^{\left\lfloor\frac{\ln r}{2\ln q}\right\rfloor},\infty\right)\cap q^{\mathbb{N}}}\{|1-a(s)|+|b(s)|\},\tag{2.5}$$

here $C>0$ is a positive constant and $\left\lfloor\frac{\ln r}{2\ln q}\right\rfloor$ is the integer part of $\frac{\ln r}{2\ln q}$. Therefore, $e(t,z)$ is analytic with respect to z in $\mathbb{C}_+:=\{z\in\mathbb{C}:\operatorname{Im}z>0\}$ and continuous in $\overline{\mathbb{C}}_+:=\{z\in\mathbb{C}:\operatorname{Im}z\ge 0\}$. From the definition of Wronskian, we easily obtain that

$$W[e(t,z),e(t,-z)]=-\frac{2i}{\sqrt{q}}\sin z,\quad z\in\left[-\frac{\pi}{2},\frac{3\pi}{2}\right]\setminus\{0,\pi\}.$$

Also, there exists an unbounded solution of the equation (2.1) in $\{q^{m_0+1},q^{m_0+2},\dots\}$ denoted by $\widehat{e}(t,z)$ fulfilling the condition

$$\lim_{t\to\infty}\widehat{e}(t,z)e^{i\frac{\ln t}{\ln q}z}\sqrt{\mu(t)}=1,\quad z\in\overline{\mathbb{C}}_+.$$

By using the bounded and unbounded solutions of (2.1), we get the Wronskian of these solutions as

$$W[e(t,z),\widehat{e}(t,z)]=-\frac{2i}{\sqrt{q}}\sin z,\quad z\in\overline{\mathbb{C}}_+,\quad t\in\left\{q^{m_0+1},q^{m_0+2},\dots\right\}.$$

3. Jost Solution and Jost Function of T

In this section, we will obtain the Jost solution and the Jost function of T. Let us define the solutions of QP (2.1)-(2.3) by y_j^- and y_j^+, $j=1,2,3$, respectively

$$\varphi_1(t,z)=\begin{cases} y_1^-(t,z)=\mathscr{M}^-(z)R(t,z)+\mathscr{N}^-(z)Q(t,z), & t\in\left\{1,q,\dots,q^{m_0-1}\right\}\\ y_1^+(t,z)=\mathscr{M}^+(z)e(t,z)+\mathscr{N}^+(z)e(t,-z), & t\in\left\{q^{m_0+1},q^{m_0+2},\dots\right\}\end{cases}$$

for $z\in\Pi$,

$$\varphi_2(t,z)=\begin{cases} y_2^-(t,z)=K^-(z)R(t,z)+L^-(z)Q(t,z), & t\in\left\{1,q,\dots,q^{m_0-1}\right\}\\ y_2^+(t,z)=K^+(z)e(t,z)+L^+(z)e(t,-z), & t\in\left\{q^{m_0+1},q^{m_0+2},\dots\right\}\end{cases}$$

for $z\in\left[-\frac{\pi}{2},\frac{3\pi}{2}\right]\setminus\{0,\pi\}$ and

$$\varphi_3(t,z)=\begin{cases} y_3^-(t,z)=P^-(z)R(t,z)+S^-(z)Q(t,z), & t\in\left\{1,q,\dots,q^{m_0-1}\right\}\\ y_3^+(t,z)=P^+(z)e(t,z)+S^+(z)\widehat{e}(t,z), & t\in\left\{q^{m_0+1},q^{m_0+2},\dots\right\}\end{cases}$$

for $z\in\Pi\setminus\{0,\pi\}$. If we consider the solution $\varphi_1(t,z)$ firstly, we get the following equation with the help of condition (2.3)

$$\begin{bmatrix}\mathscr{M}^+\\ \mathscr{N}^+\end{bmatrix}=U\begin{bmatrix}\mathscr{M}^-\\ \mathscr{N}^-\end{bmatrix},\tag{3.1}$$

where

$$U:=\begin{bmatrix}U_{11}(z) & U_{12}(z)\\ U_{21}(z) & U_{22}(z)\end{bmatrix}=H^{-1}BD\tag{3.2}$$

such that

$$D=\begin{bmatrix} R\left(q^{m_0-1},z\right) & Q\left(q^{m_0-1},z\right) \\ \bigtriangledown R\left(q^{m_0-1},z\right) & \bigtriangledown Q\left(q^{m_0-1},z\right) \end{bmatrix}$$

and

$$H^{-1}=-\frac{\sqrt{q}\mu\left(q^{m_0+1}\right)a\left(q^{m_0+1}\right)}{2i\sin z}\begin{bmatrix} \triangle e\left(q^{m_0+1},-z\right) & -e\left(q^{m_0+1},-z\right) \\ -\triangle e\left(q^{m_0+1},z\right) & e\left(q^{m_0+1},z\right) \end{bmatrix}.$$

In accordance with (3.2), the components of U are given by

$$U_{21}(t,z)=-\frac{\sqrt{q}\mu\left(q^{m_0+1}\right)a\left(q^{m_0+1}\right)}{2i\sin z}\left\{\begin{array}{c} -\triangle e\left(q^{m_0+1},z\right)\left[\delta_1 R\left(q^{m_0-1},z\right)+\delta_2\bigtriangledown R\left(q^{m_0-1},z\right)\right] \\ +e\left(q^{m_0+1},z\right)\left[\delta_3 R\left(q^{m_0-1},z\right)+\delta_4\bigtriangledown R\left(q^{m_0-1},z\right)\right] \end{array}\right\} \tag{3.3}$$

and

$$U_{22}(t,z)=-\frac{\sqrt{q}\mu\left(q^{m_0+1}\right)a\left(q^{m_0+1}\right)}{2i\sin z}\left\{\begin{array}{c} -\triangle e\left(q^{m_0+1},z\right)\left[\delta_1 Q\left(q^{m_0-1},z\right)+\delta_2\bigtriangledown Q\left(q^{m_0-1},z\right)\right] \\ +e\left(q^{m_0+1},z\right)\left[\delta_3 Q\left(q^{m_0-1},z\right)+\delta_4\bigtriangledown Q\left(q^{m_0-1},z\right)\right] \end{array}\right\}. \tag{3.4}$$

Now, we will define the following function using the solution $\varphi_1(t,z)$

$$\mathscr{F}(t,z)=\begin{cases} \mathscr{M}^-(z)R(t,z)+\mathscr{N}^-(z)Q(t,z) & , \quad t\in\left\{1,q,q^2,\ldots,q^{m_0-1}\right\} \\ e(t,z) & , \quad t\in\left\{q^{m_0+1},q^{m_0+2},\ldots\right\} \end{cases} \tag{3.5}$$

for $z\in\Pi$. $\mathscr{F}(t,z)$ is the Jost solution of QP (2.1)-(2.3). Since the function $\mathscr{F}(t,z)$ is the Jost solution of (2.1)-(2.3), we find

$$\mathscr{M}^+(z)=1, \quad \mathscr{N}^+(z)=0.$$

If we apply the point interaction (2.3) to $\mathscr{F}(t,z)$, we obtain

$$\mathscr{M}^-(z)=\frac{U_{22}(t,z)}{\det U}, \quad \mathscr{N}^-(z)=-\frac{U_{21}(t,z)}{\det U}. \tag{3.6}$$

By using the boundary condition (2.2) and (3.5), we can write the Jost function of (2.1)-(2.3) as follows:

$$\mathscr{F}(z):=\mathscr{F}(t,z)=-\frac{U_{21}(t,z)}{a_0\det U}.$$

Then, we will consider the following solution of (2.1)-(2.3) for $z\in\left[-\frac{\pi}{2},\frac{3\pi}{2}\right]\setminus\{0,\pi\}$

$$\mathscr{G}(t,z):=\begin{cases} R(t,z) & , \quad t\in\left\{1,q,q^2,\ldots,q^{m_0-1}\right\} \\ K^+(z)e(t,z)+L^+(z)e(t,-z) & , \quad t\in\left\{q^{m_0+1},q^{m_0+2},\ldots\right\}. \end{cases}$$

Since the solution $\mathscr{G}(t,z)$ satisfies the boundary condition (2.2), we get

$$K^-(z)=1, \quad L^-(z)=0.$$

By the help of (2.3), we also find

$$K^+(z)=U_{11}(t,z), \quad L^+(z)=U_{21}(t,z). \tag{3.7}$$

Lemma 3.1. *For $z\in\left[-\frac{\pi}{2},\frac{3\pi}{2}\right]\setminus\{0,\pi\}$, the Wronskian of the solutions $\mathscr{F}(t,z)$ and $\mathscr{G}(t,z)$ is given by*

$$W[\mathscr{F}(t,z),\mathscr{G}(t,z)]=\begin{cases} -\frac{\mu(1)}{\det U}L^+(z) & , \quad t\in\left\{1,q,q^2,\ldots,q^{m_0-1}\right\} \\ -\frac{2i\sin z}{\sqrt{q}}L^+(z) & , \quad t\in\left\{q^{m_0+1},q^{m_0+2},\ldots\right\}. \end{cases}$$

Proof. By the help of the definition of Wronskian, we obtain

$$W\left[\mathscr{F}(t,z),\mathscr{G}(t,z)\right]=\mu(1)\,a(1)\left\{\begin{array}{c}\left[\mathscr{M}^{-}(z)R(1,z)+\mathscr{N}^{-}(z)Q(1,z)\right]F(q,z)\\ -\left[\mathscr{M}^{-}(z)R(q,z)+\mathscr{N}^{-}(z)Q(q,z)\right]R(1,z)\end{array}\right\}$$

for $t\in\left\{1,q,q^{2},\dots,q^{m_0-1}\right\}$. Since $R(1,z)=0$, $R(q,z)=1$, $Q(1,z)=\dfrac{1}{a(1)}$ and $Q(q,z)=0$, we find

$$W\left[\mathscr{F}(t,z),\mathscr{G}(t,z)\right]=\mu(1)\,\mathscr{N}^{-}(z).$$

In view of (3.6) and (3.7), the last equation can be arranged as

$$W\left[\mathscr{F}(t,z),\mathscr{G}(t,z)\right]=-\frac{\mu(1)}{\det U}L^{+}(z),\quad t\in\left\{1,q,q^{2},\dots,q^{m_0-1}\right\}.$$

The Wronskian of these solutions also given by

$$W\left[\mathscr{F}(t,z),\mathscr{G}(t,z)\right]=-\frac{2i\sin z}{\sqrt{q}}L^{+}(z)$$

for $t\in\left\{q^{m_0+1},q^{m_0+2},\dots\right\}$ in a similar way. $\square$

4. Resolvent Operator and Continuous Spectrum of T

In this section, we will present the resolvent operator of (2.1)-(2.3) firstly and then, we will give the continuous spectrum of QP (2.1)-(2.3). It is necessary for us to define another solution $\mathscr{H}(t,z)$ of (2.1)-(2.3) for $z\in\Pi\backslash\{0,\pi\}$.

$$\mathscr{H}(t,z):=\begin{cases}R(t,z) & ,\quad t\in\left\{1,q,q^{2},\dots,q^{m_0-1}\right\}\\ P^{+}(z)\,e(t,z)+S^{+}(z)\,\widehat{e}(t,z) & ,\quad t\in\left\{q^{m_0+1},q^{m_0+2},\dots\right\},\end{cases}$$

where $\widehat{e}(t,z)$ denotes the unbounded solution of (2.1) given in Section 2. Similar to solution $\mathscr{G}(t,z)$, it is possible to write $P^{-}(z)=1$, $S^{-}(z)=0$ by using the boundary condition (2.2) and to find the coefficients $P^{+}(z)$, $S^{+}(z)$ uniquely. By using the point interaction (2.3), we get

$$\left[\begin{array}{c}P^{+}(z)\\ S^{+}(z)\end{array}\right]=Y^{-1}B\left[\begin{array}{c}R\left(q^{m_0-1},z\right)\\ \bigtriangledown R\left(q^{m_0-1},z\right)\end{array}\right],$$

where

$$Y^{-1}=-\frac{\sqrt{q}\,\mu\left(q^{m_0+1}\right)a\left(q^{m_0+1}\right)}{2i\sin z}\left[\begin{array}{cc}\triangle\widehat{e}\left(q^{m_0+1},z\right) & -\widehat{e}\left(q^{m_0+1},z\right)\\ -\triangle e\left(q^{m_0+1},z\right) & e\left(q^{m_0+1},z\right)\end{array}\right]$$

and it gives

$$P^{+}(z)=-\frac{\sqrt{q}\,\mu\left(q^{m_0+1}\right)a\left(q^{m_0+1}\right)}{2i\sin z}\left\{\begin{array}{c}\triangle\widehat{e}\left(q^{m_0+1},z\right)\left[\delta_1R\left(q^{m_0-1},z\right)+\delta_2\bigtriangledown R\left(q^{m_0-1},z\right)\right]\\ -\widehat{e}\left(q^{m_0+1},z\right)\left[\delta_3R\left(q^{m_0-1},z\right)+\delta_4\bigtriangledown R\left(q^{m_0-1},z\right)\right]\end{array}\right\}\tag{4.1}$$

and

$$S^{+}(z)=\frac{\sqrt{q}\,\mu\left(q^{m_0+1}\right)a\left(q^{m_0+1}\right)}{2i\sin z}\left\{\begin{array}{c}\triangle e\left(q^{m_0+1},z\right)\left[\delta_1R\left(q^{m_0-1},z\right)+\delta_2\bigtriangledown R\left(q^{m_0-1},z\right)\right]\\ -e\left(q^{m_0+1},z\right)\left[\delta_3R\left(q^{m_0-1},z\right)+\delta_4\bigtriangledown R\left(q^{m_0-1},z\right)\right]\end{array}\right\},\tag{4.2}$$

respectively. From (3.3), (3.7) and (4.2), it is seen that $S^{+}(z)$ can be written as $L^{+}(z)$, i.e.,

$$S^{+}(z)=L^{+}(z).\tag{4.3}$$

Similar to Lemma 3.1 and by the help of (4.3), we obtain

$$W\left[\mathscr{F}(t,z),\mathscr{H}(t,z)\right]=\begin{cases}-\dfrac{\mu(1)}{\det U}L^{+}(z) & ,\quad t\in\left\{1,q,q^{2},\dots,q^{m_0-1}\right\}\\ -\dfrac{2i\sin z}{\sqrt{q}}L^{+}(z) & ,\quad t\in\left\{q^{m_0+1},q^{m_0+2},\dots\right\}.\end{cases}$$

for $z\in\Pi\backslash\{0,\pi\}$.

Lemma 4.1. *For all $z\in\Pi\backslash\{0,\pi\}$ with $L^{+}(z)\neq 0$ and $t\neq q^{m_0}$, we define the Green function of T by*

$$G_{t,r}(z):=\begin{cases} -\dfrac{\mu\left(\frac{r}{q}\right)\mathscr{H}(r,z)\mathscr{F}(t,z)}{W[\mathscr{F},\mathscr{H}]} & , \quad r=q^{k},\ k\in\mathbb{N}_0 \\ -\dfrac{\mu\left(\frac{r}{q}\right)\mathscr{H}(t,z)\mathscr{F}(r,z)}{W[\mathscr{F},\mathscr{H}]} & , \quad r=tq^{k},\ k\in\mathbb{N} \end{cases} \tag{4.4}$$

and it gives the resolvent operator T as

$$R(T)h(t):=\sum_{r\in q^{\mathbb{N}_0}}G_{t,r}h(r),\ \ h\in\ell_2\left(q^{\mathbb{N}}\right).$$

Proof. To find the resolvent operator, we need to get the general solution of the equation

$$qa(t)y(qt,z)+b(t)y(t,z)+a\left(\frac{t}{q}\right)y\left(\frac{t}{q},z\right)-\lambda y(t,z)=h(t,z), \tag{4.5}$$

where $h\in\ell_2\left(q^{\mathbb{N}}\right)$. Since the solutions of $\mathscr{F}(t,z)$ and $\mathscr{H}(t,z)$ are linearly independent fundamental solutions of the equation (2.1), we can write the solution of (4.5) as linear combination of these solutions as follows

$$g(t,z)=m(t)\mathscr{F}(t,z)+n(t)\mathscr{H}(t,z),$$

where $m(t)$ and $n(t)$ are coefficients and are different from zero. In order to get the coefficients $m(t)$ and $n(t)$, we use the method of variation of parameters. Then, we have

$$m(t)=m(1)-\sum_{r\in q^{\mathbb{N}}}\frac{h(r,z)\mathscr{H}(r,z)\mu\left(\frac{r}{q}\right)}{W[\mathscr{F},\mathscr{H}]},\quad r\neq q^{m_0}$$

and

$$n(t)=\eta-\sum_{r\in[qt,\infty)\cap q^{\mathbb{N}}}\frac{h(r,z)\mathscr{F}(r,z)\mu\left(\frac{r}{q}\right)}{W[\mathscr{F},\mathscr{H}]},\quad r\neq q^{m_0},$$

here $\lim_{s\to\infty}n(q^{s})=\eta$ and $m(1),\ \eta$ are real numbers. It is known that the solution $g(t,z)$ is in $\ell_2\left(q^{\mathbb{N}}\right)$ and provides the boundary condition (2.2). For this reason, $m(1)$ and η are equal to zero. It completes the proof, because it gives (4.4) and it is clear to write resolvent operator by using (4.4). □

Theorem 4.2. *If (2.4) holds, then $\sigma_c(T)=\left[-2\sqrt{q},2\sqrt{q}\right]$, where $\sigma_c(T)$ denotes the continuous spectrum of T.*

Proof. To get the continuous spectrum of T, we introduce the difference operators T_0 and T_1 generated by the following q-difference expressions in $\ell_2\left(q^{\mathbb{N}}\right)$ together with (2.2) and (2.3)

$$(T_0y)(t)=qy(qt,z)+y\left(\frac{t}{q},z\right),$$

$$(T_1y)(t)=\left(a\left(\frac{t}{q}\right)-1\right)y\left(\frac{t}{q},z\right)+b(t)y(t,z)+q(a(t)-1)y(qt,z)$$

for $t\in q^{\mathbb{N}}\backslash\left\{q^{m_0-1},q^{m_0},q^{m_0+1}\right\}$, respectively. It is easily seen that $T=T_0+T_1$ and T_1 is a compact operator in $\ell_2\left(q^{\mathbb{N}}\right)$ [31]. We also obtain that T_0 is a selfadjoint operator with $\sigma_c(T_0)=\left[-2\sqrt{q},2\sqrt{q}\right]$ [2]. Then, by using Weyl Theorem [24] of a compact perturbation, we find the continuous spectrum of the operator T. □

5. Main Results

In this section, we will obtain the sets of eigenvalues and spectral singularities of T. Then, we examine the properties of these sets. By the help of (4.4) and the definition of eigenvalues [34], the set of eigenvalues of T is

$$\sigma_d(T)=\left\{\lambda\in\mathbb{C}:\lambda=2\sqrt{q}\cos z,\ z\in\Pi_+,\ L^{+}(z)=0\right\}. \tag{5.1}$$

Since spectral singularities are the poles of the kernels of resolvent operator, the set of spectral singularities of T is defined as follows

$$\sigma_{ss}(T)=\left\{\lambda\in\mathbb{C}:\lambda=2\sqrt{q}\cos z,\ z\in\left[-\frac{\pi}{2},\frac{3\pi}{2}\right]\backslash\{0,\pi\},\ L^{+}(z)=0\right\}. \tag{5.2}$$

Let S_1 denote the set of all zeros of the function L^{+} in Π_+ and S_2 denote the set of all zeros of the function L^{+} in $\left[-\frac{\pi}{2},\frac{3\pi}{2}\right]$. It is obvious that

$$S_1:=\left\{z:z\in\Pi_+,\ L^{+}(z)=0\right\}$$

$$S_2:=\left\{z:z\in\left[-\frac{\pi}{2},\frac{3\pi}{2}\right],\ L^{+}(z)=0\right\}.$$

Theorem 5.1. *The function $L^{+}(z)$ satisfies the following asymptotic for $|z| \to \infty$, under the condition (2.4)*
i) *If $\delta_1+\delta_2+\delta_3+\delta_4 \neq 0$, then*

$$L^{+}(z) = e^{4iz}[A+o(1)],$$

ii) *If $\delta_1+\delta_2+\delta_3+\delta_4 = 0$, then*

$$L^{+}(z) = e^{5iz}[B+o(1)],$$

where

$$A := \frac{\sqrt{\mu(q^{m_0+1})}a(q^{m_0+1})}{q^{\frac{m_0-2}{2}}a(q)\ldots a(q^{m_0-2})}\left\{(\delta_1+\delta_2+\delta_3+\delta_4)\rho\left(q^{m_0+1}\right)\right\}$$

and

$$B := \frac{q^{\frac{1-m_0}{2}}\mu(q^{m_0+1})a(q^{m_0+1})\rho(q^{m_0+1})}{a(q)\ldots a(q^{m_0-3})}\left\{(\delta_1+\delta_2)\frac{a(q^{m_0+1})}{a(q^{m_0-2})}+(\delta_2+\delta_4)q^{\frac{3}{2}}\right\}.$$

Lemma 5.2. *Assume (2.4). Then*
i) *The set S_1 is bounded and has at most countable number of elements and its limit points can lie only in $\left[-\frac{\pi}{2},\frac{3\pi}{2}\right]$.*
ii) *The set S_2 is compact and $\mu(S_2)=0$.*

Proof. **i)** Theorem 5.1 proves the boundedness of the sets S_1 and S_2. Furthermore, it is known that L^{+} is analytic in $\mathbb{C}_{+}$, then the set S_1 has at most countable number of elements, the limit points of the zero of L^{+} only lie in $\left[-\frac{\pi}{2},\frac{3\pi}{2}\right]$.
ii) By means of boundary uniqueness theorems of analytic functions, we find that S_2 is a closed set and Privalov Theorem [22] proves that its linear Lebesgue measure is zero. □

Let us give the following theorem as a result of (5.1), (5.2) and Lemma 5.2.

Theorem 5.3. *Assume (2.4). Then,*
i) *The set σ_d is bounded and has at most a countable number of elements and its limit points can lie only in $[-2\sqrt{q},2\sqrt{q}]$,*
ii) *The set σ_{ss} is compact and its linear Lebesgue measure is zero.*

Definition 5.4. *The multiplicity of a zero of $L^{+}(z)$ in Π is called the multiplicity of the corresponding eigenvalue or spectral singularity of T.*

Theorem 5.5. *If*

$$\sup_{t\in q^{\mathbb{N}}}\left\{\exp\left(\varepsilon\frac{\ln t}{\ln q}\right)(|1-a(t)|+|b(t)|)\right\}<\infty, \quad \varepsilon>0, \tag{5.3}$$

then the operator T has a finite number of eigenvalues and spectral singularities and each of them is of finite multiplicity.

Proof. It follows from (2.5) and (5.3) that

$$|A(t,r)| \le C\exp\left(-\frac{\varepsilon}{8}\frac{\ln r}{\ln q}\right), \quad t\in\{1,q\}, \quad r\in q^{\mathbb{N}}, \tag{5.4}$$

where C is an arbitrary constant. For the sake of simplicity, let us define

$$J(z) = L^{+}(z)\,2i\sin z. \tag{5.5}$$

In accordance with (3.3), (3.7) and (5.5), we write

$$\begin{aligned} J(z) = &-\sqrt{q}\mu\left(q^{m_0+1}\right)a\left(q^{m_0+1}\right)\mathscr{K}\left(1+\sum_{r\in q^{\mathbb{N}}}A\left(q^{m_0+2},r\right)e^{i\left(\frac{\ln r}{\ln q}\right)z}\right) \\ &-\sqrt{q}\mu\left(q^{m_0+1}\right)a\left(q^{m_0+1}\right)\mathscr{L}\left(1+\sum_{r\in q^{\mathbb{N}}}A\left(q^{m_0+1},r\right)e^{i\left(\frac{\ln r}{\ln q}\right)z}\right), \end{aligned} \tag{5.6}$$

where

$$\begin{aligned} \mathscr{K} = &-\frac{(\delta_1+\delta_2)R(q^{m_0-1},z)\rho(q^{m_0+2})e^{i(m_0+2)z}}{\sqrt{\mu(q^{m_0+2})}} \\ &+\frac{\delta_2 R(q^{m_0-2},z)\rho(q^{m_0+2})e^{i(m_0+2)z}}{\sqrt{\mu(q^{m_0+2})}} \\ &-\frac{(\delta_2+\delta_4)R(q^{m_0-2},z)\rho(q^{m_0+1})e^{i(m_0+1)z}}{\sqrt{\mu(q^{m_0+1})}} \end{aligned}$$

and

$$\mathscr{L}=\frac{(\delta_1+\delta_2+\delta_3+\delta_4)\,R\left(q^{m_0-1},z\right)\rho\left(q^{m_0+1}\right)e^{i(m_0+1)z}}{\sqrt{\mu\left(q^{m_0+1}\right)}}.$$

By the help of (5.4) and (5.6), we find that the function J has an analytic continuation to the half-plane $\operatorname{Im}z>-\frac{\varepsilon}{8}$. Thus, the limit points of all zeros of the function J in Π can not lie in $\left[-\frac{\pi}{2},\frac{3\pi}{2}\right]$. So we find that the bounded sets $\sigma_d(T)$ and $\sigma_{ss}(T)$ have no limit points by Theorem 5.3, i.e., the sets S_1 and S_2 have a finite number of elements. Analyticity of J in $\operatorname{Im}z>-\frac{\varepsilon}{8}$ proves that all zeros of J in Π have finite multiplicity. Consequently, all eigenvalues and spectral singularities of T have a finite multiplicity under the condition (5.3). □

Now, we denote the sets of all limit points of S_1 and S_2 by S_3 and S_4, respectively and the set of all zeros of J with infinite multiplicity in Π by S_5. By the help of the uniqueness theorem of analytic functions, we obtain that

$$S_1\cap S_5=\emptyset,\ S_3\subset S_2,\ S_4\subset S_2,\ S_5\subset S_2,\ S_3\subset S_5,\ S_4\subset S_5$$

and

$$\mu(S_3)=\mu(S_4)=\mu(S_5)=0.$$

In the following, we will assume that

$$\sup_{t\in q^{\mathbb{N}}}\left\{\exp\left(\varepsilon\left(\frac{\ln t}{\ln q}\right)^{\gamma}\right)(|1-a(t)|+|b(t)|)\right\}<\infty,\quad \varepsilon>0,\ \ \frac{1}{2}\le\gamma\le1 \tag{5.7}$$

which is weaker than (5.3). Under the condition (5.7), the function J is still analytic in $\mathbb{C}_+$ and infinitely differentiable on the real axis. Before giving our main result, we need following two lemmas.

Lemma 5.6. *Under the condition (5.7), the following inequality is provided*

$$\left|J^{(n)}(z)\right|\le A_n,\quad z\in\Pi,\quad n\in\mathbb{N}_0,$$

where

$$A_n\le C4^n+\widetilde{D}d^n n!n^{n\left(\frac{1}{\gamma}-1\right)}$$

and $\widetilde{D}$ and d are positive constants depending on C, ε and γ.

Proof. By means of (2.5) and (5.7), we have

$$|A(t,r)|\le C\exp\left(-\frac{\varepsilon}{8}\left(\frac{\ln r}{\ln q}\right)^{\gamma}\right),\quad t\in\{1,q\},\quad r\in q^{\mathbb{N}}. \tag{5.8}$$

By using (5.6) and (5.8), we obtain

$$\begin{aligned}J(z)=&-\widetilde{Z}\left(\frac{\delta_2\rho\left(q^{m_0+2}\right)e^{i(m_0+2)z}}{\sqrt{\mu\left(q^{m_0+2}\right)}}-\frac{(\delta_2+\delta_4)\,\rho\left(q^{m_0+1}\right)e^{i(m_0+1)z}}{\sqrt{\mu\left(q^{m_0+1}\right)}}\right)\widetilde{A}(z)\widetilde{H}(z)\\&-\widetilde{Z}\left(-\frac{(\delta_1+\delta_2)\,\rho\left(q^{m_0+2}\right)e^{i(m_0+2)z}}{\sqrt{\mu\left(q^{m_0+2}\right)}}\right)\widetilde{A}(z)\widetilde{M}(z)\\&-\widetilde{Z}\left(\frac{(\delta_1+\delta_2+\delta_3+\delta_4)\,\rho\left(q^{m_0+1}\right)e^{i(m_0+1)z}}{\sqrt{\mu\left(q^{m_0+1}\right)}}\right)\widetilde{B}(z)\widetilde{M}(z),\end{aligned} \tag{5.9}$$

where the polynomial function $r^{m_0}(z)$ is of m_0-th degree, $\widetilde{Z}=\sqrt{q}\mu\left(q^{m_0+1}\right)a\left(q^{m_0+1}\right)$,

$$\widetilde{A}(z)=\left(1+\sum_{r\in q^{\mathbb{N}}}A\left(q^{m_0+2},r\right)e^{i\left(\frac{\ln r}{\ln q}\right)z}\right),$$

$$\widetilde{B}(z)=\left(1+\sum_{r\in q^{\mathbb{N}}}A\left(q^{m_0+1},r\right)e^{i\left(\frac{\ln r}{\ln q}\right)z}\right),$$

$$\widetilde{M}(z)=\left(\frac{\left(e^{iz}+e^{-iz}\right)^{m_0-2}}{q^{m_0-2}\prod\limits_{i=1}^{m_0-2}a\left(q^i\right)}+r^{m_0-3}(z)\right)$$

and

$$\widetilde{H}(z)=\left(\frac{\left(e^{iz}+e^{-iz}\right)^{m_0-3}}{q^{m_0-3}\prod\limits_{i=1}^{m_0-3}a\left(q^i\right)}+r^{m_0-4}(z)\right).$$

In view of (5.9), we get

$$\begin{aligned}\left|J^{(n)}(z)\right| \leq{}& \frac{4^n\left|\sqrt{q}\mu\left(q^{m_0+1}\right)a\left(q^{m_0+1}\right)\right|}{\left|\sqrt{\mu\left(q^{m_0+2}\right)}q^{m_0-3}a(q)\ldots a\left(q^{m_0-3}\right)\right|}\left(\frac{\left|\left(\delta_1+\delta_2\right)\rho\left(q^{m_0+1}\right)\right|}{\left|qa\left(q^{m_0-2}\right)\right|}\right)\widetilde{A}(z)\\ &+\frac{4^n\left|\sqrt{q}\mu\left(q^{m_0+1}\right)a\left(q^{m_0+1}\right)\right|}{\left|\sqrt{\mu\left(q^{m_0+2}\right)}q^{m_0-3}a(q)\ldots a\left(q^{m_0-3}\right)\right|}\left(\left|\left(\delta_1+\delta_2\right)\rho\left(q^{m_0+1}\right)\right|\right)\widetilde{B}(z)\\ &+\frac{4^n\left|\sqrt{q}\mu\left(q^{m_0+1}\right)a\left(q^{m_0+1}\right)\right|}{\left|\sqrt{\mu\left(q^{m_0+2}\right)}q^{m_0-3}a(q)\ldots a\left(q^{m_0-3}\right)\right|}\left(\frac{\left|\delta_1+\delta_2+\delta_3+\delta_4\rho\left(q^{m_0+1}\right)\right|}{\left|qa\left(q^{m_0-2}\right)\right|}\right)\widetilde{B}(z).\end{aligned}$$

By using (5.8), we arrive at

$$\left|J^{(n)}(z)\right|\leq C4^n+C4^n\sum_{r\in q^{\mathbb{N}}}\left(\frac{\ln r}{\ln q}\right)^n e^{-\frac{\varepsilon}{8}\left(\frac{\ln r}{\ln q}\right)^{\gamma}},\quad n\in q^{\mathbb{N}_0}.$$

Moreover, if we define

$$D_n:=\sum_{m=1}^{\infty}m^n e^{-\frac{\varepsilon}{8}m^{\gamma}}$$

by using Gamma function, we estimate

$$D_n\leq\int_0^{\infty}t^n e^{-\frac{\varepsilon}{8}t^{\gamma}}dt=\frac{2^{2n+\frac{3(1+n)}{\gamma}}}{\gamma\varepsilon^{\frac{n+1}{\gamma}}}\left(\frac{n+1}{\gamma}-1\right)\Gamma\left(\frac{n+1}{\gamma}-1\right).$$

Then, using the inequalities $\left(1+\frac{1}{n}\right)^{\frac{n}{\gamma}}\leq e^{\frac{1}{\gamma}}$, $(n+1)^{\frac{1}{\gamma}-1}<e^{\frac{n}{\gamma}}$ and $n^n<n!e^n$, we write

$$D_n\leq\widetilde{D}d^n n!n^{n\left(\frac{1}{\gamma}-1\right)},\quad n\in\mathbb{N},$$

here $\widetilde{D}$ and d are positive constants depending on ε and γ. □

Lemma 5.7. *Assume that the 2π-periodic function h is analytic in $\mathbb{C}_+$, all of its derivatives are continuous in $\overline{\mathbb{C}}_+$ and*

$$\sup_{z\in\Pi}\left|h^{(n)}(z)\right|\leq A_n,\quad n\in\mathbb{N}_0.$$

If the set $V\subset[0,2\pi]$ with Lebesgue measure zero is the set of all zeros of the function h with infinity multiplicity in Π, and if

$$\int_0^w \ln t(s)\,d\mu\left(V_s\right)=-\infty,$$

where $t(s)=\inf\limits_{n\in\mathbb{N}_0}\frac{A_n s^n}{n!}$ and $\mu(V_s)$ is the Lebesque measure of the $s-$neighborhood of V and $w>0$ is an arbitrary constant, then $h\equiv0$ in $\overline{\mathbb{C}}_+$.

Theorem 5.8. *If (5.7) holds, then $S_5=\emptyset$.*

Proof. By the help of Theorem 5.3 since the function J is not equal to zero identically, it is clear that

$$\int_0^w \ln t(s)\, d\mu(S_5, s) > -\infty, \tag{5.10}$$

where $t(s) = \inf_{n \in \mathbb{N}_0} \frac{A_n s^n}{n!}$, $\mu(S_5, s)$ is the Lebesgue measure of the $s-$neighborhood of S_5 and A_n is defined in Lemma 5.6. In accordance with Lemma 5.6, we obtain

$$t(s) = \widetilde{D} \exp\left\{-\frac{1-\gamma}{\gamma} e^{-1} (ds)^{-\frac{\gamma}{1-\gamma}}\right\}. \tag{5.11}$$

From (5.10) and (5.11), we get

$$\int_0^w s^{-\frac{\gamma}{1-\gamma}} d\mu(S_5, s) < \infty.$$

Since $\frac{\gamma}{1-\gamma} \geq 1$, the integral on the left hand-side is convergent for arbitrary s if and only if

$$\mu(S_5, s) = 0,$$

i.e., $S_5 = \emptyset$. It completes the proof. □

6. Conclusions

In this study, we examine the spectral properties of the q-difference equation with point interaction given by (2.1)-(2.3). This paper is important because it is the first study to investigate the spectral properties of a q-difference equation with point interaction. Firstly, we find the Jost solution and Jost function of T. Then, we obtain the resolvent operator, Green function and continuous spectrum of (2.1)-(2.3). By the help of uniqueness theorems, we also discuss the structure and finiteness of eigenvalues and spectral singularities of the problem. In this study, the main result is to show the finiteness of eigenvalues and spectral singularities of this q-problem under sufficient conditions and to prove that their multiplicities are finite. This paper prepares a groundwork for many researchers working on spectral analysis. For next study, one can consider the impulsive condition as a matrix form that will be the general form of this problem.

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