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The Fekete-Szegö problem for certain subclass of analytic and univalent functions associated with hyperbolic sine function with the complex order

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Abstract

In this paper, we give coefficient estimates for the certain subclass of analytic and univalent functions on the open unit disk in the complex plane associated with hyperbolic sine function with the complex order. For the subclass $C_{\sinh}(\tau)$, $\tau \in \mathbb{C} - \{0\}$ defined here of analytic and univalent functions, with the quantity

$$1+\frac{1}{\tau}\left[\frac{\left(zf'(z)\right)'}{f'(z)}-1\right]$$

subordinated to $1+\sinh z$, we obtain coefficient estimates for initial two coefficients and examine the Fekete-Szegö problem.

Keywords: Convex function, sine hyperbolic function, coefficient estimate, Fekete-Szegö problem, complex order

1. Introduction

The main focus of this section is to give some basic information that we will use in the proof of the main results.

Let $H(U)$ be the class of all analytic functions in $U = \{z \in \mathbb{C}: |z < 1|\}$. By A, we will denote the class of functions $f \in H(U)$ given by the following series expansion, explicitly satisfying the conditions $f(0)=0$ and $f'(0)-1=0$

$$\begin{aligned} f(z) &= z + a_2 z^2 + a_3 z^3 + \cdots \\ &= z + \sum_{n=2}^{\infty} a_n z^n, a_n \in \mathbb{C}. \end{aligned} \tag{1.1}$$

Moreover, let S be a subclass of all univalent functions of A. The class S was first time introduced by Köebe (Köebe 1909) and has become a core component of research in this area. Bieberbach (Bieberbach 1916) published a paper in which the famous coefficient hypothesis was proposed. This conjecture states that if $f \in S$ and has the series form (1.1), then $|a_n| \le n$ for all $n \ge 2$. Many mathematicians worked hard to solve this problem. But for the first time in 1985 it was De-Branges (De-Branges 1985) who settled this long-lasting conjecture.

It is well-know that a univalent function $f \in S$ is called a convex function, if this function maps open unit disk U onto the convex shaped domain of the complex plane. The set of all convex functions which satisfies the following condition is denoted by C

$$\operatorname{Re}\left(\frac{\left(zf'(z)\right)'}{f'(z)}\right) > 0,\ z \in U;$$

that is,

$$C = \left\{ f \in S:\ \operatorname{Re}\left(\frac{\left(zf'(z)\right)'}{f'(z)}\right) > 0,\ z \in U \right\}.$$

Some of the important and well-investigated subclass of S include the class $C(\alpha)$ given below, which called the class of convex functions of order $\alpha\left(\alpha \in [0,1)\right)$

$$C(\alpha) = \left\{ f \in S:\ \operatorname{Re}\left(\frac{\left(zf'(z)\right)'}{f'(z)}\right) > \alpha,\ z \in U \right\}.$$

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2. *Materials and Methods*

It is well-known that an analytical function ω satisfying the conditions $\omega(0)=0$ and $|\omega(z)|<1$ is called Schwartz function. Two analytic functions f and g in U are said that f is subordinate to g and denoted by $f \prec g$, if there exists a Schwartz function ω, such that $f(z)=g(\omega(z))$.

In 1994, Ma and Minda (Ma and Minda 1994) using subordination terminology was defined the class $C(\varphi)$ as follows

$$C(\varphi)=\left\{ f \in S: \ \frac{(zf'(z))'}{f'(z)} \prec \varphi(z), z \in U \right\},$$

where $\varphi(z)$ is a univalent function with $\varphi(0)=1$, $\varphi'(0)>0$ and the region $\varphi(U)$ is star-shaped about the point $\varphi(0)=1$ and symmetric with respect to real axis. Such a function has a series expansion of the following form

$$\varphi(z)=1+b_1z+b_2z^2+b_3z^3+\cdots$$
$$=1+\sum_{n=1}^{\infty} b_n z^n, \ b_1>0 \ .$$

In the past few years, numerous subclasses of the collection S have been introduced as special choices of the function φ (see for example (Sokol 2011, Janowski 1970, Arif, et al 2019, Brannan 1969, Sokol and Stankiewcz et al. 2021, Sharma et al. 2016, Kumar and Arora 2020, Mendiratta et a.l 2015, Shi et al 2019, Bano and Raza 2020, Alotaibi et al. 2020, Ullah et al. 2021, Cho et al. 2019, Mustafa et al.2022, Mustafa 2017, Xu et al, 2012)).

Finding bounds for the function coefficients in a given collection is one of the most fundamental problems in geometric function theory.

As known that the first order of Hankel determinant of the function $f \in S$ defined as follows

$$H_{2,1}(f)=\begin{vmatrix} 1 & a_2 \\ a_2 & a_3 \end{vmatrix}=a_3-a_2^2.$$

The functional $H_{2,1}(f,\mu)=a_3-\mu a_2^2$ is known as the generalized Fekete-Szegö functional, where μ is a complex or real number (Duren 1983). Estimating the upper bound of $|a_3-\mu a_2^2|$ is known as the Fekete-Szegö problem in the theory of analytic functions.

Now by using the definition of subordination, we introduce a new subclass of analytic and univalent functions associated with sine hyperbolic function as follows.

Definition 2.1. For $\tau \in \mathbb{C}-\{0\}$ a function $f \in S$ is said to be in the class $C_{\sinh}(\tau)$, if the following condition is satisfied

$$1+\frac{1}{\tau}\left[\frac{(zf'(z))'}{f'(z)}-1\right] \prec 1+\sinh z, z \in U \ ;$$

that is,

$$C_{\sinh}(\tau) \equiv C(\tau, 1+\sinh z)$$
$$=\left\{ f \in S: \ \frac{(zf'(z))'}{f'(z)} \prec 1+\sinh z \right\}, \ z \in U.$$

Remark 2.1. In the cases $\tau=1$, we have class $C_{\sinh} \equiv C(1+\sinh z)$ which reviewed in (Mustafa et al. 2023).

Let P be the class of analytic functions in U satisfied the conditions $p(0)=1$ and $\mathrm{Re}(p(z))>0$, $z \in U$, which from the subordination principle easily can written

$$\mathrm{P}=\left\{ p \in A: \ p(z) \prec \frac{1+z}{1-z}, z \in U \right\},$$

where $p(z)$ has the series expansion of the form

$$\begin{aligned} p(z) &= 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots \\ &= 1 + \sum_{n=1}^{\infty} p_n z^n, \ z \in U \ . \end{aligned} \tag{2.1}$$

The class P defined above is known as the class Caratheodory functions (Miller 1975).

Let's present some necessary lemmas known in the literature for the proof of our main results.

Lemma 2.1. (Duren 1983). Let the function $p(z)$ belong in the class P. Then,

$|p_n| \le 2$ for each $n \in \mathbb{N}$ and $|p_n - \lambda p_k p_{n-k}| \le 2$

for $n, k \in \mathbb{N}, n > k$ and $\lambda \in [0,1]$.

The equalities hold for

$$p(z) = \frac{1+z}{1-z}.$$

Lemma 2.2. (Duren 1983) Let the analytic function $p(z)$ be of the form (1.2), then

$$2p_2 = p_1^2 + \left(4 - p_1^2\right)x,$$

$$4p_3 = p_1^3 + 2\left(4 - p_1^2\right)p_1 x - \left(4 - p_1^2\right)p_1 x^2 + 2\left(4 - p_1^2\right)\left(1 - |x|^2\right)y$$

for $x, y \in \mathbb{C}$ with $|x| \le 1$ and $|y| \le 1$.

3. *Results*

In this section, we give upper bound estimates for initial two coefficients for the function class $C_{\sinh}(\tau)$ and examine Fekete-Szegö problem for this class.

Let us first give the following theorem.

Theorem 3.1. Let the function $f \in A$ given by (1.1) belong to the class $C_{\sinh}(\tau)$. Then,

$$|a_2| \le \frac{|\tau|}{2}$$

and

$$|a_3| \le \frac{|\tau|}{6} \begin{cases} 1 & \text{if} \quad |\tau| \le 1, \\ |\tau| & \text{if} \quad |\tau| \ge 1. \end{cases}$$

Proof. Let $f \in C_{\sinh}(\tau)$. Then, there exists a Schwartz function $\omega(z)$, such that

$$\frac{\left(zf'(z)\right)'}{f'(z)} = 1 + \tau \cdot \sinh \omega(z), \ z \in U \ .$$

By writing the Caratheodory function $p \in \mathrm{P}$ in terms of Schwartz function ω, we have

$$p(z) = \frac{1+\omega(z)}{1-\omega(z)} = 1 + p_1 z + p_2 z^2 + \cdots \ .$$

It follows from that

$$\begin{aligned} \omega(z) &= \frac{p(z)-1}{p(z)+1} \\ &= \frac{1}{2} p_1 z + \frac{1}{2}\left(p_2 - \frac{p_1^2}{2}\right) z^2 + \cdots \ . \end{aligned} \tag{3.1}$$

From the series expansion (1.1) of the function $f(z)$, we can write

$$\frac{\left(zf'(z)\right)'}{f'(z)} = 1 + 2a_2 z + 2\left(3a_3 - 2a_2^2\right) z^2 + \cdots \ . \tag{3.2}$$

Since

$$\sinh z = z + \frac{1}{3!} z^3 + \frac{1}{5!} z^5 + \cdots , \tag{3.3}$$

from the series expansion (3.1) of the function $\omega(z)$, we have

$$\begin{aligned} &1 + \tau \cdot \sinh \omega(z) \\ &= 1 + \frac{\tau}{2} p_1 z + \frac{\tau}{2}\left(p_2 - \frac{p_1^2}{2}\right) z^2 + \cdots \ . \end{aligned} \tag{3.4}$$

Equalizing (3.2) and (3.4), then comparing the coefficients of the same degree terms on the right and left sides, we obtain the following equalities for two initial coefficients of the function $f(z)$

$$a_2 = \frac{\tau}{4} p_1, \tag{3.5}$$

$$a_3 = \frac{\tau(\tau-1)}{24} p_1^2 + \frac{\tau}{12} p_2. \tag{3.6}$$

Using Lemma 2.1, from the equalities (3.5) we can easily see that

$$|a_2| \le \frac{|\tau|}{2}.$$

Applying the Lemma 2.2, the equality (3.6) we can write as follows

$$a_3 = \frac{\tau}{24}\left(\tau p_1^2 + \left(4 - p_1^2\right)x\right), \tag{3.7}$$

where $x \in \mathbb{C}$ with $|x| \le 1$. Applying triangle inequality, from this equality to (3.7) we obtain

$$|a_3| \le \frac{|\tau|}{24}\left(|\tau| t^2 + \left(4 - t^2\right)\xi\right),$$

where $\xi = |x|$ and $t = |p_1|$. If we maximize the function φ: [0,1] → ℝ defined as follows

$$\varphi(\xi) = |\tau| t^2 + \left(4 - t^2\right)\xi, \ \xi \in [0,1],$$

we write

$$|a_3| \le \frac{|\tau|}{24}\left(\left(|\tau| - 1\right)t^2 + 4\right), \ t \in [0,2].$$

From this, we obtain desired estimate for $|a_3|$.

Thus, the proof of the theorem is completed.

Taking $\tau = 1$ in Theorem 3.1, we obtain the following result for $|a_2|$ and $|a_3|$ obtained in (Mustafa et al. 2023).

Theorem 3.2. Let the function $f \in A$ given by (1.1) belong to the class $C_{\sinh}$.
Then, we have

$$|a_2| \le \frac{1}{2} \text{ and } |a_3| \le \frac{1}{6}.$$

Let us now give the following theorem on the Fekete-Szegö inequality.

Theorem 3.3. Let the function $f \in A$ given by (1.1) belong to the class $C_{\sinh}(\tau)$ and $\mu \in \mathbb{C}$. Then,

$$\left|a_3 - \mu a_2^2\right| \le \frac{|\tau|}{6}\begin{cases} 1 & \text{if } \ |2-3\mu||\tau| \le 2, \\ \dfrac{|2-3\mu||\tau|}{2} & \text{if } \ |2-3\mu||\tau| \ge 2. \end{cases}$$

Proof. Let $f \in C_{\sinh}(\tau)$ and $\mu \in \mathbb{C}$. From the equalities (3.5) and (3.6), using Lemma 2.2 we can write

$$a_3 - \mu a_2^2 = \frac{\tau}{16}\left[\left(\frac{2}{3} - \mu\right)\tau p_1^2 + \frac{2}{3}\left(4 - p_1^2\right)x\right], \tag{3.8}$$

where $x \in \mathbb{C}$ with $|x| \le 1$. From this equality, applying triangle inequality we can write

$$\left|a_3 - \mu a_2^2\right| \le \frac{|\tau|}{16}\left[\left|\frac{2}{3} - \mu\right||\tau| t^2 + \frac{2}{3}\left(4 - t^2\right)\xi\right], \ \xi \in [0,1] \tag{3.9}$$

with $t = |p_1| \in [0,2]$ and $\xi = |x|$.

By maximizing the function

$$\psi(\xi) = \left|\frac{2}{3} - \mu\right||\tau| t^2 + \frac{2}{3}\left(4 - t^2\right)\xi, \ \xi \in [0,1],$$

from the inequality (3.9), we obtain the following inequality

$$\left|a_3 - \mu a_2^2\right| \le \frac{|\tau|}{16}\left\{\left[\left|\frac{2}{3} - \mu\right||\tau| - \frac{2}{3}\right]t^2 + \frac{8}{3}\right\}, \ t \in [0,2]. \tag{3.10}$$

From here, we obtained the desired result of the theorem.

Thus, the proof of theorem is completed.

Taking $\tau = 1$ in Theorem 3.3, we obtain the following result for the Fekete-Szegö inequality obtained in (Mustafa et al. 2023).

Theorem 3.4 . Let the function $f \in A$ given by (1.1) belong to the class $C_{\sinh}$ and $\mu \in \mathbb{C}$. Then,

$$\left|a_3 - \mu a_2^2\right| \le \frac{1}{6}\begin{cases} 1 & \text{if } \ |2-3\mu| \le 2, \\ \dfrac{|2-3\mu|}{2} & \text{if } \ |2-3\mu| \ge 2. \end{cases}$$

In case $\mu \in \mathbb{R}$, Theorem 3.3 is given as below.

Theorem 3.5. Let the function $f \in A$ given by (1.1) belong to the class $C_{\sinh}(\tau)$ and $\mu \in \mathbb{R}$. Then,

$$\left|a_3 - \mu a_2^2\right| \le \frac{|\tau|}{6}\begin{cases} 1 & \text{if } \quad \mu \in \left[\dfrac{2(|\tau|-1)}{3|\tau|}, \dfrac{2(|\tau|+1)}{3|\tau|}\right], \\ \dfrac{|2-3\mu||\tau|}{2} & \text{if } \ \mu \le \dfrac{2(|\tau|-1)}{3|\tau|} \text{ or } \mu \ge \dfrac{2(|\tau|+1)}{3|\tau|}. \end{cases}$$

The proof of this theorem is done similarly to the proof of Theorem 3.3.

Taking $\mu = 0$ and $\mu = 1$ in Theorem 3.5, we get the estimate for $|a_3|$ obtained in Theorem 3.1 and the following result, respectively.

Corollary 3.1. If the function $f \in A$ given by (1.1) belong to the class $C_{\sinh}(\tau)$, then

$$\left|a_3 - a_2^2\right| \le \frac{|\tau|}{6}\begin{cases} 1 & \text{if } \ |\tau| \le 2, \\ \dfrac{|\tau|}{2} & \text{if } \ |\tau| \ge 2. \end{cases}$$

Taking $\tau = 1$ in Corollary 3.1, we obtain the following result obtained in (Mustafa et al. 2023).

Corollary 3.2. If the function $f \in A$ given by (1.1) belong to the class $C_{\sinh}$, then

$$\left|a_3 - a_2^2\right| \le \frac{1}{6}.$$

4. *Discussion*

In this study, we obtained the results obtained in (Mustafa et al. 2023) for wider function classes.

Really, it can be easily seen the results in (Mustafa et al. 2023) are obtained if we take $\tau = 1$ in our study.

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