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A Robust Approach About Compact Operators on Some Generalized Fibonacci Difference Sequence Spaces

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Abstract: In this new study, which deals with the different properties of $\ell_p(\widehat{F}(r,s))$ $(1 \leq p < \infty)$ and $\ell_\infty(\widehat{F}(r,s))$ spaces defined by Candan and Kara in 2015 by using Fibonacci numbers according to a certain rule, we have tried to review all the qualities and features that the authors of the previous editions have found most useful. This document provides everything needed to characterize the matrix class $(\ell_1, \ell_p(\widehat{F}(r,s)))$ $(1 \leq p < \infty)$. Using the Hausdorff measure of non-compactness, we simultaneously provide estimates for the norms of the bounded linear operators L_A defined by these matrix transformations and identify requirements to derive the corresponding subclasses of compact matrix operators. The results of the current research can be regarded as to be more inclusive and broader when compared to the similar results available in the literature.

Keywords: Sequence spaces, Fibonacci numbers, compact operators, Hausdorff measure of noncompactness.

1. Elementary Classical Concepts

As always, our aim is to use the matrix domain and to remind readers about the information they will need to use calculus effectively in their work in later sections. To achieve this, we retained the paper's mathematical level, the orientation of the new sequence space to the Hausdorff measure, its concentration on previous works, and variety of the theorems, and continued to adapt some of the methods used in measurement theory. Although many of the presentations in this new paper are noticeably more general than those in earlier articles, the level of rigor is about the same. As part of the overall review plan, it is going to be helpful for beginners to review the five notable books given in [1–5] with accessible material, without sacrificing the standards or scope their users want to see. Let us start by trying to explain some of the essentials without exaggerating the obvious. The history of numbers is almost as old as the existence of humanity and was created to

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meet the mathematical needs of all people and scientists. This was true in the beginnings of the subject, and it is true today. In much of our work, the sequences will have domains and ranges that are sets of naturel numbers $\mathbb{N} = \{0, 1, 2, ...\}$ and real numbers $\mathbb{R}$, respectively. We will write $\lim_k$, $\sup_k$, $\inf_k$ and $\sum_k$ instead of $\lim_{k\to\infty}$, $\sup_{k\in\mathbb{N}}$, $\inf_{k\in\mathbb{N}}$ and $\sum_{k=0}^{\infty}$, respectively.

We will now consider two related topics that will be used in the next sections: infinite sequences and infinite series. An infinite sequence of numbers is a function whose domain is the set of natural numbers. The word series always implies an infinite number of term to be combined by adding in a definite order. The vector space of all real sequences is expressed by ω. We are quite familiar with that each subspace of ω is said a sequence space. In order to use in this work, a few additional notations concerning sequences are needed. The sets of all finite sequences, bounded sequences, convergent sequences, and null sequences, respectively, should be denoted by, φ, ℓ_∞, c and c_0. For any real number p with $1 \le p < \infty$, the sequence space $\left\{x \in \omega : \sum_k |x_k|^p < \infty\right\}$ is denoted by the notation ℓ_p. In addition to these, the sequence $(1, 1, ...)$ and for each natural number n, the sequence with 1 only in the n^{th} term and 0 in all other terms is denoted by the notations e and $e^{(n)}$, respectively. The sum $\sum_{k=0}^{n} x_k e^{(k)}$ is indicated by $x^{[n]}$ and is referred to as the n-section of any sequence x. Series whose partial sums sequence are convergent and bounded are also shown with cs and bs notations, respectively.

A complete normed space is referred to as a $B - space$. A $K - space$, on the other hand, is a topological sequence space in which all coordinate functionals π_k, given by $\pi_k(x) = x_k$, are continuous. A $BK - space$ is essentially a Banach space with continuous coordinates, meeting the requirements of both a $K - space$ and a $B - space$. If all sequences $x = (x_k) \in X$ share the same representation, then a $BK - space$ denoted as $X \supset \varphi$ is said to possess AK, where $x = \sum_k x_k e^{(k)}$. To provide an example, the sequence space ℓ_p $(1 \le p < \infty)$ can be regarded as a $BK - space$ with the norm $\|x\|_p = \left(\sum_k |x_k|^p\right)^{1/p}$. Furthermore, c_0, c, and ℓ_∞ also qualify as $BK - spaces$, possessing the norm $\|x\|_\infty = \sup_k |x_k|$. Additionally, the $BK - spaces$ c_0 and ℓ_p exhibit AK, where $1 \le p < \infty$.

If there exists a singular sequence (α_n) consisting of scalars such that $x = \sum_n \alpha_n b_n$, meaning that $\lim_m \|x - \sum_{n=0}^{m} \alpha_n b_n\| = 0$, then the sequence (b_n) in a normed space X is referred to as a Schauder basis for all $x \in X$.

The β-dual of a sequence space X is defined as follows:

$$X^\beta = \{a = (a_k) \in \omega : ax = (a_k x_k) \in cs \text{ for all } x = (x_k) \in X\}.$$

An infinite matrix of real numbers, denoted by $A = (a_{nk})_{n,k=0}^{\infty}$, where $n, k \in \mathbb{N}$, can be represented as A_n, which denotes the sequence in the n^{th} row of A. Furthermore, if $x = (x_k)_{k=0}^{\infty} \in$

ω, the A-transform of x is defined as the sequence $Ax = \{A_n(x)\}_{n=0}^{\infty}$, where

$$A_n(x) = \sum_{k=0}^{\infty} a_{nk} x_k \quad (n \in \mathbb{N}), \tag{1}$$

provided that the series on the right-hand side converges for each $n \in \mathbb{N}$.

We denote (X, Y) as the class of all infinite matrices that map from X to Y, where X and Y are subsets of ω. In other words, $A \in (X, Y)$ if and only if $A_n \in X^{\beta}$ for every $n \in N$ and $Ax \in Y$ for every $x \in X$.

One way to create a new sequence space is by utilizing the matrix domain, and a thorough comprehension of it requires substantial expertise. Let X be any sequence space. Then the domain X_A of an infinite matrix A in X is defined by

$$X_A = \{x = (x_k) \in \omega : Ax \in X\}. \tag{2}$$

Let us also mention here that X_A is also a sequence space. The reader can refer to the recent papers [6–10] on the domains of certain triangles in the classical sequence spaces and related topics.

The following results are fundamental and often used [11, 12].

Lemma 1.1 *Let $X \supset \phi$ and Y be a BK–space.*
(a) Therefore, for any matrix $A \in (X, Y)$, we get $(X, Y) \subset B(X, Y)$, so indicating that for any $x \in X$, $L_A(x) = Ax$ describes an operator $L_A \in B(X, Y)$.
(b) If X has AK, and after that $B(X, Y) \subset (X, Y)$, meaning that there is a $A \in (X, Y)$ with any operator having $L \in B(X, Y)$ and $L(x) = Ax$ for every $x \in X$.

2. The Hausdorff Measure of Non-Compactness

In this part, our aim is to describe the Hausdorff measure used in theory and practice that characterizes compact operators between Banach spaces. For this purpose, this section stars with clear expressions of relative definitions, guidelines and theorems together with explanatory and other demonstrative subject. It follows proven and supplementary theorems. The proven theorems give to demonstrate and magnify the theory, and to reiterate the fundamental principles that are crucial for effective learning. The concept of Hausdorff measure of non-compactness appears in some branches of mathematics. Recently, this concept has been used to characterize compact matrix operators between BK–spaces under certain conditions.

The Hausdorff measure of non-compactness χ concept stems largely from the investigations of Goldenštein, Gohberg and Markus [13] and in the following years this concept was taken up and studied by Goldenštein and Markus [14]. Yet some of its ideas date back to the time of Kuratowski

[15]. Later, Darbo [16] took this measure and generalized another concept besides the classical Schauder fixed point principle.

In the context of infinite-dimensional Banach spaces X and Y, it is important to restate the definition of a compact operator. A linear operator L that maps from X to Y is considered compact if it encompasses the entire domain of X and, in addition, if the sequence $(L(x_n))$ representing the images of all bounded sequences (x_n) in X under L has a convergent subsequence. In the field of functional analysis, the collection of all compact operators in $B(X,Y)$ is denoted by $C(X,Y)$.

Let (X,d) be a metric space. We define the open ball $B(x,r)$ as the set $\{x \in X : d(x,x_0) < r\}$, where r represents the radius and x_0 denotes the center. Furthermore, let $M(X)$ denote the collection of all bounded subsets of X. If $Q \in M(X)$, then the Hausdorff measure of non-compactness of the set Q, denoted by $\chi(Q)$, is defined as follows:

$$\chi(Q) = \inf\left\{\epsilon > 0 : Q \subset \bigcup_{k=1}^{n} B(x_k, r_k), x_k \in X,\ r_k < \epsilon\ (k = 1,2,...),\ n \in \mathbb{N}\right\}.$$

The Hausdorff measure of non-compactness is defined as the function $\chi : M_X \to [0,\infty)$.

In previous works such as [11, 17–20], the applications of the Hausdorff measure theorems to condensing operators, compact matrix operators on some BK-spaces, and measures of non-compactness in Banach spaces are further explored.

The objective of this paragraph is to provide a concise description of the Hausdorff measure of non-compactness operators between Banach spaces. Let X and Y be Banach spaces, and let χ_1 and χ_2 be the Hausdorff measures of non-compactness on X and Y, respectively. If $L(Q) \in M(Y)$ for all $Q \in M(X)$, and if there exists $C \geq 0$ such that $\chi_2(L(Q)) \leq C\chi_1(Q)$ for all $Q \in M(X)$, then the operator $L : X \to Y$ is referred to as (χ_1, χ_2)-bounded. The quantity

$$|L|(\chi_1, \chi_2) = \inf\{C \geq 0 : \chi_2(L(Q)) \leq C\chi_1(Q) \text{ for all } Q \in \mathcal{M}X\}$$

is defined as the (χ_1, χ_2)–measure of non-compactness of L if the operator L is $(\chi_1,\ \chi_2)$-bounded.

It is important to note that if both χ_1 and χ_2 are denoted as χ, then $|L|(\chi_1, \chi_2) = |L|_\chi$.

Our primary objective in this context is to provide a comprehensive explanation of the applications of the Hausdorff measure of non-compactness in characterizing compact operators between Banach spaces. Let X and Y be Banach spaces, and let L be an element of $B(X,Y)$, indicating that L is a bounded linear operator from X to Y. If L is non-compact, the Hausdorff measure of non-compactness of L, denoted as $\|L\|_\chi$, is defined as follows ([20, Theorem 2.25]):

$$\|L\|_\chi = \chi(L(SX)). \tag{3}$$

Furthermore, L is characterized as a compact operator if and only if the Hausdorff measure of non-compactness $\|L\|_\chi$ is equal to zero, as expressed in ([20, Corollary 2.26]):

$$\|L\|_\chi = 0. \tag{4}$$

The determination of the Hausdorff measure of non-compactness, denoted as $\chi(Q)$, for bounded sets Q in a Banach space X is based on the identities presented in (3) and (4). These identities simplify the characterization of compact operators $L \in B(X,Y)$. Estimates, or even identities, for $\chi(Q)$ can be obtained when X possesses a Schauder basis.

Theorem 2.1 *([13] or [20, Theorem 2.23]) Let X be a Banach space with a Schauder basis $(b_k)_{k=0}^\infty$, $Q \in M_X$, $P_n : X \to X$ will be the projectors onto the linear span of $\{b_0, b_1, \ldots, b_n\}$ and $R_n = I - P_n$ for $n = 0, 1, \ldots$, in which I indicates the identity map on X. Under these conditions, the following inequality is satisfied*

$$\frac{1}{a} \cdot \limsup_{n\to\infty} \left(\sup_{x\in Q} \|\mathcal{R}_n(x)\| \right) \le \chi(Q) \le \limsup_{n\to\infty} \left(\sup_{x\in Q} \|\mathcal{R}_n(x)\| \right),$$

in which $a = \limsup_{n\to\infty} \|R_n\|$.

The following result, in especially, demonstrates how to calculate the Hausdorff measure of non-compactness in the BK-spaces with AK, c_0 and ℓ_p $(1 \le p < \infty)$.

Theorem 2.2 *([20, Theorem 2.15]) A bounded subset of the normed space X, in which X is ℓ_p for $1 \le p < \infty$ or c_0, is defined as Q. We can have*

$$\chi(Q) = \lim_{n\to\infty} \left(\sup_{x\in Q} \|\mathcal{R}_n(x)\| \right) \tag{5}$$

if $P_n : X \to X$ is the operator described by $P_n(x) = x^{[n]}$ for every $x = (x_k)_{k=0}^\infty \in X$ and $R_n = I - P_n$ for $n = 0, 1, \ldots$.

It is highly reasonable to deduce both necessary and sufficient criteria for matrix operators between a Schauder basis and a BK-space by employing the aforementioned discoveries, as well as the Hausdorff measure of non-compactness. Matrix mappings across BK-spaces give rise to bounded linear operators between these Banach spaces, rendering AK as compact operators. Presently, numerous researchers have embraced this approach in multiple research publications (see, for instance, [21–31]. The significance of these concepts will become evident in subsequent discussions. In this work, we provide a description of the matrix classes $(\ell_1, \ell_p(\widehat{F}(r,s)))$ $(1 \le p < \infty)$. Moreover, we establish conditions for deriving the relevant subclasses of compact matrix operators through the utilization of the Hausdorff measure of non-compactness. Additionally, we derive an identity for the norms of the bounded linear operators L_A that are determined by these matrix transformations.

3. The Fibonacci Difference Sequence Spaces $\ell_p(\widehat{F}(r,s))$ and $\ell_\infty(\widehat{F}(r,s))$

Although infinite sequences were used extensively in the early history of the calculus, especially, they have appeared in the history of mathematics since antiquity. In the middle ages the mathematician Fibonacci, in his work Liber Abaci (1202) used sequences of numbers $1,1,2,3,5,\ldots$. You may already be familiar with Fibonacci sequences, but if not, you will understand the following formula easy follow. For convenience, the steps in the sequence are usually labeled $1,1,2,3,5,\ldots$ and so on. In a much clearer way, the *Fibonacci sequences* $f=(f_n)$ starts with $f_0=f_1=1$ and uses the recursion formula

$$f_n = f_{n-1} + f_{n-2}; \quad n \geq 2.$$

The use of Fibonacci sequences is widely available and give opportunity for hands-on experience. When the most striking differences in art and architecture, plants and some living things in nature were carefully examined, it was seen that they were related to the Fibonacci numbers. Let me also point out here that, many applications of Fibonacci sequences are beyond the scope of this work, but the material in this section can prepare you for later study as well as provide knowledge that you can use as needed. Reference number [32] can be examined for a lot of information about Fibonacci numbers, including the Golden ratio.

Let $1 \leq p \leq \infty$ and q represent the conjugate of p throughout, that is, $q = p/(p-1)$ for $1 < p < \infty$, that is, $q = p/(p-1)$ for $1 < p < \infty$, $q = \infty$ for $p = 1$ or $q = 1$ for $p = \infty$.

In 2015, right after Kara [33], Candan and Kara [34] introduced the generalized Fibonacci difference sequence spaces $\ell_p(\widehat{F}(r,s))$ and $\ell_\infty(\widehat{F}(r,s))$, as follows;

$$\ell_p(\widehat{F}(r,s)) = \left\{ x = (x_n) \in \omega : \sum_n \left| r\frac{f_n}{f_{n+1}}x_n + s\frac{f_{n+1}}{f_n}x_{n-1} \right|^p < \infty \right\}; \ 1 \leq p < \infty$$

and

$$\ell_\infty(\widehat{F}(r,s)) = \left\{ x = (x_n) \in \omega : \sup_{n\in\mathbb{N}} \left| r\frac{f_n}{f_{n+1}}x_n + s\frac{f_{n+1}}{f_n}x_{n-1} \right| < \infty \right\}.$$

When we use the equivalent notation of (2) for the sequence spaces $\ell_p(\widehat{F}(r,s))$ and $\ell_\infty(\widehat{F}(r,s))$, related sequence spaces becomes

$$\ell_p(\widehat{F}(r,s)) = (\ell_p)_{\widehat{F}(r,s)} \ (1 \leq p < \infty) \ \text{ and also } \ \ell_\infty(\widehat{F}(r,s)) = (\ell_\infty)_{\widehat{F}(r,s)}, \tag{6}$$

in which the matrix $\widehat{F}(r,s) = (\widehat{f}_{nk}(r,s))$ is described by

$$\widehat{f}_{nk}(r,s) = \begin{cases} s\frac{f_{n+1}}{f_n} & (k = n-1) \\ r\frac{f_n}{f_{n+1}} & (k = n) \\ 0 & (0 \le k < n-1) \text{ or } (k > n) \end{cases} \quad (n,k \in \mathbb{N}). \tag{7}$$

To signal the fact that the sequence spaces $\ell_p(\widehat{F}(r,s))$ and $\ell_\infty(\widehat{F}(r,s))$ are BK-spaces according to the

$$\|x\|_{\ell_p(\widehat{F}(r,s))} = \left(\sum_n |y_n(x)|^p\right)^{1/p} \quad (1 \le p < \infty) \text{ and } \|x\|_{\ell_\infty(\widehat{F}(r,s))} = \sup_{n\in\mathbb{N}} |y_n(x)|, \tag{8}$$

norms defined on them, respectively, in which the sequence $y = (y_n) = (\widehat{F}(r,s)_n(x))$ which is the $\widehat{F}(r,s)$-transform of any sequence $x = (x_n)$, is used. That is

$$y_n = \widehat{F}(r,s)_n(x) = \begin{cases} r\frac{f_0}{f_1}x_0 = rx_0 & (n = 0) \\ r\frac{f_n}{f_{n+1}}x_n + s\frac{f_{n+1}}{f_n}x_{n-1} & (n \ge 1) \end{cases} \quad (n \in \mathbb{N}). \tag{9}$$

It should be emphasized that the findings of this study are more comprehensive than those of Alotaibi et al. [35] in 2015.

4. Main Results

Many applications of compact operators are beyond the scope of this paper, but the material in this section can prepare you to understand the subject and help you remember information you can use when needed. From a historical perspective, the current concept of the Hausdorff measure represents a culmination of the collective efforts of numerous individuals. However, the notion of non-compactness' Hausdorff measure was originally introduced in 1957 by Goldenštein, Gohberg, and Markus, and was subsequently further explored by Goldenštein and Markus. In the study [36], the sequence spaces Y, ℓ_∞, c_0 and c were considered, enabling the characterization of the classes $(\ell_p(\widehat{F}), Y)$, $(\ell_\infty(\widehat{F}), Y)$, $(\ell_1(\widehat{F}), Y)$, as well as the compact operators $(\ell_p(\widehat{F}), \ell_1)$ and $(\ell_1(\widehat{F}), \ell_p)$. In this study, we introduce the classes $B(\ell_1, \ell_p^\lambda)$ for $1 \le p < \infty$ and compute the operator norms in $B(\ell_1, \ell_p^\lambda)$. Furthermore, leveraging the findings from the previous section, we describe the classes $C(\ell_1, \ell_p)$ for $1 \le p < \infty$ and determine the Hausdorff measure of non-compactness for operators in $B(\ell_1, \ell_p^\lambda)$.

Let $1 \le p < \infty$. We now provide a characterization of $B(\ell_1, \ell_p(\widehat{F}(r,s)))$, along with the computation of the operator norms in $B(\ell_1, \ell_p(\widehat{F}(r,s)))$. Additionally, we can utilize the results presented in the previous section to both specify the Hausdorff measure of non-compactness for operators in $B(\ell_1, \ell_p(\widehat{F}(r,s)))$ and characterize the classes $C(\ell_1, \ell_p(\widehat{F}(r,s)))$ for $1 \le p < \infty$.

The following result is particularly advantageous in certain proofs.

Lemma 4.1 *([20, Theorem 3.8]) T is a triangular matrix and with it X and Y being any two sequence spaces; for the matrix A to be an element of the (X, Y_T) class, the necessary and sufficient condition is that $C = T \cdot A$ and the matrix C belongs to the class (X, Y). In addition, if the X and Y are BK-spaces, and also if the matrix A is an element of the class (X, Y_T), then*

$$\|L_A\| = \|L_C\|. \tag{10}$$

We then define the identities for the operator norm and the characterizations of the classes $B(\ell_1, \ell_p(\widehat{F}(r,s)))$ for $1 \le p < \infty$.

Theorem 4.2 *Let $1 \le p < \infty$.*

(a) We have $L \in B(\ell_1, \ell_p(\widehat{F}(r,s)))$ if and only if there exists an infinite matrix $A \in (\ell_1, \ell_p(\widehat{F}(r,s)))$ such that

$$\|A\| = \sup_k \left(\sum_n \left| r\frac{f_n}{f_{n+1}} a_{nk} + s\frac{f_{n+1}}{f_n} a_{n-1,k} \right|^p \right)^{1/p} < \infty \tag{11}$$

and

$$L(x) = Ax \text{ for all } x \in \ell_1. \tag{12}$$

(b) If $L \in B(\ell_1, \ell_p(\widehat{F}(r,s)))$, then

$$\|L\| = \|A\|. \tag{13}$$

Proof For (a), when we keep in mind that ℓ_1 is a BK-space with AK, for $L \in B(\ell_1, \ell_p(\widehat{F}(r,s)))$ from Lemma 1.1 under the condition $1 \le p < \infty$ hypothesis condition; the necessary and sufficient condition is that there is an infinite matrix A such that $A \in (\ell_1, \ell_p(\widehat{F}(r,s)))$ provided that the condition (12) is met. If we denote the product of the matrices $\widehat{F}(r,s) = (\widehat{f}_{nk}(r,s))$ and $A = (a_{nk})$ by $C = (c_{nk})$, that is, we can express it clearly as follows

$$c_{nk} = r\frac{f_n}{f_{n+1}} a_{nk} + s\frac{f_{n+1}}{f_n} a_{n-1,k}.$$

Now it is quiet easy to say that from Lemma 4.1 (a) that the necessary and sufficient condition $A \in (\ell_1, \ell_p(\widehat{F}(r,s)))$ is $C \in (\ell_1, \ell_p)$. If the Example 8.4.1D in the reference [12] is used at this stage of the proof, it is seen that the necessary and sufficient condition for $C \in (\ell_1, \ell_p)$ is

$$\|C\| = \sup_k \left(\sum_{n=0}^{\infty} |c_{nk}|^p \right)^{1/p} < \infty,$$

which proves the claim. □

(b) First, we show that $\|L\| \leq \|A\|$. Let $L \in B(\ell_1, \ell_p^{\lambda})$. It is seen from (10) that $\|L\| = \|L_C\|$ for $L_C \in B(\ell_1, \ell_p)$ is presented by the equation $L_C(x) = Cx$ for every $x \in \ell_1$. Now, we can write by the Minkowsky's inequality that, we can write the following expressions

$$\begin{aligned}
\|L_C(x)\|_p &= \left(\sum_{n=0}^{\infty}\left|\sum_{k=0}^{\infty} c_{nk}x_k\right|^p\right)^{1/p} \\
&\leq \sum_{k=0}^{\infty}|x_k|\left(\sum_{n=0}^{\infty}|c_{nk}|^p\right)^{1/p} \\
&\leq \|C\| \cdot \|x\| \\
&= \|A\| \cdot \|x\|
\end{aligned}$$

and from here we can write the following inequality

$$\|L\| \leq \|A\| \tag{14}$$

for the norms of L and A. Now, let us prove the other side of the inequality. For this, when $e^{(k)} \in S_{\ell_1}$ $(k \in N)$ is taken, it is seen that

$$\|L\| \geq \|A\| \tag{15}$$

from the equation below

$$\left\|L_C(e^{(k)})\right\| = \left(\sum_{n=0}^{\infty}|c_{nk}|^p\right)^p.$$

When (14) and (15) are considered together, it is proved that (13).

The Hausdorff measure of the non-compactness of operators in $B(\ell_1, \ell_p(\widehat{F}(r,s)))$ will be established in the expression below. Another closely related result to be used in the first come proof is given below.

Lemma 4.3 *([37, Theorem 4.2]) Let T be a triangle and χ and χ_T be the Hausdorff measures of non-compactness on M_X and M_{X_T}, respectively. Assume that X is a linear metric space with a translation invariant metric. If $Q \in M_{X_T}$, then $\chi_T(Q) = \chi(TQ)$.*

Theorem 4.4 *Let $L \in B(\ell_1, \ell_p(\widehat{F}(r,s)))$ with $1 \leq p < \infty$ and A demonstrate the matrix which stands for L. In that case we get*

$$\|L\|_{\chi_{\ell_p(\widehat{F}(r,s))}} = \lim_{m\to\infty}\left(\sup_k \sum_{n=m}^{\infty}\left|r\frac{f_n}{f_{n+1}}a_{jk} + s\frac{f_{n+1}}{f_n}a_{j-1,k}\right|^p\right)^{1/p}.$$

Proof First of all, we briefly write $S = S_{\ell_1}$, also $C^{[m]}$ $(m \in \mathbb{N})$ for the matrix with the rows $C_n^{[m]} = 0$ for $0 \le n \le m$ and $C_n^{[m]} = C_n$ for $n \ge m+1$. In this case, if we use both Lemma 4.3 and together with (3), (5), (11) and (13) the following equations can easily be calculated

$$\begin{aligned}
\|L\|_{\chi_{\ell_p(\widehat{F}(r,s))}} &= \chi_{\ell_p(\widehat{F}(r,s))}(L(S)) \\
&= \chi_{\ell_p}(L_C(S)) \\
&= \lim_{m\to\infty}\left(\sup_{x\in S}\|\mathcal{R}_m(Cx)\|_p\right) \\
&= \lim_{m\to\infty}\left(\sup_{x\in S}\|C^{[m]}x\|_p\right) \\
&= \lim_{m\to\infty}\|C^{[m]}\| \\
&= \lim_{m\to\infty}\left(\sup_k \sum_{n=m}^{\infty}\left|r\frac{f_n}{f_{n+1}}a_{jk} + s\frac{f_{n+1}}{f_n}a_{j-1,k}\right|^p\right)^{1/p}.
\end{aligned}$$

∎

This is the desired result.

We are now ready to give the following theorem, which obtains the characterization of $C(\ell_1, \ell_p(\widehat{F}(r,s)))$ by coordinating the condition given in (4) and Theorem 4.4.

Theorem 4.5 *If $L \in B(\ell_1, \ell_p(\widehat{F},\ 1 \le p < \infty(r,s)))$ and at the same time the matrix A is the matrix representing L, a necessary and sufficient condition for L to be compact is that the following limit is equal to zero, that is*

$$\lim_{m\to\infty}\left(\sup_k \sum_{n=m}^{\infty}\left|r\tfrac{f_n}{f_{n+1}}a_{jk} + s\tfrac{f_{n+1}}{f_n}a_{j-1,k}\right|^p\right) = 0.$$

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Declaration of Ethical Standards

The author declares that the materials and methods used in his study do not require ethical committee and/or legal special permission.

Conflicts of Interest

The author declares no conflict of interest.

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