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ON $\theta-$CONVEX CONTRACTIVE MAPPINGS WITH APPLICATION TO INTEGRAL EQUATIONS

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Abstract. The fundamental goal of our paper is to study $\theta-$convex contractive mappings in metric spaces. We demonstrate some fixed point results for such mappings. Also, we give an application to integral equations of our results. Consequently, our results encompass numerous generalizations of the Banach contraction principle on metric space.

1. Introduction and Preliminaries

Banach [1] initially gave the Banach contraction principle which is an outstanding result in fixed point theory. Due to its significance, over the years, abounding researchers extended and generalized this contraction in many ways.

The notion of almost contraction was introduced by Berinde [2]. Also almost contraction was compared with other contractions and Berinde [2], [3], [4] demonstrated some fixed point theorems related to almost contraction.

Firstly Jleli [5] gave an attractive contraction called $\theta-$contraction and researched the uniqueness and existence of these mappings in complete metric spaces. After Jleli's first article [5], some different fixed point theorems were introduced Jleli [6], Hussain [7] and Imdad [8] by changing and relaxing the conditions of $\mho$.

In recent years, a remarkable generalization of the Banach contraction principle is the theorem by Istratescu [9]. Again, Istarescu studied convex contractions in [9], [10], [11]. Since Istratescu's fixed point theorems, many authors studied numerous generalizations and applications of the result of Istratescu (see [12]- [24]).

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Ciric [25] used the concept of orbitally continuous for proving the uniqueness and existence of the fixed point mappings. Afterwards, Bisht [15] proved some fixed point theorems by replacing the continuity condition with orbital continuity.

Merging the ideas of Istratescu [9] and Jleli [5], we introduce a generalization of convex type contractions. The goal of our paper is to introduce generalized $\theta-$convex contractive mappings and to demonstrate some fixed point theorems. Theorems that have been demonstrated in our paper are generalizations of a variety of results in the literature.

Now, at first we mention some fundemental definitions and notions related to our work.

$F(h) = \{t \in W : ht = t\}$ is fixed point of h.

Bisht [15] gave the following definition instead of continuity condition to be used their theorems.

Definition 1. *[15] Let (W, ϱ) be a metric space and h be a self mapping on W. We say that h is orbitally continuous at a point $u \in W$ if $\lim\limits_{j\to\infty} h^{n_j}t = u$ implies that $\lim\limits_{j\to\infty} h^{n_j}t = hu$.*

Berinde [2], [3], [4] gave the concepts of almost contraction, multivalued almost contraction and the continuity of almost contractions.

Definition 2. *[2] Let (W, ϱ) be a metric space and h be a self mapping on W. h is called an almost contraction if there exists a constant $\zeta \in (0, 1)$ and $L \geq 0$ such that*

$$\varrho(ht, hs) \leq \zeta\varrho(t, s) + L\varrho(s, ht)$$

for all $t, s \in W$.

Firstly, Jleli [5] gave the concept of $\theta-$contraction mappings and the following family.

Let $\mho$ denotes the set of all mappings $\theta : (0, \infty) \to (1, \infty)$ which hold the following conditions:

(1) θ is strictly increasing;
(2) for all sequence $\{\eta_n\} \subset (0, \infty)$, $\lim_{n\to\infty} \theta(\eta_n) = 1$ if and only if $\lim_{n\to\infty} \eta_n = 0$;
(3) there exist $\ell \in (0, \infty]$ and $r \in (0, 1)$ such that $\lim_{n\to\infty} \frac{\theta(\eta)-1}{(\eta)^r} = \ell$.

Υ be the set of nondecreasing functions $\varsigma : [0, +\infty) \to [0, +\infty)$ such that $\sum\limits_{j=1}^{+\infty} \varsigma^j(\eta) < +\infty$ for each $\eta > 0$, where ς^j is the $j-$th iterate of ς.

Remark 1. *Each function $\varsigma \in \Upsilon$ satisfies $\lim_{n\to\infty} \varsigma^n(\eta) = 0$ and $\varsigma(\eta) < \eta$ for all $\eta > 0$.*

Firstly, Jleli [5] gave the definition of $\theta-$contraction as follows.

Definition 3. *Let (W,ϱ) be a metric space and $h:W\to W$ be a self-mapping. Then h is called $\theta-$contraction if there exist $\kappa\in(0,1)$ such that*

$$\theta(\varrho(ht,hs))\leq[\theta(\varrho(t,s))]^{\kappa}$$

for all $t,s\in W$, with $ht\neq hs$.

Istratescu [9], [10] gave the following definitions.

Definition 4. *Let (W,ϱ) be a metric space and $h:W\to W$ be a self-mapping. Then h is called convex contraction of order 2 if there exist $d_1,d_2\in(0,1)$ such that $d_1+d_2<1$ and*

$$\varrho(h^2t,h^2s)\leq d_1\varrho(ht,hs)+d_2\varrho(t,s)$$

for all $t,s\in W$.

Definition 5. *Let (W,ϱ) be a metric space and $h:W\to W$ be a self-mapping. Then h is called two-sided convex contraction mappings if there exist $d_1,d_2,d_3,d_4\in(0,1)$ such that $d_1+d_2+d_3+d_4<1$ and*

$$\varrho(h^2t,h^2s)\leq d_1\varrho(t,ht)+d_2\varrho(ht,h^2t)+d_3\varrho(s,hs)+d_4\varrho(hs,h^2s)$$

for all $t,s\in W$.

2. Main Results

In this chapter, we give concept of generalized $\theta-$convex contractions in metric spaces. We demonstrate some fixed point results for such contractions on metric spaces. The following Theorem's hypothesis are basically weaker than the set of contraction type mappings.

Now, we will give the definition of generalized $\theta-$convex contractive mappings.

Definition 6. *Let (W,ϱ) be a metric space and $h:W\to W$ be a self-mapping. Then h is called generalized $\theta-$convex contraction if there exist $L\geq 0$, $\varsigma\in\Upsilon$ and $\kappa\in(0,1)$ such that*

$$\varrho(h^2t,h^2s)>0\Rightarrow\theta\left(\varrho(h^2t,h^2s)\right)\leq\left[\theta\left(\varsigma(M_I(t,s))\right)\right]^{\kappa}+LN_I(t,s)\tag{1}$$

where $\theta\in\mho$ and

$$\begin{aligned}M_I(t,s)&=\max\left\{\varrho(t,s),\varrho(ht,hs),\varrho(t,ht),\varrho(s,hs),\varrho(ht,h^2t),\varrho(hs,h^2s)\right\},\\N_I(t,s)&=\min\left\{\varrho(t,ht),\varrho(s,hs),\varrho(t,hs),\varrho(s,ht),\varrho(ht,h^2t),\varrho(hs,h^2s)\right\},\end{aligned}$$

for all $t,s\in W$.

Remark 2. *Every convex contraction of order 2 and two-sided convex contraction are a generalized $\theta-$convex contraction. Also, every $\theta-$contraction is a generalized $\theta-$convex contraction. But the reverse doesn't have to be true.*

Since, our novel class of contractive type mappings is more general, it will be more advantageous to work using this new class.

The following theorem is our first result related to generalized $\theta-$convex contractive mappings.

Theorem 1. *Let (W,ϱ) be a complete metric space and $h:W\to W$ be a generalized $\theta-$convex contraction. If h is either orbitally continuous on W or h is continuous, then h has a unique fixed point.*

Proof. Starting at the point $t_0\in W$, the sequence $\{t_n\}$ is constructed by $t_n=ht_{n-1}=h^nt_0$, $n\geq 1$. If $t_{n_0+1}=t_{n_0}$ for any $n_0\in\mathbb{N}\cup\{0\}$, then it is clear that, t_{n_0} is a fixed point of h. Consequently, assume that $t_{n_0+1}\neq t_{n_0}$ for all $n_0\in\mathbb{N}\cup\{0\}$. Setting $m=\max\{\varrho(t_0,t_1),\varrho(t_1,t_2)\}$. First of all, we show that $\{\varrho(t_n,t_{n+1})\}$ is a strictly nonincreasing sequence in W. Since h is a generalized $\theta-$convex contraction, using Remark 1 and from the first axiom of θ, we have

$$\begin{aligned}
\theta(\varrho(t_2,t_3)) &= \theta\left(\varrho\left(h^2t_0,h^2t_1\right)\right)\\
&\leq \left[\theta\left(\varsigma\left(\max\left\{\begin{array}{c}\varrho(t_0,t_1),\varrho(ht_0,ht_1),\varrho(t_0,ht_0),\\ \varrho\left(ht_0,h^2t_0\right),\varrho(t_1,ht_1),\varrho\left(ht_1,h^2t_1\right)\end{array}\right\}\right)\right)\right]^{\kappa}\\
&\quad +L\min\left\{\begin{array}{c}\varrho(t_0,ht_0),\varrho(t_1,ht_1),\varrho(t_0,ht_1),\\ \varrho(t_1,ht_0),\varrho\left(ht_0,h^2t_0\right),\varrho\left(ht_1,h^2t_1\right)\end{array}\right\}\\
&= [\theta(\varrho(\max\{\varrho(t_0,t_1),\varrho(t_1,t_2),\varrho(t_2,t_3)\}))]^{\kappa}\\
&\leq [\theta(\max\{m,\varrho(t_2,t_3)\})]^{\kappa}.
\end{aligned}$$

If $\max\{m,\varrho(t_2,t_3)\}=\varrho(t_2,t_3)$, then we have

$$\theta(\varrho(t_2,t_3))\leq[\theta(\varrho(t_2,t_3))]^{\kappa}.$$

If we take ln two both sides of the inquality, then we have

$$\ln\theta(\varrho(t_2,t_3))\leq\kappa\ln[\theta(\varrho(t_2,t_3))]$$

which is a contradiction. Hence, we get

$$\max\{m,\varrho(t_2,t_3)\}=m=\max\{\varrho(t_0,t_1),\varrho(t_1,t_2)\}.$$

Since $\varsigma(\eta)<\eta$ for all $\eta>0$, we have

$$\begin{aligned}
\theta(\varrho(t_3,t_4)) &\leq \left[\theta\left(\varsigma\left(\max\left\{\begin{array}{c}\varrho(t_1,t_2),\varrho(ht_1,ht_2),\varrho(t_1,ht_1),\\ \varrho(ht_1,h^2t_1),\varrho(t_2,ht_2),\varrho\left(h_2,h^2t_2\right)\end{array}\right\}\right)\right)\right]^{\kappa}\\
&\quad +L\min\left\{\begin{array}{c}\varrho(t_1,ht_1),\varrho(t_2,ht_2),\varrho(t_1,ht_2),\\ \varrho(t_2,ht_1),\varrho(ht_1,h^2t_1),\varrho\left(ht_2,h^2t_2\right)\end{array}\right\}\\
&\leq [\theta(\max\{\varrho(t_1,t_2),\varrho(t_2,t_3),\varrho(t_3,t_4)\})]^{\kappa}.
\end{aligned}$$

If $\max\{\varrho(t_1,t_2),\varrho(t_2,t_3),\varrho(t_3,t_4)\}=\varrho(t_3,t_4)$, then we obtain

$$\theta(\varrho(t_3,t_4))\leq[\theta(\varrho(t_3,t_4))]^{\kappa}.$$

If we take ln two both sides of the inequality, then we have

$$\ln\theta(\varrho(t_3,t_4))\leq\kappa\ln[\theta(\varrho(t_3,t_4))].$$

This is one more contradiction, from which it is concluded that $\max\{\varrho(t_1,t_2),\varrho(t_2,t_3)\}>\varrho(t_3,t_4)$. Thus, $m>\varrho(t_2,t_3)>\varrho(t_3,t_4)$. Hence, by induction one can get

$\{\varrho(t_n,t_{n+1})\}$ is a strictly nonincreasing sequence in W. This implies that

$$\begin{aligned}\theta(\varrho(t_n,t_{n+1})) &\leq \left[\theta\left(\varsigma\left(\max\left\{\begin{array}{c}\varrho(t_{n-2},t_{n-1}),\varrho(t_{n-1},t_n),\varrho(t_{n-2},t_{n-1}),\\ \varrho(t_{n-1},t_n),\varrho(t_{n-1},t_n),\varrho(t_n,t_{n+1})\end{array}\right\}\right)\right)\right]^{\kappa}\\ &\quad +L\min\left\{\begin{array}{c}\varrho(t_{n-2},t_{n-1}),\varrho(t_{n-1},t_n),\varrho(t_{n-2},t_n),\\ \varrho(t_{n-1},t_{n-1}),\varrho(t_{n-1},t_n),\varrho(t_n,t_{n+1})\end{array}\right\}\\ &\leq [\theta(\max\{\varrho(t_{n-2},t_{n-1}),\varrho(t_{n-1},t_n),\varrho(t_n,t_{n+1})\})]^{\kappa}.\end{aligned}$$

If $\max\{\varrho(t_{n-2},t_{n-1}),\varrho(t_{n-1},t_n),\varrho(t_n,t_{n+1})\}=\varrho(t_n,t_{n+1})$ then we get

$$\theta(\varrho(t_n,t_{n+1}))\leq[\theta(\varrho(t_n,t_{n+1}))]^{\kappa},$$

which is once again contradiction. Therefore, we have

$$\theta(\varrho(t_n,t_{n+1}))\leq[\theta(\max\{\varrho(t_{n-2},t_{n-1}),\varrho(t_{n-1},t_n)\})]^{\kappa}$$

and

$$\begin{aligned}\theta(\varrho(t_n,t_{n+1})) &\leq [\theta(\varrho(t_{n-1},t_n))]^{\kappa}\\ &\leq [\theta(\varrho(t_{n-2},t_{n-1}))]^{\kappa^2}\\ &\;\;\vdots\\ &\leq [\theta(m)]^{\kappa^l},\end{aligned}$$

whenever $l=2n$ or $l=2n+1$, for $l\geq 1$. Hence, we have

$$1\leq\theta(\varrho(t_n,t_{n+1}))\leq[\theta(m)]^{\kappa^l},\text{ for all } l\geq 1. \tag{2}$$

Letting $n\to\infty$, following two cases arise.

Case 1. $1\leq\theta(\varrho(t_n,t_{n+1}))\leq[\theta(m)]^{\kappa^n}$, for all $n\geq 2$ and n is even.

Case 2. $1\leq\theta(\varrho(t_n,t_{n+1}))\leq[\theta(m)]^{\kappa^{n-1}}$, for all $n\geq 3$ and n is odd.

From Case 1 and Case 2 we get $\lim_{n\to\infty}\theta(\varrho(t_n,t_{n+1}))=1$. By the second axiom of θ, we get $\lim_{n\to\infty}\varrho(t_n,t_{n+1})=0$. From the third axiom of θ, there exist $\ell\in(0,\infty]$ and $r\in(0,1)$

$$\lim_{n\to\infty}\frac{\theta(\varrho(t_n,t_{n+1}))-1}{[\varrho(t_n,t_{n+1})]^r}=\ell.$$

Assume that $\ell<\infty$ and $\top=\frac{\ell}{2}>0$. From the limit definition, there exists $n_0\in\mathbb{N}$ such that

$$\left|\frac{\theta(\varrho(t_n,t_{n+1}))-1}{[\varrho(t_n,t_{n+1})]^r}-\ell\right|\leq\top\text{ for all } n\geq n_0$$

which implies that

$$\frac{\theta\varrho(t_n,t_{n+1})-1}{[\varrho(t_n,t_{n+1})]^r}\leq\ell-\top=\top\text{ for all } n\geq n_0.$$

Therefore, we have

$$n[\varrho(t_n,t_{n+1})]^r\leq\Bbbk n[\theta(\varrho(t_n,t_{n+1}))-1]\ \text{ for all } n\geq n_0$$

where $\Bbbk = \frac{1}{\top}$. Assume that $\top > 0$ is an arbitrary number and $\ell = \infty$. From the limit definition, there exists $n_0 \in \mathbb{N}$ such that

$$\frac{\theta\left(\varrho\left(t_n, t_{n+1}\right)\right) - 1}{\left[\varrho\left(t_n, t_{n+1}\right)\right]^r} \geq \top \text{ for all } n \geq n_0$$

which implies that

$$n\left[\varrho\left(t_n, t_{n+1}\right)\right]^r \leq \Bbbk n\left[\theta\left(\varrho\left(t_n, t_{n+1}\right)\right) - 1\right] \text{ for all } n \geq n_0 \tag{3}$$

where $\Bbbk = \frac{1}{\top}$. Therefore, in two cases there exists $n \geq n_0$ and $\Bbbk > 0$ such that (2.3) is satisfied. Using (2.2), we get

$$n\left[\varrho\left(t_n, t_{n+1}\right)\right]^r \leq \Bbbk n\left(\left[\theta\left(m\right)\right]^{k^l} - 1\right) \text{ for all } l \geq 2n_0 + 1 \text{ or } l \geq 2n_0.$$

Letting $n \to \infty$, we get $\lim\limits_{n\to\infty} n\left[\varrho\left(t_n, t_{n+1}\right)\right]^r = 0$. Hence, there exists $n_1 \in \mathbb{N}$ such that

$$\varrho\left(t_n, t_{n+1}\right) \leq \frac{1}{n^{\frac{1}{r}}} \text{ for all } n \geq n_1.$$

Now, we will demostrate that $\{t_n\}$ is a Cauchy sequence. For all $p > q \geq n_1$, we get

$$\begin{aligned}
\varrho\left(t_p, t_q\right) &\leq \varrho\left(t_p, t_{p-1}\right) + \varrho\left(t_{p-1}, t_{p-2}\right) + \cdots + \varrho\left(t_{q+1}, t_q\right) \\
&\leq \sum_{j=q}^{p-1} \varrho\left(t_j, t_{j+1}\right) \\
&< \sum_{j=q}^{\infty} \varrho\left(t_j, t_{j+1}\right) \\
&\leq \sum_{j=q}^{\infty} \frac{1}{j^{\frac{1}{r}}}.
\end{aligned}$$

Since $\sum_{j=q}^{\infty} \frac{1}{j^{\frac{1}{r}}}$ is convergent, $\lim\limits_{p,q\to\infty} \varrho\left(t_p, t_q\right) = 0$. Hence, we get that $\{t_n\}$ is a Cauchy sequence in W. Since (W, ϱ) is a complete metric space, there exists $u \in W$ such that $t_n \to u$. Assume that h is continuous. Since $t_n \to u \in W$ and W is complete metric space, we get

$$\varrho\left(u, hu\right) = \lim_{n\to\infty} \varrho\left(t_n, ht_n\right) = \lim_{n\to\infty} \varrho\left(t_n, t_{n+1}\right) = 0.$$

Therefore $u \in F\left(h\right)$. Again, assume that h is orbitally continuous on W, then

$$t_{n+1} = ht_n = h\left(h^n t_0\right) \to hu \text{ as } n \to \infty.$$

Since W is complete metric space, $hu = u$ that is $u \in F\left(h\right)$. Now, assume that u and v are arbitrary two fixed point of h. Then we get

$$\theta\left(\varrho\left(u, v\right)\right) = \theta\left(\varrho\left(h^2 u, h^2 v\right)\right) \leq \left[\theta\left(\varsigma\left(M_I\left(u, v\right)\right)\right)\right]^{\kappa} + LN_I\left(u, v\right)$$

$$
\begin{aligned}
&\leq \left[\theta\left(\varsigma\left(\max\left\{\begin{array}{c} \varrho(u,v), \varrho(hu,hv), \varrho(u,hu), \varrho(v,hv), \\ \varrho\left(hu,h^2u\right), \varrho\left(hv,h^2v\right) \end{array}\right\}\right)\right)\right]^{\kappa} \\
&\quad + L \min\left\{\begin{array}{c} \varrho(u,hu), \varrho(v,hv), \varrho(u,hv), \varrho(v,hu), \\ \varrho\left(hu,h^2u\right), \varrho\left(hv,h^2v\right) \end{array}\right\} \\
&\leq [\theta(\varrho(u,v))]^{\kappa}.
\end{aligned}
$$

Thus we get

$$\theta(\varrho(u,v)) \leq [\theta(\varrho(u,v))]^{\kappa}.$$

If we take ln two both sides of the inquality, then we obtain

$$\ln\theta(\varrho(u,v)) \leq \kappa \ln\theta(\varrho(u,v)).$$

Since $\kappa \in (0,1)$, it is a contradiction. Hence $u = v$, that is, h has a unique fixed point in W. □

Now, we shall give an example to illustrate the generality of Theorem 1.

Example 1. *Let (W, ϱ) be a metric space, h be a self mapping on W and $\theta(t) = e^{\sqrt{t}}$ for $t > 0$, that is, $\theta \in \mho$. Assume that h is a convex contraction of type-2 for all $t, s \in W$ with $\varrho(h^2t, h^2s) > 0$, $B = \sum_{j=1}^{6} d_j < 1$ and $d_j \geq 0$ for all $j = 1, 2, \ldots, 6$.*

$$
\begin{aligned}
\varrho(h^2t,h^2s) &\leq d_1\varrho(t,s) + d_2\varrho(ht,hs) + d_3\varrho(t,ht) + d_4\varrho(s,hs) \\
&\quad + d_5\varrho(ht,h^2t) + d_6\varrho(hs,h^2s) \\
&\leq \sum_{j=1}^{6} d_j \max\left\{\varrho(t,s), \varrho(ht,hs), \varrho(t,ht), \varrho(s,hs), \varrho(ht,h^2t), \varrho(hs,h^2s)\right\} \\
&\leq B \max\left\{\varrho(t,s), \varrho(ht,hs), \varrho(t,ht), \varrho(s,hs), \varrho(ht,h^2t), \varrho(hs,h^2s)\right\},
\end{aligned}
$$

where $t, s \in W$ with $\varrho(h^2t, h^2s) > 0$. We obtain that

$$\varrho\left(h^2t, h^2s\right) \leq BM_I(t,s).$$

Taking $\varsigma(t) = B^{\frac{1}{2}}t$, we have

$$e^{\sqrt{\rho(h^2t,h^2s)}} \leq e^{B^{\frac{1}{4}}\sqrt{M_I(t,s)}} = \left[e^{\sqrt{\varrho(M_I(t,s))}}\right]^{\kappa}$$

where $\kappa = B^{\frac{1}{4}}$. Since $\theta(t) = e^{\sqrt{t}}$ for $t > 0$, we deduce that

$$
\begin{aligned}
\theta\left(\varrho(h^2t,h^2s)\right) &\leq [\theta(\varsigma(M_I(t,s)))]^{\kappa} \\
&\leq [\theta(\varsigma(M_I(t,s)))]^{\kappa} + LN_I(t,s),
\end{aligned}
$$

where $L \geq 0$. This shows that, h is a generalized $\theta-$convex contractive mapping.

Remark 3. *Above example show that our contraction condition generalizes Istratescu's contraction conditions [9], [10].*

Definition 7. *Let* (W,ϱ) *be a metric space. A self-mapping* $h:W\to W$ *is called an almost* $\theta-$*convex contraction if there exist* $L\geq 0$ *and* $\kappa\in(0,1)$ *such that*

$$\varrho(h^2t,h^2s)>0\Rightarrow\theta(\varrho(h^2t,h^2s)\leq[\theta(M_I(t,s))]^{\kappa}+LN_I(t,s)$$

where $\theta\in\mho$ *and*

$$\begin{aligned}M_I(t,s) &= \max\left\{\varrho(t,s),\varrho(ht,hs),\varrho(t,ht),\varrho(s,hs),\varrho(ht,h^2t),\varrho(hs,h^2s)\right\},\\ N_I(t,s) &= \min\left\{\varrho(t,ht),\varrho(s,hs),\varrho(t,hs),\varrho(s,ht),\varrho(ht,h^2t),\varrho(hs,h^2s)\right\},\end{aligned}$$

for all $t,s\in W$.

Definition 6 and Definition 7 generalize and merge the results derived by Jleli [5] and Istratescu [9], [10], and some other connected results in the literature. Also, our novel contractions can be considered as an attracted generalization of Darbo's fixed point problem [26], [27].

Corollary 1. *Let* (W,ϱ) *be a complete metric space and* $h:W\to W$ *be an almost* $\theta-$*convex contraction. If* h *is either orbitally continuous on* W *or* h *is continuous, then* h *has a unique fixed point that is* $u=hu$, $u\in W$.

If we take $L=0$ in Theorem 1, then we obtain the following corollary.

Corollary 2. *Let* (W,ϱ) *be a metric space and* $h:W\to W$ *be a self-mapping. If there exist* $\varsigma\in\Upsilon$ *and* $\kappa\in(0,1)$ *such that*

$$\varrho(h^2t,h^2s)>0\Rightarrow\theta(\varrho(h^2t,h^2s))\leq[\theta(\varsigma(M_I(t,s)))]^{\kappa} \tag{4}$$

where $\theta\in\mho$ *and*

$$M_I(t,s)=\max\left\{\varrho(t,s),\varrho(ht,hs),\varrho(t,ht),\varrho(s,hs),\varrho(ht,h^2t),\varrho(hs,h^2s)\right\},$$

for all $t,s\in W$. *Also, assume that* h *is either orbitally continuous on* W *or* h *is continuous, then* $u=hu$, $u\in W$.

By taking $L=0$ and not considering $\varsigma\in\Upsilon$ in Theorem 1, we deduce the following corollary.

Corollary 3. *Let* (W,ϱ) *be a metric space and a self-mapping* h *on* W. *If there exist* $\kappa\in(0,1)$ *such that*

$$\varrho(h^2t,h^2s)>0\Rightarrow\theta(\varrho(h^2t,h^2s))\leq[\theta(M_I(t,s))]^{\kappa} \tag{5}$$

where $\theta\in\mho$ *and*

$$M_I(t,s)=\max\left\{\varrho(t,s),\varrho(ht,hs),\varrho(t,ht),\varrho(s,hs),\varrho(ht,h^2t),\varrho(hs,h^2s)\right\},$$

for all $t,s\in W$. *Also, assume that* h *is either orbitally continuous on* W *or* h *is continuous, then* $u=hu$, $u\in W$.

We get the following results as shown in Example 1.

Corollary 4. *Let (W, ϱ) be a metric space and a self-mapping h on W. For all $t, s \in W$,*

$$\varrho(h^2 t, h^2 s) \leq B \max\left\{\varrho(t,s), \varrho(ht,hs), \varrho(t,ht), \varrho(s,hs), \varrho(ht,h^2 t), \varrho(hs,h^2 s)\right\}$$

where $B \in [0,1)$. Also, assume that h is either orbitally continuous on W or h is continuous, then $u = hu$, $u \in W$.

Corollary 5. *Let (W, ϱ) be a metric space and h is a convex contraction of type-2 on W. Also, assume that h is either orbitally continuous on W or h is continuous, then $u = hu$, $u \in W$.*

3. Application

Now, we give an application of our result for nonlinear integral equations.

$$t(u) = \vartheta(u) + \int_e^f K(u, v, t(v))\, dv \tag{6}$$

where $e, f \in \mathbb{R}$, $C[e,f] = \{h : [e,f] \to \mathbb{R} \text{ continuous functions}\}$, $t \in C([e,f], \mathbb{R})$, $K : [e,f] \times [e,f] \times \mathbb{R} \to \mathbb{R}$ and $\vartheta : [e,f] \to \mathbb{R}$.

Theorem 2. *Consider the integral equation* (3.1). *Assume that the following conditions satisfy:*

(i) $K : [e,f] \times [e,f] \times \mathbb{R} \to \mathbb{R}$ and $\vartheta : [e,f] \to \mathbb{R}$ are continuous functions;
(ii) there exists $\gamma \in [0,1)$ such that

$$|K(u,v,ht(v)) - K(u,v,hs(v))| \leq \gamma \frac{\max\left\{\begin{array}{c} |t(v) - s(v)|, |ht(v) - hs(v)|, \\ |t(v) - ht(v)|, \left|ht(v) - h^2 t(v)\right|, \\ |s(v) - hs(v)|, \left|hs(v) - h^2 s(v)\right| \end{array}\right\}}{f - e}$$

for all $t, s \in C([e,f], \mathbb{R})$ and $u, v \in [e,f]$.

Then nonlinear integral equation (3.1) has a unique solution.

Proof. $W = C[e,f]$, $\varrho(h,g) = |h - g| = \max_{t \in [e,f]} |ht - gt|$, for all $h, g \in W$, and (W, ϱ) is a complete metric space. $h : W \to W$ be a continuous operator defined by

$$ht(u) = \vartheta(u) + \int_e^f K(u,v,t(v))\, dv.$$

Starting at the point $t_0 \in W$, the sequence $\{t_n\}$ is constructed by $t_n = ht_{n-1} = h^n t_0$, $n \geq 1$. From (3.1), we get

$$t_{n+1} = ht_n(u) = \vartheta(u) + \int_e^f K(u,v,t_n(v))\, dv.$$

Now, we will demonstrate that h is a generalized $\theta-$convex contractive mapping. We can write

$$\begin{aligned}\left|h^2t(u)-h^2s\left(u\right)\right| &= \left|\int_e^f K\left(u,v,ht\left(v\right)\right)dv-\int_e^f K\left(u,v,hs\left(v\right)\right)dv\right| \\ &\leq \int_e^f \left|K\left(u,v,ht\left(v\right)\right)-K\left(u,v,hs\left(v\right)\right)\right|dv \\ &\leq \frac{\gamma}{f-e}\int_e^f \max\left\{\begin{array}{c}\left|t(v)-s\left(v\right)\right|,\left|ht(v)-hs\left(v\right)\right|, \\ \left|t(v)-ht\left(v\right)\right|,\left|ht(v)-h^2t\left(v\right)\right|, \\ \left|s(v)-hs\left(v\right)\right|,\left|hs(v)-h^2s\left(v\right)\right|\end{array}\right\}dv\end{aligned}$$

and

$$\begin{aligned}\varrho\left(h^2t,h^2s\right) &= \max_{u\in[e,f]}\left|h^2t(u)-h^2s\left(u\right)\right| \\ &\leq \frac{\gamma}{f-e}\max_{u\in[e,f]}\int_e^f \max\left\{\begin{array}{c}\left|t(v)-s\left(v\right)\right|,\left|ht(v)-hs\left(v\right)\right|, \\ \left|t(v)-ht\left(v\right)\right|,\left|ht(v)-h^2t\left(v\right)\right|, \\ \left|s(v)-hs\left(v\right)\right|,\left|hs(v)-h^2s\left(v\right)\right|\end{array}\right\}dv \\ &\leq \frac{\gamma}{f-e}\max\left[\max_{c\in[e,f]}\left\{\begin{array}{c}\left|t(c)-s\left(c\right)\right|,\left|ht(c)-hs\left(c\right)\right|, \\ \left|t(c)-ht\left(c\right)\right|,\left|ht(c)-h^2t\left(c\right)\right|, \\ \left|s(c)-hs\left(c\right)\right|,\left|hs(c)-h^2s\left(c\right)\right|\end{array}\right\}\right]\int_e^f dv \\ &\leq \gamma\max\left\{\varrho(t,s),\varrho(ht,hs),\varrho(t,ht),\varrho(s,hs),\varrho(ht,h^2t),\varrho(hs,h^2s)\right\} \\ &\leq \gamma M_I(t,s).\end{aligned}$$

Thus

$$\varrho\left(h^2t,h^2s\right)\leq\gamma M_I(t,s).$$

Define $\theta(t)=e^{\sqrt{t}}$ for $t>0$ and $\varsigma(t)=\gamma^{\frac{1}{2}}t$. We have

$$e^{\sqrt{\varrho(h^2t,h^2s)}}\leq e^{\gamma^{\frac{1}{4}}\sqrt{M_I(t,s)}}=\left[e^{\sqrt{\varrho(M_I(t,s))}}\right]^{\kappa}$$

where $\kappa=\gamma^{\frac{1}{4}}$. Thus, we get

$$\theta\left(\varrho(h^2t,h^2s\right)\leq\left[\theta\left(\varsigma\left(M_I(t,s)\right)\right)\right]^{\kappa}+LN_I(t,s)$$

where $L\geq 0$. This shows that, h is a generalized $\theta-$convex contractive mapping. That is, the conditions of Theorem 1 are hold. Thus, h has a unique fixed point in W, and so, the nonlinear integral equation (3.1) has a unique solution. □

4. Conclusion

We present generalized $\theta-$convex contractive mappings in this paper. This contractive condition not only extends several existing contraction definitions but also merge some existing contractions. Afterward, we investigate the existence of a fixed point for our novel type contraction, we state some consequences. Our results generalize and merge the results derived by Istratescu [9], [10] and Jleli [5], and some other connected results in the literature. Our new contraction can be considered as an interesting generalization of Darbo's fixed point problem [26], [27]. As well as the corollaries in this paper, to underline the novelty of our given results, we show an example that shows that Theorem 1 is a genuine generalization of Istratescu's results [9]. Moreover, as a possible application, we applied our main results to study the existence of a solution for a nonlinear integral equation. The new concept allows for further studies and applications. By choosing the appropriate auxiliary function such as simulation function and others, one can get several more results. Also, one can get the analogue of our result in the set-up of cyclic mappings.

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