



Interval Linear Programming And Fuzzy DEA-BCC Models With Ranking Of DMU New Approach

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Abstract

In this paper, a new approach for obtaining Ranking efficiency with the fuzzy data numbers are being considered. Most Fuzzy DEA models are introduced in the literary words which are parametric models structured on alpha cuts. yet , the model is introduced in this study is parametric and is used trapezoidal fuzzy number. From the theoretical perspective, the objective of this study is to develop a simple and effective Fuzzy DEA-BCC model. The most and maximum possible efficiency scores of each DMU are estimated a few α -level, this model can be applied to determine many issues are associated with qualitative factors. It is checked by applying the proposed method in two numerical examples are compared with the results of eight current models of fuzzy DEA.

Keywords: Fuzzy DEA, Efficiency Measurement, Ranking Model, α -Level, Decision Making Unit.

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Aralıklı Lineer Programlama ve Bulanık VZA-BCC Modelleri ile KVB Sıralamalı Yeni Yaklaşım

Özet

Bu yazıda, bulanık veri sayıları ile Sıralamada etkinlik elde etmek için yeni bir yaklaşım ele alınmaktadır. Literatürde sunulan bulanık KVB modellerinin çoğu, alfa kesimler üzerinde yapılandırılmış parametrik modellerdir. Ancak bu çalışmada tanıtılan model parametrik ve trapezoid bulanık sayı kullanmaktadır. Teori perspektifinden, bu çalışmanın amacı, basit ve etkili bir Bulanık KVB-BCC modeli geliştirmektir. Her bir KVB'nun mümkün olan en fazla ve en fazla verim puanının birkaç α seviyesinde olduğu tahmin edilmektedir, bu model, nitel faktörlerle ilişkili birçok sorunu belirlemek için uygulanabilir. İki sayısal örnekte önerilen yöntemi uygulayarak bulanık KVB'nın sekiz güncel modelinin sonuçları ile karşılaştırılmıştır.

Anahtar Kelimeler: Bulanık KVB, Etkinlik Ölçümü, Sıralama Modeli, α -Düzeyi, Karar Verme Birimi.

1. Introduction and DEA Preliminaries

Data envelopment analysis (DEA) is a linear programming method (LP), which measures the relative efficiency of the associated with the decision-making units (DMUs) when multiple inputs and outputs are present. To examine the radial technical efficiency of a given DMU_p, Charnes et al. [6] proposed the constant RTS model (CRS or CCR). Assume that there are n DMUs to be evaluated, where every DMU_j ($j = 1, 2, \dots, n$), produces s outputs, y_{rj} ($r = 1, 2, \dots, s$), using m inputs, x_{ij} ($i = 1, 2, \dots, m$). The CCR model is proposed to evaluate the efficiency of a specific DMU_p [6]. And to see the other basic models of DEA, the defendant is able to see [3,7]. The Advanced DEA models divided DMUs into two efficient and inefficient groups while in practice. So, there are a variety of researches classified ranking methods and Fuzzy Data envelopment analysis (FDEA) are presented in the DEA literature [1,21]. This method removes the unit under the assessment from a group of DMUs, and measures the distance of DMU from the new efficient frontier. The α -level approach is the most popular fuzzy DEA models. Many articles are published about this method in the literary DEN search.

From this approach, the key two ideas are to transform the fuzzy CCR model in some parametric programs in order to find the lower and upper limits or interval of the α -level membership functions of the efficiency scores. And find the fuzzy efficiency scores of the DMUs using fuzzy linear programs which require ranking fuzzy sets. It is created by the membership function for spaces of input and output on the most basis of the interpretation of the tolerance limits.

Kao and Liu [36] followed up on the basic idea of transforming a fuzzy DEA model to a family of conventional crisp DEA models and developed a solution procedure to measure the efficiencies of the DMUs with fuzzy observations in the BCC model. Kao and Liu [36] proposed a pair of two-level mathematical models to calculate the lower bound and upper bound of the fuzzy efficiency score for a specific α -level and used the ranking fuzzy numbers method of Chen and Klein to rank the obtained fuzzy efficiencies [8,9]. Saati et al. [52] suggested a fuzzy CCR model as a possibilistic programming problem and transformed it into an interval programming problem using α -level based approach. The resulting interval programming problem could be solved as a crisp LP model for a given α with some variable substitutions, use triangular fuzzy inputs and the triangular fuzzy outputs, and x'_{ij} and y'_{rj} are the decision variables obtained from variable substitutions used to transform the original fuzzy model proposed into a parametric LP model with $\alpha \in [0,1]$. Saati and Memariani [54] suggested a procedure for determining a common set of weights in fuzzy DEA based on the α -level method proposed by Saati et al. [52] with triangular fuzzy data. In this method, the upper bounds of the input and output weights were determined by solving some fuzzy LP models and then a common set of weights were obtained by solving another fuzzy LP model. Hatami-Marbini and Saati [20] developed a fuzzy BCC model which considered fuzziness in the input and output data as well as the u_0 variable. Consequently, they obtained the stability of the fuzzy u_0 as an interval by means of the method proposed by Saati et al. [52]. Hatami-Marbini et al. [16] used the method of Saati et al. [52] and proposed a four-phase fuzzy DEA framework based on the theory of displaced ideal. Liu et al. [43] developed a modified fuzzy DEA model to handle fuzzy and incomplete information on weight indices in product design evaluation transformed fuzzy information into trapezoidal fuzzy numbers and considered incomplete information on indices weights as constraints. They used an α -level approach to convert their fuzzy DEA model into a family of conventional crisp DEA models.

Jahanshahloo et al. [33] developed a fuzzy l_1 -norm model with trapezoidal fuzzy inputs/outputs that was initially suggested by Jahanshahloo et al. [29] for solving the crisp data in DEA. Wang et al. [59] proposed a fuzzy DEA-Neural approach with a self-organizing map for classification in their neural network.

They used the upper and lower bounds or enterval of efficiency score at different possibilistic levels in their model. Hosseinzadeh Lotfi et al. [24] developed two methods for solving fuzzy CCR model with respect to fuzzy, ordinal and exact data. They used an analogue function to transform the fuzzy data into exact values in the first method.

In the second approach (fuzzy ranking approach) of efficiency measurement was initially developed by Guo and Tanaka [15]. Furthermore, they proposed a fuzzy BCC model using the fuzzy number ranking method proposed by Yao and Wu [61] for trapezoidal fuzzy data in DEA.

In this study, we determine the maximum possible efficiency scores of each DMU at some α -levels. The obtained score of each DMU at level α can be represented by an interval. A number of these crisp intervals are used to represent the scores as a fuzzy interval. Finally, DMUs are ranked according to their fuzzy scores. So far, several papers have been presented on the theoretical development of DEA whit fuzzy data (FDEA models).

The defuzzification approach was developed by Lertworasirikul [41,42]. In this approach the fuzzy inputs and outputs are defuzzified into crisp values. This approach is simple but ignores the uncertainty in data. The α -level based approach was introduced by Kao and Liu [36], and Lertworasirikul [41]. In this approach, the FDEA model is solved by parametric programming using α -cuts. Solving the model at a given α -level produces a corresponding interval efficiency for the target DMU. A number of such intervals can be used to construct the corresponding fuzzy efficiency. This approach provides fuzzy efficiency but requires the ranking of fuzzy efficiency sets. The fuzzy ranking approach was developed by Guo and Tanaka [14]. In this approach, Both fuzzy inequalities and equalities in the fuzzy CCR model are defined by ranking methods so that the resulting model is α -level PLP model. This approach provides fuzzy efficiency for an evaluated DMU at a specified α -level. Guo and Tanaka compare fuzzy efficiencies using only one number at a given α -level. Lertworasirikul

et al. [41,42] show that for the special case, in which fuzzy membership functions of fuzzy data are trapezoidal types.

Charnes et al. [6] proposed the constant RTS model (CRS or CCR) to evaluate the radial technical efficiency of a given DMU_p. Assume that there are n DMUs to be evaluated, where every DMU_j ($j = 1, 2, \dots, n$), produces s outputs, y_{rj} ($r = 1, 2, \dots, s$), using m inputs, x_{ij} ($i = 1, 2, \dots, m$). The BCC and CCR models differ only in that the former includes an additional convexity constraint, $\sum_{j=1}^n \lambda_j = 1$, the additional variable, u_0 , in the dual BCC model as shown in this model (1).

$$\max w_p = \sum_{r=1}^s u_r y_{rp} - u_0$$

such that

$$\sum_{i=1}^m v_i x_{ip} = 1,$$

$$\sum_{r=1}^s u_r y_{rj} - \sum_{i=1}^m v_i x_{ij} - u_0 \leq 1 \quad (\forall j), \quad (1)$$

$$u_r, v_i \geq 0 \quad (\forall r, i).$$

2. Fuzzy Set Theory

One way to describe the vagueness and lack of precision of data is fuzzy sets theory which was introduced by Zadeh [63] in 1965. Here some of the definitions of this theory are presented [8,64].

Definition 2.1: (Fuzzy set) If X is a collection of objects denoted by x , then a fuzzy set \tilde{x} in X is a set of ordered pairs:

$$\tilde{x} = \{(x; \mu_{\tilde{x}}(x)) / x \in X\}$$

such that $\mu_{\tilde{x}}(x)$ is called the membership function which for each $x \in X$ associates a number in $[0,1]$, indicating to what degree x is a member of X .

Definition 2.2: (Convex fuzzy set) A fuzzy set \tilde{x} is convex if

$$\mu_{\tilde{x}}(\lambda x_1 + (1-\lambda)x_2) \geq \min\{\mu_{\tilde{x}}(x_1), \mu_{\tilde{x}}(x_2)\} \quad (x_1, x_2 \in X \text{ and } \lambda \in [0;1]). \quad (2)$$

Definition 2.3: (Normal fuzzy set) A fuzzy set \tilde{x} in X is said to be normal if there exist $x \in X$ such that $\mu_{\tilde{x}}(x) = 1$.

Definition 2.4: (Fuzzy number) A fuzzy number \tilde{x} is a convex normalized fuzzy set \tilde{x} of the real line \mathbb{R} such that

1. There exist exactly one $x_0 \in \mathbb{R}$ with $\mu_{\tilde{x}}(x_0) = 1$ (*unimodal*).
2. $\mu_{\tilde{x}}(x)$ is piecewise continuous.

Definition 2.5: (Positive fuzzy number) A fuzzy number \tilde{x} is called positive (negative), denoted by $\tilde{x} > 0$ ($\tilde{x} < 0$), if its membership function, $\mu_{\tilde{x}}(x)$ satisfies, $\mu_{\tilde{x}}(x) = 0, x < 0$ ($x > 0$).

Definition 2.6: (LR fuzzy number) A fuzzy number \tilde{x} is said to be LR if

$$\mu_{\tilde{x}}(x) = \begin{cases} L\left(\frac{a-x}{\sigma}\right) & x < a, \sigma > 0 \\ R\left(\frac{x-b}{\beta}\right) & x > b, \beta > 0 \end{cases}, \quad (3)$$

where σ and β are left and right spreads, respectively, and a function $L(\cdot)$ is the left shape function satisfying:

1. $L(x) = L(-x)$,
2. $L(0) = 1$ and $L(1) = 0$,
3. $L(x)$ is non-decreasing on $[0, \infty)$.

Naturally, a right shape function $R(\cdot)$ is similarly defined as $L(\cdot)$.

Definition 2.7: (LR fuzzy interval) A fuzzy set \tilde{x} is said to be an LR fuzzy interval if

$$\mu_{\tilde{x}}(x) = \begin{cases} L\left(\frac{a-x}{\sigma}\right) & x < a, \sigma > 0 \\ 1 & a \leq x \leq b \\ R\left(\frac{x-b}{\beta}\right) & x > b, \beta > 0 \end{cases}, \quad (4)$$

where $[a, b]$ is the peak or core of \tilde{x} and a and b are left and right spreads, respectively, and the functions $L(\cdot)$ and $R(\cdot)$ are the same as the functions of LR fuzzy number.

Definition 2.8: (Triangular fuzzy number) A LR fuzzy number \tilde{x} is said to be triangular if $L(\cdot)$ and $R(\cdot)$ be linear functions.

Remark: A membership function of triangular fuzzy number $\tilde{x} = (L; M; R)$ ($L \leq M \leq R$) is as follows:

$$\mu_{\tilde{x}}(x) = \begin{cases} \left(\frac{x-L}{M-L}\right) & L \leq x < M \\ 1 & x = M \\ \left(\frac{x-R}{M-R}\right) & M < x \leq R \end{cases} . \quad (5)$$

Definition 2.9: (Trapezoidal fuzzy number) $\tilde{x} = (x_0, y_0, \sigma, \beta)$ with two defuzzifier x_0, y_0 and left fuzziness $\sigma > 0$ and right fuzziness $\beta > 0$ is a fuzzy set where the membership function is as:

$$\mu_{\tilde{x}}(x) = \begin{cases} \frac{1}{\sigma}(x - x_0 + \sigma) & x_0 - \sigma \leq x \leq x_0 \\ 1 & x \in [x_0, y_0] \\ \frac{1}{\beta}(y_0 - x + \beta) & y_0 \leq x \leq y_0 + \beta \\ 0 & otherwise \end{cases} . \quad (6)$$

4. Introduction to Theory

In this section, we review some basic definitions of fuzzy sets [11,35,65].

Definition 3.1: A fuzzy number \tilde{x} in parametric form is a pair $(\tilde{x}^i, \tilde{x}^d)$ of functions $\tilde{x}^i(r), \tilde{x}^d(r), 0 \leq r \leq 1$, which satisfy the following requirements:

1. \tilde{x}^i is a bounded monotonic increasing left continuous function,
2. \tilde{x}^d is a bounded monotonic decreasing left continuous function,
3. $\tilde{x}^i \leq \tilde{x}^d, 0 \leq r \leq 1$

and its parametric form is

$$\tilde{x}^i(r) = (x_0 - \sigma + \sigma r), \quad \tilde{x}^d(r) = (y_0 + \beta - \beta r).$$

Provided that, $x_0 = y_0$ then \tilde{x} is a triangular fuzzy number, and we write $\tilde{x} = (x_0, \sigma, \beta)$.

Definition 3.2: The support of fuzzy number \tilde{x} is defined as follows:

$$\text{Supp}(\tilde{x}) = \overline{\{x \mid \mu_{\tilde{x}}(x) > 0\}},$$

where $\overline{\{x \mid \mu_{\tilde{x}}(x) > 0\}}$ is closure of set $\{x \mid \mu_{\tilde{x}}(x) > 0\}$.

Definition 3.3: The addition and scalar multiplication of fuzzy numbers are defined by the extension principle and can be equivalently represented in [8,62,65] as follows. For arbitrary $\tilde{x} = (\tilde{x}^i, \tilde{x}^d)$, $\tilde{y} = (\tilde{y}^i, \tilde{y}^d)$ we define addition $\tilde{x} + \tilde{y}$ and multiplication by scalar $k > 0$ as:

$$\begin{aligned} (\tilde{x}^i + \tilde{y}^i)(r) &= \tilde{x}^i(r) + \tilde{y}^i(r) , (\tilde{x}^d + \tilde{y}^d)(r) = \tilde{x}^d(r) + \tilde{y}^d(r) , \\ (k\tilde{x})^i(r) &= k\tilde{x}^i(r) , (k\tilde{x})^d(r) = k\tilde{x}^d(r) , \\ (k\tilde{y})^i(r) &= k\tilde{y}^i(r) , (k\tilde{y})^d(r) = k\tilde{y}^d(r). \end{aligned} \quad (7)$$

To emphasis the collection of all fuzzy numbers with addition and multiplication as defined by (7) is denoted by E, which is a convex cone. The image (opposite) of $\tilde{x} = (x_0, y_0, \sigma, \beta)$ can be defined by $-\tilde{x} = (-x_0, -y_0, \beta, \sigma)$, see [26,65].

Definition 3.4: (α -cut of fuzzy set) A α -cut of fuzzy set \tilde{x} is a crisp subset of X which, denoted by:

$$\begin{aligned} \tilde{x}^\alpha = \{x \in X / \mu_{\tilde{x}}(x) \geq \alpha\} &= [x^{\alpha L}; x^{\alpha U}] = [X^{\min} \{x \in X / \mu_{\tilde{x}}(x) \geq \alpha\} \\ & ; X^{\max} \{x \in X / \mu_{\tilde{x}}(x) \geq \alpha\}] \end{aligned} \quad (8)$$

The α -cuts of \tilde{x} and \tilde{y} are defined as:

$$\tilde{x}^\alpha = \{x \in X / \mu_{\tilde{x}}(x) \geq \alpha\} = [x^{\alpha L}; x^{\alpha U}], \quad (9)$$

and

$$\tilde{y}^\alpha = \{x \in X / \mu_{\tilde{y}}(x) \geq \alpha\} = [y^{\alpha L}; y^{\alpha U}] \quad (10)$$

We can draw a membership function of triangular fuzzy number with concept of local α -cut .

Definition 3.5: The trapezoidal fuzzy number $\tilde{x} = (x_0, y_0, \sigma, \beta)$ is reduced to a real number \tilde{x} if $x_0 = y_0 = \sigma = \beta$. Conversely, a real number u can be written as a trapezoidal fuzzy number $\tilde{x} = (x, x, x, x)$. Similarly, the α -level $\tilde{x} = (x_0, y_0, \sigma, \beta)$ can easily be determined as:

$$[\tilde{x}]^\alpha = [\alpha x_0 + (1 - \alpha)\sigma, \alpha y_0 + (1 - \alpha)\beta], \quad (11)$$

where $\alpha \in [0,1]$. If $x = x_0 = y_0$, then $\tilde{x} = (x_0, \sigma, \beta)$ is called a triangular fuzzy number.

Definition 3.6: A real fuzzy number \tilde{x} denoted by $\tilde{x} = (x_0, y_0, \sigma, \beta, w)$ is described as any fuzzy subset of the real line \mathbb{R} with a membership function $\mu_{\tilde{x}}$, which satisfies the following properties:

$\mu_{\tilde{x}}$ is a semicontinuous mapping from \mathbb{R} to the closed interval $[0,w]$ ($0 \leq w \leq 1$),

$\mu_{\tilde{x}}(x) = 0$ for all $x \in [-\infty, \sigma]$,

$\mu_{\tilde{x}}$ is increasing on $[\sigma, x_0]$,

$\mu_{\tilde{x}}(x) = w$ for all $x \in [x_0, y_0]$, where w is a constant and $0 < w \leq 1$,

$\mu_{\tilde{x}}$ is decreasing on $[y_0, \beta]$,

$\mu_{\tilde{x}}(x) = 0$ for all $x \in [\beta, \infty]$,

where x_0, y_0, σ and β are real numbers.

Unless elsewhere specified, it is assumed that \tilde{x} is convex and bounded, i.e., $-\infty < \sigma, \beta < \infty$. If $w = 1$, \tilde{x} is a normal fuzzy number, and if $0 < w < 1$, \tilde{x} is a nonnormal fuzzy number.

Definition 3.7: Suppose that we have two positive trapezoidal fuzzy numbers $\tilde{A} = (x_a, y_a, \sigma_a, \beta_a)$ and $\tilde{B} = (x_b, y_b, \sigma_b, \beta_b)$, then the arithmetic operations of these two trapezoidal fuzzy numbers are defined as follows:

$$\tilde{A}(+) \tilde{B} = (x_a + x_b, y_a + y_b, \sigma_a + \sigma_b, \beta_a + \beta_b),$$

$$\tilde{A}(-) \tilde{B} = (x_a - x_b, y_a - y_b, \sigma_a - \sigma_b, \beta_a - \beta_b),$$

$$\tilde{A}(\times) \tilde{B} = (x_a x_b, y_a y_b, \sigma_a \sigma_b, \beta_a \beta_b),$$

$$k\tilde{A} = (kx_a, ky_a, k\sigma_a, k\beta_a) \quad (\forall k \in \mathbb{R}^+),$$

$$k\tilde{B} = (kx_b, ky_b, k\sigma_b, k\beta_b) \quad (\forall k \in \mathbb{R}^+),$$

$$(\tilde{A})^{-1} = \left(\frac{1}{y_a}, \frac{1}{x_a}, \frac{1}{\beta_a}, \frac{1}{\sigma_a}\right), \quad (\tilde{B})^{-1} = \left(\frac{1}{y_b}, \frac{1}{x_b}, \frac{1}{\beta_b}, \frac{1}{\sigma_b}\right),$$

$$\tilde{A}(\div) \tilde{B} = \tilde{A}(\times) \tilde{B}^{-1} = \left(\frac{x_a}{y_b}, \frac{y_a}{x_b}, \frac{\sigma_a}{\beta_b}, \frac{\beta_a}{\sigma_b}\right).$$

Definition 3.8: An alternative way of fuzzy arithmetic can be defined based on the interval of the arithmetic of α -level intervals. The interval arithmetic If \tilde{A} and \tilde{B} be two fuzzy numbers with α -level intervals $\tilde{A}^\alpha = [\tilde{A}^{\alpha L}, \tilde{A}^{\alpha U}]$ and $\tilde{B}^\alpha = [\tilde{B}^{\alpha L}, \tilde{B}^{\alpha U}]$ then the Definition 3.7 can be achieved as follows:

$$\tilde{A}^\alpha(+) \tilde{B}^\alpha = [\tilde{A}^{\alpha L} + \tilde{B}^{\alpha L}, \tilde{A}^{\alpha U} + \tilde{B}^{\alpha U}],$$

$$\tilde{A}^\alpha(-) \tilde{B}^\alpha = [\tilde{A}^{\alpha L} - \tilde{B}^{\alpha L}, \tilde{A}^{\alpha U} - \tilde{B}^{\alpha U}],$$

$$\begin{aligned}\tilde{A}^\alpha(\times)\tilde{B}^\alpha &= \left[\begin{array}{l} \min\{\tilde{A}^{\alpha L}\tilde{B}^{\alpha L}, \tilde{A}^{\alpha L}\tilde{B}^{\alpha U}, \tilde{A}^{\alpha U}\tilde{B}^{\alpha L}, \tilde{A}^{\alpha U}\tilde{B}^{\alpha U}\}, \\ \max\{\tilde{A}^{\alpha L}\tilde{B}^{\alpha L}, \tilde{A}^{\alpha L}\tilde{B}^{\alpha U}, \tilde{A}^{\alpha U}\tilde{B}^{\alpha L}, \tilde{A}^{\alpha U}\tilde{B}^{\alpha U}\} \end{array} \right], \\ (\tilde{A}^\alpha)^{-1} &= \left[\frac{1}{\tilde{A}^{\alpha L}}, \frac{1}{\tilde{A}^{\alpha U}} \right], (\tilde{B}^\alpha)^{-1} = \left[\frac{1}{\tilde{B}^{\alpha L}}, \frac{1}{\tilde{B}^{\alpha U}} \right], \\ \tilde{A}^\alpha(\div)\tilde{B}^\alpha &= \tilde{A}(\times)\tilde{B}^{-1} = \tilde{A}^\alpha(\times)\frac{1}{\tilde{B}^\alpha}.\end{aligned}$$

Definition 3.9: The minimum t -norm is usually applied in fuzzy linear programming to assess a linear combination of fuzzy quantities. Therefore, a given set of trapezoidal fuzzy numbers $\tilde{x}_j = (x_{0j}, y_{0j}, \sigma_j, \beta_j)$ ($j = 1, 2, \dots, n$) and $\lambda_j \geq 0, \sum_{j=1}^n \lambda_j \tilde{x}_j$ are defined as follows:

$$\sum_{j=1}^n \lambda_j \tilde{x}_j = \left(\sum_{j=1}^n \lambda_j x_{0j}, \sum_{j=1}^n \lambda_j y_{0j}, \sum_{j=1}^n \lambda_j \sigma_j, \sum_{j=1}^n \lambda_j \beta_j \right), \quad (12)$$

where $\sum_{j=1}^n \lambda_j \tilde{x}_j$ denotes the combination $\lambda_1 \tilde{x}_1 \oplus \lambda_2 \tilde{x}_2 \oplus \dots \oplus \lambda_n \tilde{x}_n$.

A trapezoidal fuzzy number is widely used for solving practical problems. Hence, for the sake of simplicity and without loss of generality, we assume that all fuzzy numbers used throughout in this section are trapezoidal fuzzy numbers.

4. Fuzzy BCC Model

In this section, we propose an alternative fuzzy BCC model for evaluating the efficiency of a set of DMUs with fuzzy inputs and outputs derived from the α -level approach proposed by Saati et al. [52]. Let us consider n DMUs, each of which uses m different fuzzy inputs to generate a different fuzzy outputs. And the model (1) standard form of BCC model for assessing DMUs is written with typical fuzzy BCC model with fuzzy data can be expressed as:

$$\max w_p = \frac{\sum_{r=1}^s u_r \tilde{y}_{rp} - u_0}{\sum_{i=1}^m v_i \tilde{x}_{ip}}$$

such that

$$\begin{aligned}\frac{\sum_{r=1}^s u_r \tilde{y}_{rp} - u_0}{\sum_{i=1}^m v_i \tilde{x}_{ij}} &\leq 1 \quad (\forall j), \\ u_r, v_i &\geq 0 \quad (\forall r, i).\end{aligned} \quad (13)$$

Therefore, model (13) can be rewritten as follows:

$$\max w_p = \frac{\sum_{r=1}^s u_r (x_{rp}, y_{rp}, \sigma_{rp}, \beta_{rp}) - u_0}{\sum_{i=1}^m v_i (x_{ip}, y_{ip}, \sigma_{ip}, \beta_{ip})}$$

such that

$$\frac{\sum_{r=1}^s u_r (x_{rj}, y_{rj}, \sigma_{rj}, \beta_{rj}) - u_0}{\sum_{i=1}^m v_i (x_{ij}, y_{ij}, \sigma_{ij}, \beta_{ij})} \leq 1 \quad (\forall j), \quad (14)$$

$$u_r, v_i \geq 0 \quad (\forall r, i).$$

The right-hand side of the first constraint in model (13) must be equal to 1 because of the normalization of the efficiency scores of the DMUs.

For all DMUs as well as for the measures of both the lower and upper bound efficiencies [62]:

Let

$$E_j = \frac{\sum_{r=1}^s u_r \tilde{y}_{rj}}{\sum_{i=1}^m v_i \tilde{x}_{ij}} - u_0^* \quad (j = 1, 2, \dots, n),$$

$$E_j = \frac{\sum_{r=1}^s u_r [\tilde{y}_{rj}^L; \tilde{y}_{rj}^U]}{\sum_{i=1}^m v_i [\tilde{x}_{ij}^L; \tilde{x}_{ij}^U]} - u_0^* = \frac{[\sum_{r=1}^s u_r \tilde{y}_{rj}^L; \sum_{r=1}^s u_r \tilde{y}_{rj}^U]}{[\sum_{i=1}^m v_i \tilde{x}_{ij}^L; \sum_{i=1}^m v_i \tilde{x}_{ij}^U]} - u_0^*$$

$$= \left[\frac{\sum_{r=1}^s u_r \tilde{y}_{rj}^L}{\sum_{i=1}^m v_i \tilde{x}_{ij}^U} - u_0^*, \frac{\sum_{r=1}^s u_r \tilde{y}_{rj}^U}{\sum_{i=1}^m v_i \tilde{x}_{ij}^L} - u_0^* \right] \quad (15)$$

should also be an interval number, which we denote by $[E_j^L, E_j^U] \subseteq (0,1]$ ($j = 1, 2, \dots, n$).

Then

$$\frac{\sum_{r=1}^s u_r \tilde{y}_{rj}^U}{\sum_{i=1}^m v_i \tilde{x}_{ij}^L} \leq 1 \quad (j = 1, 2, \dots, n),$$

$$\frac{\sum_{r=1}^s u_r \tilde{y}_{rj}^L}{\sum_{i=1}^m v_i \tilde{x}_{ij}^U} > 0 \quad (j = 1, 2, \dots, n).$$

Hence the score of DMU₀.

5. The Proposed Method

In this section, we develop the fuzzy BCC model in order to measure the crisp efficiency of each DMU under consideration for different α -values (α -levels), for all DMUs as well as for the measures of both the lower and upper bound efficiencies.

We can applying the α -cuts of fuzzy DEA, Using the equations (8-10) in model FBCC (13) the following model would be achieved:

$$\max E_{\alpha} = \frac{\sum_{r=1}^s u_r [\tilde{y}_{rp}^{\alpha L}; \tilde{y}_{rp}^{\alpha U}] - u_0}{\sum_{i=1}^m v_i [\tilde{x}_{ip}^{\alpha L}; \tilde{x}_{ip}^{\alpha U}]}$$

such that

$$\frac{\sum_{r=1}^s u_r [\tilde{y}_{rj}^{\alpha L}; \tilde{y}_{rj}^{\alpha U}] - u_0}{\sum_{i=1}^m v_i [\tilde{x}_{ij}^{\alpha L}; \tilde{x}_{ij}^{\alpha U}]} \leq 1 \quad (\forall j), \quad (16)$$

$$u_r, v_i \geq 0 \quad (\forall r, i).$$

Now interval DEA model is developed for measuring the upper and lower bounds of the best relative efficiency of each DMU with interval input and output data.

$$\max E^{\alpha U} = \sum_{r=1}^s u_r [\tilde{y}_{rp}^{\alpha U}] - u_0$$

such that

$$\sum_{i=1}^m v_i [\tilde{x}_{ip}^{\alpha L}] = 1,$$

$$\sum_{r=1}^s u_r [\tilde{y}_{rj}^{\alpha U}] - \sum_{i=1}^m v_i [\tilde{x}_{ij}^{\alpha L}] - u_0 \leq 0,$$

$$\sum_{r=1}^s u_r [\tilde{y}_{rj}^{\alpha L}] - \sum_{i=1}^m v_i [\tilde{x}_{ij}^{\alpha U}] - u_0 \leq 0 \quad (j = 1, 2, \dots, n; j \neq 0), \quad (17)$$

$$u_r, v_i \geq 0 \quad (\forall r, i).$$

$$\max E^{\alpha L} = \sum_{r=1}^s u_r [\tilde{y}_{rp}^{\alpha L}] - u_0$$

such that

$$\sum_{i=1}^m v_i [\tilde{x}_{ip}^{\alpha U}] = 1,$$

$$\sum_{r=1}^s u_r [\tilde{y}_{rj}^{\alpha L}] - \sum_{i=1}^m v_i [\tilde{x}_{ij}^{\alpha U}] - u_0 \leq 0,$$

$$\sum_{r=1}^s u_r [\tilde{y}_{rj}^{\alpha U}] - \sum_{i=1}^m v_i [\tilde{x}_{ij}^{\alpha L}] - u_0 \leq 0 \quad (j = 1, 2, \dots, n; j \neq 0), \quad (18)$$

$$u_r, v_i \geq 0 \quad (\forall r, i).$$

In model (17) and (18), the Trading and rotation variables:

$$u_r(x_{rp}, y_{rp}, \sigma_{rp}, \beta_{rp})^{\alpha} = \bar{y}_{rp} \quad (r = 1, 2, \dots, s; p = 1, 2, \dots, n),$$

$$v_i(x_{ip}, y_{ip}, \sigma_{ip}, \beta_{ip})^{\alpha} = \bar{x}_{ip} \quad (i = 1, 2, \dots, m; p = 1, 2, \dots, n),$$

$$u_r(x_{rj}, y_{rj}, \sigma_{rj}, \beta_{rj})^{\alpha} = \bar{y}_{rj} \quad (r = 1, 2, \dots, s; j = 1, 2, \dots, n), \quad (19)$$

$$v_i(x_{ij}, y_{ij}, \sigma_{ij}, \beta_{ij})^\alpha = \bar{x}_{ij} \quad (i = 1, 2, \dots, m; j = 1, 2, \dots, n)$$

are introduced to receive the following program:

$$\max E^{\alpha U} = \sum_{r=1}^s \bar{y}_{rp}^U - u_0$$

such that

$$\begin{aligned} \sum_{i=1}^m \bar{x}_{ip}^L &= 1, \\ \sum_{r=1}^s \bar{y}_{rj}^U - \sum_{i=1}^m \bar{x}_{ij}^L - u_0 &\leq 0, \\ \sum_{r=1}^s \bar{y}_{rj}^L - \sum_{i=1}^m \bar{x}_{ij}^U - u_0 &\leq 0 \quad (j = 1, 2, \dots, n; j \neq 0) \\ u_r, v_i &\geq 0 \quad (\forall r, i), u_0 \text{ uncertain.} \end{aligned} \quad (20)$$

$$\max E^{\alpha L} = \sum_{r=1}^s \bar{y}_{rp}^L - u_0$$

such that

$$\begin{aligned} \sum_{i=1}^m \bar{x}_{ip}^U &= 1, \\ \sum_{r=1}^s \bar{y}_{rj}^L - \sum_{i=1}^m \bar{x}_{ij}^U - u_0 &\leq 0, \\ \sum_{r=1}^s \bar{y}_{rj}^U - \sum_{i=1}^m \bar{x}_{ij}^L - u_0 &\leq 0 \quad (j = 1, 2, \dots, n; j \neq 0) \\ u_r, v_i &\geq 0 \quad (\forall r, i), u_0 \text{ uncertain.} \end{aligned} \quad (21)$$

The fuzzy linear programming problem given by (20) and (21) is equivalent to a crisp parametric linear programming problem. Using the optimal value, we can determine the situation for returns to scale RTS when a DMUP is efficient. Similar to the conventional DEA model of Banker and Thrall [3].

From (15) the lower and upper bounds of the efficiency score for a given α can be reduced as:

$$E_j = \left[\frac{\sum_{r=1}^s u_r^* [\alpha x_{rp} + (1-\alpha)\sigma_{rp}]}{\sum_{i=1}^m v_i^* [\alpha y_{ip} + (1-\alpha)\beta_{ip}]} - u_0^*, \frac{\sum_{r=1}^s u_r^* [\alpha y_{rp} + (1-\alpha)\beta_{rp}]}{\sum_{i=1}^m v_i^* [\alpha x_{ip} + (1-\alpha)\sigma_{ip}]} - u_0^* \right].$$

We can applying the α -level of fuzzy DEA. Using the equations (11) in model FBCC (13) the following model would be achieved:

$$\max w_p = \frac{\sum_{r=1}^s u_r [\alpha x_{rp} + (1-\alpha)\sigma_{rp}, \alpha y_{rp} + (1-\alpha)\beta_{rp}] - u_0}{\sum_{i=1}^m v_i [\alpha x_{ip} + (1-\alpha)\sigma_{ip}, \alpha y_{ip} + (1-\alpha)\beta_{ip}]}$$

such that

$$\frac{\sum_{r=1}^s u_r [\alpha x_{rj} + (1-\alpha)\sigma_{rj}, \alpha y_{rj} + (1-\alpha)\beta_{rj}] - u_0}{\sum_{i=1}^m v_i [\alpha x_{ij} + (1-\alpha)\sigma_{ij}, \alpha y_{ij} + (1-\alpha)\beta_{ij}]} \leq 1 \quad (\forall j), \quad (22)$$

$$u_r, v_i \geq 0 \quad (\forall r, i), u_0 \text{ uncertain},$$

where $\tilde{y}_{rj} = (x_{rj}, y_{rj}, \sigma_{rj}, \beta_{rj})$ and $\tilde{x}_{ij} = (x_{ij}, y_{ij}, \sigma_{ij}, \beta_{ij})$ are the r th fuzzy output and i th fuzzy input values of the j th DMU, respectively, are characterized as trapezoidal fuzzy numbers.

Model (22) is an interval programming model that can be solved by standard optimization methods. Hence, we transform the interval model (22) into a programming model using the following interval alteration variables:

$$\begin{aligned} [\alpha x_{rp} + (1-\alpha)\sigma_{rp}, \alpha y_{rp} + (1-\alpha)\beta_{rp}] &= \hat{y}_{rp} \quad (\forall r, p) \\ [\alpha x_{ip} + (1-\alpha)\sigma_{ip}, \alpha y_{ip} + (1-\alpha)\beta_{ip}] &= \hat{x}_{ip} \quad (\forall i, p) \\ [\alpha x_{rj} + (1-\alpha)\sigma_{rj}, \alpha y_{rj} + (1-\alpha)\beta_{rj}] &= \hat{y}_{rj} \quad (\forall r, j) \\ [\alpha x_{ij} + (1-\alpha)\sigma_{ij}, \alpha y_{ij} + (1-\alpha)\beta_{ij}] &= \hat{x}_{ij} \quad (\forall i, j). \end{aligned} \quad (23)$$

The substitutions of the above interval alteration variables in model (22) will result in the following programming model:

$$\max w_p = \frac{\sum_{r=1}^s u_r \hat{y}_{rp} - u_0}{\sum_{i=1}^m v_i \hat{x}_{ip}}$$

such that

$$\begin{aligned} \frac{\sum_{r=1}^s u_r \hat{y}_{rj} - u_0}{\sum_{i=1}^m v_i \hat{x}_{ij}} &\leq 1 \quad (\forall j), \\ u_r, v_i &\geq 0 \quad (\forall r, i). \end{aligned} \quad (24)$$

In model (24), the alternation variables as:

$$\begin{aligned} u_r \hat{y}_{rp} &= \bar{y}_{rp} \quad (r = 1, 2, \dots, s, p = 1, 2, \dots, n), \\ v_i \hat{x}_{ip} &= \bar{x}_{ip} \quad (i = 1, 2, \dots, m, p = 1, 2, \dots, n), \\ u_r \hat{y}_{rj} &= \bar{y}_{rj} \quad (r = 1, 2, \dots, s, j = 1, 2, \dots, n), \end{aligned} \quad (25)$$

$$v_i \hat{x}_{ij} = \bar{x}_{ij} \quad (i = 1, 2, \dots, m ; j = 1, 2, \dots, n)$$

are introduced to receive the following program:

$$\max \sum_{r=1}^s \bar{y}_{rp} - u_0$$

such that

$$\begin{aligned} \sum_{i=1}^m \bar{x}_{ip} &= 1, \\ \sum_{r=1}^s \bar{y}_{rj} - \sum_{i=1}^m \bar{x}_{ij} - u_0 &\leq 0 \quad (j \neq p), \\ u_r(\alpha x_{rj} + (1-\alpha)\sigma_{rj}) &\leq \bar{y}_{rj} \leq u_r(\alpha y_{rj} + (1-\alpha)\beta_{rj}) \quad (\forall r, j), \\ v_i(\alpha x_{ij} + (1-\alpha)\sigma_{ij}) &\leq \bar{x}_{ij} \leq v_i(\alpha y_{ij} + (1-\alpha)\beta_{ij}) \quad (\forall i, j), \\ u_r, v_i &\geq 0 \quad (\forall r, i), \quad u_0 \text{ uncertain.} \end{aligned} \quad (26)$$

Model (26) is equivalent to a interval programming model with $\alpha \in [0,1]$. Therefore, analyzing the efficiency of DMUs with the proposed method, for a set of n different values of α , e.g. α_i ($i = 1, 2, \dots, n$). Therefore, it will be necessary to obtain an integrated efficiency score for DMUs to rank them. The fuzzy linear programming problem given by (26) is equivalent to a crisp interval linear programming problem. Using the optimal value, we can determine the situation for returns to scale RTS for BCC and CCR when a DMU_p is efficient. Similar to the conventional DEA model of Banker and Thrall [3].

From (15) the interval of the efficiency score for a given α can be reduced as:

$$E_j = \left[\frac{\sum_{r=1}^s u_r^* [\alpha x_{rj} + (1-\alpha)\sigma_{rj}]}{\sum_{i=1}^m v_i^* [\alpha y_{ij} + (1-\alpha)\beta_{ij}]} - u_0^*, \frac{\sum_{r=1}^s u_r^* [\alpha x_{rj} + (1-\alpha)\sigma_{rj}]}{\sum_{i=1}^m v_i^* [\alpha x_{ij} + (1-\alpha)\sigma_{ij}]} - u_0^* \right]$$

$$\left[\frac{\sum_{r=1}^s u_r^* [\alpha y_{rj} + (1-\alpha)\beta_{rj}]}{\sum_{i=1}^m v_i^* [\alpha y_{ij} + (1-\alpha)\beta_{ij}]} - u_0^*, \frac{\sum_{r=1}^s u_r^* [\alpha y_{rj} + (1-\alpha)\beta_{rj}]}{\sum_{i=1}^m v_i^* [\alpha x_{ij} + (1-\alpha)\sigma_{ij}]} - u_0^* \right]$$

6. Numerical Examples

In this section, we use two numerical examples, Saati et. al. [53] and Guo and Tanaka [15] are solved by the proposed method and the results are compared with previously presented methods.

Example 1. Consider 4 DMUs with inputs and outputs which are presented in Table 1, this problem results are compared with two previously presented Kao and Liu [36] and Razavi, et. al. [51].

Table 1: Inputs and outputs of 4 DMUs.

DMU	Input	α -cut	Output	α -cut
A	(11,12,14)	$[11+\alpha, 14-2\alpha]$	10	[10,10]
B	30	[30,30]	(12,13,14,16)	$[12+\alpha, 16-2\alpha]$
C	40	[40,40]	11	[11,11]
D	(45,47,52,55)	$[45+2\alpha, 55-3\alpha]$	(12,15,19,22)	$[12+3\alpha, 22-3\alpha]$

Considering the DMU D, by FBCC models (20,21) for upper and lower objective models are solved:

$$\max E^{\alpha U}(D) = \max (22 - 3\alpha)u_1 - u_0$$

such that

$$\begin{aligned} (45 + 2\alpha)v_1 &= 1 \\ 10u_1 - (11 + \alpha)v_1 - u_0 &\leq 0 \\ (16 - 2\alpha)u_1 - 30v_1 - u_0 &\leq 0 \\ 11u_1 - 40v_1 - u_0 &\leq 0 \\ (22 - 3\alpha)u_1 - (45 + 2\alpha)v_1 - u_0 &\leq 0 \\ 10u_1 - (14 - 2\alpha)v_1 - u_0 &\leq 0 \\ (12 + \alpha)u_1 - 30v_1 - u_0 &\leq 0 \\ 11u_1 - 40v_1 - u_0 &\leq 0 \\ (22 - 3\alpha)u_1 - (55 + 3\alpha)v_1 - u_0 &\leq 0 \\ u_1, v_1 &\geq 0, u_0 : \text{unrestricted} . \end{aligned}$$

$$\max E^{\alpha L}(D) = \max (12 + 3\alpha)u_1 - u_0$$

such that

$$\begin{aligned} (55 + 3\alpha)v_1 &= 1 \\ 10u_1 - (14 - 2\alpha)v_1 - u_0 &\leq 0 \\ (12 + \alpha)u_1 - 30v_1 - u_0 &\leq 0 \\ 11u_1 - 40v_1 - u_0 &\leq 0 \\ (22 - 3\alpha)u_1 - (55 + 3\alpha)v_1 - u_0 &\leq 0 \\ 10u_1 - (11 + \alpha)v_1 - u_0 &\leq 0 \\ (16 - 2\alpha)u_1 - 30v_1 - u_0 &\leq 0 \end{aligned}$$

$$11u_1 - 40v_1 - u_0 \leq 0$$

$$(22 - 3\alpha)u_1 - (45 + 2\alpha)v_1 - u_0 \leq 0$$

$$u_1, v_1 \geq 0, u_0 : \text{unrestricted}.$$

The results for these models are solved for different α -values in Table 2.

Table 2: The α -cuts of the efficiency at four α -values on the proposed method and two previously presented.

		A $[E^{\alpha L}, E^{\alpha U}]$			B $[E^{\alpha L}, E^{\alpha U}]$		
A	Proposed BCC	Kao and Liu	Razavi et.al.	Proposed BCC	Kao and Liu	Razavi and et.al.	
0.0	[0.98,1.0]	[1.0,1.0]	0.9497	[0.72,1.0]	[0.71,1.0]	[0.71,0.95]	
0.5	[0.98,1.0]	[1.0,1.0]	0.9725	[0.80,1.0]	[0.79,1.0]	[0.79,0.95]	
0.75	[0.98,1.0]	[1.0,1.0]	0.9873	[0.82,1.0]	[0.83,1.0]	[0.83,0.95]	
1.0	[0.98,1.0]	[1.0,1.0]	1.0	[0.90,1.0]	[0.88,1.0]	[0.88,0.95]	

		C $[E^{\alpha L}, E^{\alpha U}]$			D $[E^{\alpha L}, E^{\alpha U}]$		
A	Proposed BCC	Kao and Liu	Razavi and et.al.	Proposed BCC	Kao and Liu	Razavi and et.al.	
0.0	[0.58,1.0]	[0.54,0.91]	0.5436	[0.72,1.0]	[0.74,1.0]	[0.54,1.0]	
0.5	[0.61,1.0]	[0.58,0.85]	0.5890	[0.79,1.0]	[0.89,1.0]	[0.65,1.0]	
0.75	[0.64,1.0]	[0.60,0.82]	0.6086	[0.91,1.0]	[0.96,1.0]	[0.70,1.0]	
1.0	[0.70,1.0]	[0.63,0.79]	0.6395	[0.96,1.0]	[1.0,1.0]	[0.78,1.0]	

Table 3: The results for model (20,21) and the ranking of each DMU for different α -levels.

Interval Score								
	$\alpha = 0$	Rank	$\alpha = 0.5$	Rank	$\alpha = 0.75$	Rank	$\alpha = 1.0$	Rank
A	[0.98,1.0]	1	[0.98,1.0]	1	[0.98,1.0]	1	[0.98,1.0]	1
B	[0.72,1.0]	2	[0.80,1.0]	2	[0.82,1.0]	3	[0.90,1.0]	3
C	[0.58,1.0]	4	[0.61,1.0]	4	[0.64,1.0]	4	[0.70,1.0]	4
D	[0.72,1.0]	3	[0.79,1.0]	3	[0.91,1.0]	2	[0.96,1.0]	2

Example 2. Consider 5 DMUs with 2 inputs and 2 outputs which are presented in Table 4, this problem results are compared with six previously presented Guo and Tanaka [15], Lertworasirikul et al. [41,42], Saati et al. [52] and Majid et. al. [45].

Table 4: 2 Inputs and 2 outputs of 5 DMUs.

DMU	Input 1	Input 2	Output 1	Output 2
A	(3.5,4,4.5)	(1.9,2.1,2.3)	(2.4,2.6,2.8)	(3.8,4.1,4.4)
B	(2.9,2.9,2.9)	(1.4,1.5,1.6)	(2.2,2.2,2.2)	(3.3,3.5,3.7)
C	(4.4,4.9,5.4)	(2.2,2.6,3.2)	(2.7,3.2,3.7)	(4.3,5.1,5.9)
D	(3.4,4.1,4.8)	(2.2,2.3,2.4)	(2.5,2.9,3.3)	(5.5,5.7,5.9)
E	(5.9,6.5,7.1)	(3.6,4.1,4.6)	(4.4,5.1,5.8)	(6.5,7.4,8.3)

Considering the DMU D, by FBCC model (26) for interval of the objective models are solved:

$$\max E^\alpha(A) = \max (\bar{y}_{11} + \bar{y}_{21}) - u_0$$

such that

$$\bar{x}_{11} + \bar{x}_{21} = 1$$

$$(\bar{y}_{11} + \bar{y}_{21}) - (\bar{x}_{11} + \bar{x}_{21}) - u_0 \leq 0$$

$$(\bar{y}_{12} + \bar{y}_{22}) - (\bar{x}_{12} + \bar{x}_{22}) - u_0 \leq 0$$

$$(\bar{y}_{13} + \bar{y}_{23}) - (\bar{x}_{13} + \bar{x}_{23}) - u_0 \leq 0$$

$$(\bar{y}_{14} + \bar{y}_{24}) - (\bar{x}_{14} + \bar{x}_{24}) - u_0 \leq 0$$

$$(\bar{y}_{15} + \bar{y}_{25}) - (\bar{x}_{15} + \bar{x}_{25}) - u_0 \leq 0$$

$$u_1(\alpha(2.4) + (1 - \alpha)2.6) \leq \bar{y}_{11} \leq u_1(\alpha(2.4) + (1 - \alpha)2.8)$$

$$u_1(\alpha(2.2) + (1 - \alpha)2.2) \leq \bar{y}_{12} \leq u_1(\alpha(2.2) + (1 - \alpha)2.2)$$

$$u_1(\alpha(2.7) + (1 - \alpha)3.2) \leq \bar{y}_{13} \leq u_1(\alpha(2.7) + (1 - \alpha)3.7)$$

$$u_1(\alpha(2.5) + (1 - \alpha)2.9) \leq \bar{y}_{14} \leq u_1(\alpha(2.5) + (1 - \alpha)3.3)$$

$$u_1(\alpha(4.4) + (1 - \alpha)5.1) \leq \bar{y}_{15} \leq u_1(\alpha(4.4) + (1 - \alpha)5.8)$$

$$\begin{aligned}
u_2(\alpha(3.8) + (1 - \alpha)4.1) &\leq \bar{y}_{11} \leq u_2(\alpha(3.8) + (1 - \alpha)4.4) \\
u_2(\alpha(3.3) + (1 - \alpha)3.5) &\leq \bar{y}_{12} \leq u_2(\alpha(3.3) + (1 - \alpha)3.7) \\
u_2(\alpha(4.3) + (1 - \alpha)5.1) &\leq \bar{y}_{13} \leq u_2(\alpha(4.3) + (1 - \alpha)5.9) \\
u_2(\alpha(5.5) + (1 - \alpha)5.7) &\leq \bar{y}_{14} \leq u_2(\alpha(5.5) + (1 - \alpha)5.9) \\
u_2(\alpha(6.5) + (1 - \alpha)7.4) &\leq \bar{y}_{15} \leq u_2(\alpha(6.5) + (1 - \alpha)8.3)
\end{aligned}$$

$$\begin{aligned}
v_1(\alpha(3.5) + (1 - \alpha)4.0) &\leq \bar{x}_{11} \leq v_1(\alpha(3.5) + (1 - \alpha)4.5) \\
v_1(\alpha(2.9) + (1 - \alpha)2.9) &\leq \bar{x}_{12} \leq v_1(\alpha(2.9) + (1 - \alpha)2.9) \\
v_1(\alpha(4.4) + (1 - \alpha)4.9) &\leq \bar{x}_{13} \leq v_1(\alpha(4.4) + (1 - \alpha)5.4) \\
v_1(\alpha(3.4) + (1 - \alpha)4.1) &\leq \bar{x}_{14} \leq v_1(\alpha(3.4) + (1 - \alpha)4.8) \\
v_1(\alpha(5.9) + (1 - \alpha)6.5) &\leq \bar{x}_{15} \leq v_1(\alpha(5.9) + (1 - \alpha)7.1)
\end{aligned}$$

$$\begin{aligned}
v_2(\alpha(1.9) + (1 - \alpha)2.1) &\leq \bar{x}_{11} \leq v_2(\alpha(1.9) + (1 - \alpha)2.3) \\
v_2(\alpha(1.4) + (1 - \alpha)1.5) &\leq \bar{x}_{12} \leq v_2(\alpha(1.4) + (1 - \alpha)1.6) \\
v_2(\alpha(2.2) + (1 - \alpha)2.6) &\leq \bar{x}_{13} \leq v_2(\alpha(2.2) + (1 - \alpha)3.0) \\
v_2(\alpha(2.2) + (1 - \alpha)2.3) &\leq \bar{x}_{14} \leq v_2(\alpha(2.2) + (1 - \alpha)2.4) \\
v_2(\alpha(3.6) + (1 - \alpha)4.1) &\leq \bar{x}_{15} \leq v_2(\alpha(3.6) + (1 - \alpha)4.6)
\end{aligned}$$

$u_1, v_1, u_2, v_2 \geq 0, u_0 : \text{unrestricted} .$

$$\max E^\alpha(B) = \max (\bar{y}_{12} + \bar{y}_{22}) - u_0$$

such that

$$\bar{x}_{12} + \bar{x}_{22} = 1$$

$$(\bar{y}_{11} + \bar{y}_{21}) - (\bar{x}_{11} + \bar{x}_{21}) - u_0 \leq 0$$

$$(\bar{y}_{12} + \bar{y}_{22}) - (\bar{x}_{12} + \bar{x}_{22}) - u_0 \leq 0$$

$$(\bar{y}_{13} + \bar{y}_{23}) - (\bar{x}_{13} + \bar{x}_{23}) - u_0 \leq 0$$

$$(\bar{y}_{14} + \bar{y}_{24}) - (\bar{x}_{14} + \bar{x}_{24}) - u_0 \leq 0$$

$$(\bar{y}_{15} + \bar{y}_{25}) - (\bar{x}_{15} + \bar{x}_{25}) - u_0 \leq 0$$

$$\begin{aligned}
u_1(\alpha(2.4) + (1 - \alpha)2.6) &\leq \bar{y}_{11} \leq u_1(\alpha(2.4) + (1 - \alpha)2.8) \\
u_1(\alpha(2.2) + (1 - \alpha)2.2) &\leq \bar{y}_{12} \leq u_1(\alpha(2.2) + (1 - \alpha)2.2) \\
u_1(\alpha(2.7) + (1 - \alpha)3.2) &\leq \bar{y}_{13} \leq u_1(\alpha(2.7) + (1 - \alpha)3.7) \\
u_1(\alpha(2.5) + (1 - \alpha)2.9) &\leq \bar{y}_{14} \leq u_1(\alpha(2.5) + (1 - \alpha)3.3) \\
u_1(\alpha(4.4) + (1 - \alpha)5.1) &\leq \bar{y}_{15} \leq u_1(\alpha(4.4) + (1 - \alpha)5.8)
\end{aligned}$$

$$\begin{aligned}
u_2(\alpha(3.8) + (1 - \alpha)4.1) &\leq \bar{y}_{11} \leq u_2(\alpha(3.8) + (1 - \alpha)4.4) \\
u_2(\alpha(3.3) + (1 - \alpha)3.5) &\leq \bar{y}_{12} \leq u_2(\alpha(3.3) + (1 - \alpha)3.7) \\
u_2(\alpha(4.3) + (1 - \alpha)5.1) &\leq \bar{y}_{13} \leq u_2(\alpha(4.3) + (1 - \alpha)5.9) \\
u_2(\alpha(5.5) + (1 - \alpha)5.7) &\leq \bar{y}_{14} \leq u_2(\alpha(5.5) + (1 - \alpha)5.9) \\
u_2(\alpha(6.5) + (1 - \alpha)7.4) &\leq \bar{y}_{15} \leq u_2(\alpha(6.5) + (1 - \alpha)8.3)
\end{aligned}$$

$$\begin{aligned}
v_1(\alpha(3.5) + (1 - \alpha)4.0) &\leq \bar{x}_{11} \leq v_1(\alpha(3.5) + (1 - \alpha)4.5) \\
v_1(\alpha(2.9) + (1 - \alpha)2.9) &\leq \bar{x}_{12} \leq v_1(\alpha(2.9) + (1 - \alpha)2.9) \\
v_1(\alpha(4.4) + (1 - \alpha)4.9) &\leq \bar{x}_{13} \leq v_1(\alpha(4.4) + (1 - \alpha)5.4) \\
v_1(\alpha(3.4) + (1 - \alpha)4.1) &\leq \bar{x}_{14} \leq v_1(\alpha(3.4) + (1 - \alpha)4.8) \\
v_1(\alpha(5.9) + (1 - \alpha)6.5) &\leq \bar{x}_{15} \leq v_1(\alpha(5.9) + (1 - \alpha)7.1)
\end{aligned}$$

$$\begin{aligned}
v_2(\alpha(1.9) + (1 - \alpha)2.1) &\leq \bar{x}_{11} \leq v_2(\alpha(1.9) + (1 - \alpha)2.3) \\
v_2(\alpha(1.4) + (1 - \alpha)1.5) &\leq \bar{x}_{12} \leq v_2(\alpha(1.4) + (1 - \alpha)1.6) \\
v_2(\alpha(2.2) + (1 - \alpha)2.6) &\leq \bar{x}_{13} \leq v_2(\alpha(2.2) + (1 - \alpha)3.0) \\
v_2(\alpha(2.2) + (1 - \alpha)2.3) &\leq \bar{x}_{14} \leq v_2(\alpha(2.2) + (1 - \alpha)2.4) \\
v_2(\alpha(3.6) + (1 - \alpha)4.1) &\leq \bar{x}_{15} \leq v_2(\alpha(3.6) + (1 - \alpha)4.6)
\end{aligned}$$

$$u_1, v_1, u_2, v_2 \geq 0, u_0 : \text{unrestricted}.$$

The results for these model are solved for the different α -value in Table 5.

Table 5: The efficiency at four α -values on the proposed method and six previously presented.

Method	α -level	DMUs				
		A	B	C	D	E
Guo and Tanaka (2001) CCR model	0.00	(0.66,0.81,0.99)	(0.88,0.98,1.09)	(0.60,0.82,1.12)	(0.71,0.3,1.25)	(0.61,0.79,1.02)
	0.50	(0.75,0.83,0.92)	(0.94,0.97,1.00)	(0.12,0.83,0.14)	(0.85,0.97,1.12)	(0.72,0.82,0.93)
	0.75	(0.77,0.81,0.99)	(0.80,0.98,1.09)	(0.22,0.82,0.30)	(0.71,0.93,1.25)	(0.61,0.79,1.02)
	1.00	(0.85,0.85,0.94)	(1.00,1.00,1.00)	(0.86,0.86,0.86)	(1.00,1.00,1.00)	(1.00,1.00,1.00)
Lertworasirikul et.al. (2003a)CCR model	0.00	1.107	1.238	1.276	1.520	1.296
	0.50	0.963	1.112	1.035	1.258	1.159
	0.75	0.904	1.055	0.932	1.131	1.095
	1.00	1.000	0.855	1.000	0.861	1.000
Saati et al. (2002) CCR model	0.00	1.000	1.000	1.000	1.000	1.000
	0.50	0.954	1.000	1.000	1.000	1.000
	0.75	0.901	1.000	0.929	1.000	1.000
	1.00	0.855	1.000	0.862	1.000	1.000
Lertworasirikul et al. (2003b) BCC model	0.00	1.299	1.247	1.699	1.692	∞
	0.50	1.062	1.119	1.243	1.300	∞
	0.75	0.969	1.059	1.074	1.142	∞
	1.00	0.889	1.000	0.935	1.000	1.000
Majid et.al. (2011)CCR model	0.00	0.915	1.000	0.948	1.000	0.991
	0.50	0.909	1.000	0.945	1.000	0.991
	0.75	0.903	1.000	0.941	1.000	0.840
	1.00	1.000	1.000	1.000	1.000	1.000
Marbini et.al (2012)BCC model	0.00	1.000	1.000	1.000	1.000	1.000
	0.50	1.000	1.000	1.000	1.000	1.000
	0.75	0.982	1.000	1.000	1.000	1.000
	1.00	0.918	1.000	0.960	1.000	1.000
<i>Proposed</i>	0.00	(0.71,0.85,0.99)	1.000	(0.60,0.80,1.00)	(0.70,0.85,1.00)	1.000
	0.50	(0.76,0.86,0.96)	1.000	(0.62,0.83,0.94)	(0.85,0.97,1.12)	1.000
	0.75	(0.76,0.87,0.98)	1.000	(0.62,0.85,0.98)	(0.71,0.93,1.25)	1.000
	1.00	(0.83,0.89,0.95)	1.000	(0.86,0.86,0.86)	(1.00,1.00,1.00)	1.000

Table 6: The results for model (26) and the ranking of each DMU for different α levels.

	Interval Score							
	$\alpha = 0$	Rank	$\alpha = 0.5$	Rank	$\alpha = 0.75$	Rank	$\alpha = 1.0$	Rank
A	(0.71,0.85,0.99)	3	(0.76,0.86,0.96)	4	(0.76,0.87,0.98)	3	(0.83,0.89,0.95)	5
B	1	1	1	1	1	1	1	1
C	(0.60,0.80,1.00)	5	(0.62,0.83,0.94)	5	(0.62,0.85,0.98)	5	(0.86,0.86,0.86)	4
D	(0.70,0.85,1.00)	4	(0.85,0.97,1.12)	3	(0.71,0.93,1.25)	4	(1.00,1.00,1.00)	3
E	1	2	1	2	1	2	1	2

7. Conclusions

The development of fuzzy DEA with imprecise and ambiguous data has evolved the scope of its application to efficient measurement in real-life problems. In the real world there are many problems which have fuzzy parameters. In this study, we have developed a fuzzy DEA framework and guideline with a CCR and BCC model and used an α -cut approach to convert the fuzzy BCC model into a parametric programming model. Instead of comparing the equality (or inequality) of these two intervals, we have defined in a variable in the period to meet our limitations and to maximize the efficiency value. We did present a numerical example to show the similarities and dissimilarities between our solution and the solutions extracted from two fuzzy DEA methods in other ways.

We use these models (20,,2126) proposed in this study, in order to find the efficiencies of the DMUs presented in table mentioned as 1 and 4. On top of that, the optimal values of u_0 for the successful DMUs are reported in Table 2 and 5 to look for the efficiency. For example, DMU B and C are successful when α is equal to 0, 0. 5, 0.75 and 1.0. Also our model can be used when the number of DMUs is less used than the blended number of inputs and outputs, and can be used full fuzzy numbers. This model can be extended to the other subject areas and types of DEA.

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