Hacettepe Journal of
Mathematics & Statistics

Hacet. J. Math. Stat.
Volume XX (x) (XXXX), 1 – 12
DOI : 10.15672/hujms.xx

Research Article

Left-ray right-ray hybrid topologies on the real line

Sami Lazaar[1], Tom Richmond*[2], Khadhri Sabrine[3]

[1] Faculty of Sciences, University of Taibah, Medina, Saudi Arabia and Faculty of Sciences of Tunis, Tunis El-Manar University, Tunis, Tunisia
[2] Western Kentucky University, Bowling Green, KY 42101, USA
[3] Faculty of Sciences of Sfax, Sfax, Tunisia

Abstract

Given a non-empty set $A \subseteq \mathbb{R}$, we consider the smallest topology on $\mathbb{R}$ which contains the open left rays containing points $a \in A$ and the open right rays containing points $b \in \mathbb{R} - A$. We present a natural model for this hybrid topology and show that it is quasi-metrizable. We investigate other variations of this topology.

Mathematics Subject Classification (2020). Primary 54A05; Secondary 54E35

Keywords. hybrid topology, Hattori topology, quasi-metric

1. Introduction

Given a set $A \subseteq \mathbb{R}$, the Hattori space $H(A)$ is $\mathbb{R}$ with the topology having the base $\{(a-\varepsilon, a+\varepsilon) : a \in A, \varepsilon > 0\} \cup \{[b, b+\varepsilon) : b \in \mathbb{R} - A, \varepsilon > 0\}$. These spaces were introduced in [7] and studied further in [1–4,9,12]. The Hattori topology may be viewed as a hybrid of the Euclidean and lower-limit topologies on $\mathbb{R}$. In [12], hybrid topologies based on various combinations of the lower-limit, upper-limit, left-ray, discrete, and Euclidean topologies were studied. Here, we consider hybrid topologies of the left-ray and right-ray topologies. After some basic properties in Section 1, in Section 2 we investigate the subposet of the hybrids of left-ray and right-ray topologies in the lattice $TOP(\mathbb{R})$ of all topologies on $\mathbb{R}$. The left-ray and right-ray topologies each arise from a quasi-metric. In Section 3, we show that our hybrids of left-ray and right-ray topologies arise from hybrid quasi-metrics. In Section 4, we consider variations using closed rays and using rays $(-\infty, a+\varepsilon)$ where $\varepsilon > 0$ is bounded above by 1. In the last section, we consider hybrids of the Euclidean and right-ray topologies, which are closely related.

A quasi-metric on X is a function $q : X \times X \to [0, \infty)$ satisfying (a) $x = y$ if and only if $q(x, y) = 0 = q(y, x)$ and (b) $q(x, y) + q(y, z) \geq q(x, z)$, for all $x, y, z \in X$. The left-ray topology arises from the quasi-metric $q_{lr}(x, y) = y - x$ if $y \geq x$ and $q_{lr}(x, y) = 0$ if $y < x$. The right-ray quasi-metric is defined similarly. There are several quasi-metrization theorems [5,6,8,10], but it is often difficult to exhibit a specific quasi-metric for a quasi-metrizable space. For example, Fletcher and Lindgren [5] show that the Niemytzki tangent

*Corresponding Author.
Email addresses: SAZAR@taibahu.edu.sa (S. Lazaar), tom.richmond@wku.edu (T. Richmond), khadhrisabrine@gmail.com (K. Sabrine)
Received: 05.06.2023; Accepted: 27.12.2023

disk topology on the closed half-plane (see [14]) is quasi-metrizable, but to our knowledge, no explicit quasi-metric has been exhibited. Further results on quasi-metrics can be found in the works of H.-P. Künzi and of F. Yildiz, including [11]. Standard topological concepts can be found in [13]. Recall that a set C in a poset P is *order dense* if $a, b \in P$ and $a < b$ imply that there exists $c \in C$ with $a < c < b$. For the poset $(\mathbb{R}, \leq)$, C is order dense in $\mathbb{R}$ if and only if C is dense in $\mathbb{R}$ with the Euclidean topology. Throughout, we assume $A \subseteq \mathbb{R}$ and $B = \mathbb{R} - A$. We use $C - D$ to represent the relative complement of D in C.

2. Basic Properties

Given $A \subseteq \mathbb{R}$, by $L(A)$, we denote the hybrid of the left-ray and right-ray topology on $\mathbb{R}$ having a subbasis

$$\mathcal{S} = \{(-\infty, a+\varepsilon) : a \in A, \varepsilon > 0\} \cup \{(b-\varepsilon, \infty) : b \in \mathbb{R} - A, \varepsilon > 0\}.$$

Note that $\mathcal{S}$ is not a basis, since the intersection of two oppositely directed rays will never contain a ray.

Let $B = \mathbb{R} - A$. Assume that A, B are non-empty (or else we simply have the right-ray or left-ray topology). If there exists $a < b$ with $a \in A, b \in B$, then a has the neighborhood base $(a-\varepsilon, a+\varepsilon) = (b-(b-a+\varepsilon), \infty) \cap (-\infty, a+\varepsilon)$ and similarly, b has the neighborhood base $(b-\varepsilon, b+\varepsilon)$. If $[a, \infty) \subseteq A$ (so $a \in A, a \geq \sup B$), then a has the neighborhood base $\{(b-\varepsilon, a+\varepsilon) : b < a, b \in B, \varepsilon > 0\}$. If $(-\infty, b] \subseteq B$ (so $b \leq \inf A$), then b has the neighborhood base $\{(b-\varepsilon, a+\varepsilon) : b < a, a \in A, \varepsilon > 0\}$. We tabulate this in Table 1. Figure 1 suggests the neighorhoods.

$x = a \in A$ or $x = b \in B$	form of the neighborhood base at x ($\varepsilon > 0$)
$x = a < \sup B$	$(x-\varepsilon, x+\varepsilon)$
$x = a \geq \sup B$	$(b-\varepsilon, x+\varepsilon)$, $b \leq \sup B \leq a = x$
$\inf A < b = x$	$(x-\varepsilon, x+\varepsilon)$
$x = b \leq \inf A$	$(x-\varepsilon, \inf A+\varepsilon)$

Table 1. Neighborhoods of points in $L(A)$.

Some of the results below also hold for $A = \emptyset$ or $A = \mathbb{R}$, but care must be taken in these cases. Since $B = \mathbb{R} - A$, if $\emptyset \subset A \subset \mathbb{R}$, then $\inf A \leq \sup B$. However, if $A = \emptyset$, then $\inf A = \infty \geq \sup B$ and if $A = \mathbb{R}$, $\sup B = -\infty \leq \inf A$.

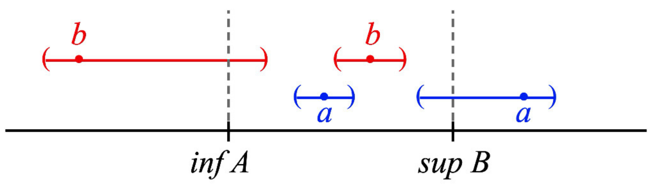

Figure 1. $L(A)$ neighborhoods of $a \in A$ and $b \in B$, based on the position of a, b.

Example 2.1. A model for $L(A)$. For real numbers $i < j$, consider the set $X = \{(i, y) \in \mathbb{R}^2 : y \geq 0\} \cup \{(x, 0) \in \mathbb{R}^2 : i \leq x \leq j\} \cup \{(j, y) \in \mathbb{R}^2 : y \geq 0\}$ as suggested in Figure 2. Think of the x-axis as being on the ground and the y-axis extending vertically above the ground. It takes energy to travel on the ground (horizontally) and upward, but no added energy to travel downward—gravity supplies that required force. This suggests a quasi-metric on X defined by taking the ε-ball around $(r,s) \in X$ to be the set of points accessible from (r,s) if you have enough energy to move a distance of ε units. We will delay the verification that this is a quasi-metric until Section 4, but we can recognize that the topology required is $L(A)$, with $A = \{i\} \cup [j,\infty)$, so $[i,j] = [\inf A, \sup B]$. If exactly one of $\inf A$ and $\sup B$ is infinite, then only one of the vertical poles of Figure 2 is present.

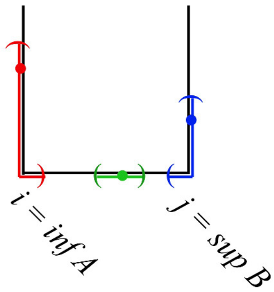

Figure 2. $L(A)$ models distances required to move overland or upward.

We list some immediate facts about $L(A)$ for reference.

Theorem 2.2. (a) *$L(A)$ is the Euclidean topology if and only if $A \neq \mathbb{R}$, $\inf A = -\infty$, and $\sup B = \infty$.*

(b) *For any $A \subseteq \mathbb{R}$, $L(A)$ is T_0.*

(c) *$L(A)$ is T_1 if and only if $A \neq \mathbb{R}$, $\inf A = -\infty$, and $\sup B = \infty$.*

(d) *If $\emptyset \subset A \subset \mathbb{R}$, the constant sequence $(x)_{n=1}^{\infty}$ in $L(A)$ has the unique limit x if $x \in (\inf A, \sup B)$, converges to all $y \in (-\infty, x]$ if $x \leq \inf A$, and converges to all $z \in [x,\infty)$ if $x \geq \sup B$.*

(e) *$L(A)$ is always connected.*

(f) *$L(A)$ is separable.*

Proof. (a) follows from the definition of $L(A)$.

(b) If $x \neq y$ then either $(-\infty, \frac{x+y}{2})$ or $(\frac{x+y}{2}, \infty)$ is a neighborhood of one of the points which excludes the other.

(c) If $\inf A = -\infty$ and $\sup B = \infty$, then by (a), $L(A)$ is Euclidean, which is T_1. If $\inf A > -\infty$, then for $x < y < \inf A$, every neighborhood of x includes y, so $L(A)$ is not T_1. The dual argument shows that if $\sup B < \infty$, $L(A)$ is not T_1.

(d) is immediate.

(e) If $A = \emptyset$ or $\mathbb{R}$, then $L(A)$ is the left- or right-ray topology, which is connected. Otherwise, $(\inf A, \sup B)$ inherits the Euclidean topology and is connected. If U, V is a separation of $L(A)$, for $y \in (\inf A, \sup B) \cap U$, U must contain the connected set $(\inf A, \sup B)$. If $x \leq \inf A$, every open set containing x must intersect $(\inf A, \sup B)$, so $x \in U$. Similarly, every open neighborhood of $y > \sup B$ must intersect $(\inf A, \sup B)$, so $y \in U$. This gives the contradiction that $U = \mathbb{R}$.

(f) In $L(A)$, every open set intersects $\mathbb{Q}$. □

Combining (a) and (c), we see that for $A \neq \mathbb{R}$, $L(A)$ is T_1 if and only if it is T_j for any $j \in \{2, 3, 3.5, 4\}$. Note that if A and B are order dense (or equivalently, dense in the Euclidean topology), then $L(A)$ is the Euclidean topology.

From Figure 1, we see that if $i = \inf A < \sup B = j$, the closed and bounded sets are arbitrary intersections of Euclidean closed sets $[i, c_n] \cup [d_n, j] \subseteq [i,j]$ which are contained in the Euclidean subspace $[i,j]$ of $L(A)$, and thus are compact. If $i = \inf A = \sup B = j$, the closed sets are of form $\emptyset, \mathbb{R}, (-\infty, c]$ for $c < i = j$, and $[d, \infty)$ for $d > i = j$, so $\emptyset$ is the only closed and bounded set. Thus, in all cases, the closed and bounded sets in $L(A)$ are compact. The converse fails. In $L(A)$, compact sets need not be closed, as the following example shows.

Example 2.3. Let $A = (-1, 0)$. For $x < \inf A = -1$, $cl\{x\} = (-\infty, x]$ so $\{x\}$ is not closed. As a finite set, $\{x\}$ is compact.

Or, with $A = (0, \infty)$, consider $S = (2, 3]$. Any open set covering 3 covers $[0, 3] = [\sup B, 3]$ and thus covers S. Thus, S is compact, but S is not closed in $L(A)$ nor in the Euclidean topology.

A compact set S in $L(A)$ must be bounded. In particular, $\mathbb{R}$ is not $L(A)$-compact. This is implied by the following characterization of the compact subsets of $L(A)$.

Theorem 2.4. *Suppose $A \neq \emptyset$ and $B \neq \emptyset$. Then a non-empty set $S \subseteq \mathbb{R}$ is $L(A)$-compact if and only if $(-\infty, \sup B) \cap S$ is empty or has a smallest element, $(\inf A, \infty) \cap S$ is empty or has a largest element, and there is no sequence (s_n) in S converging (in the Euclidean topology) to $x \in (\inf A, \sup B) - S$.*

Proof. Suppose $(-\infty, \sup B) \cap S \neq \emptyset$ and has no smallest element. Then there exists a strictly decreasing sequence (s_n) in S with $S \subseteq \bigcup\{(s_n, \infty) : n \geq 1\}$ and $s_n < b$ for some $b \in B$ and all $n \in \mathbb{N}$. Now $\{(s_n, \infty) : n \in \mathbb{N}\}$ is an open cover of S with no finite subcover. The dual argument covers the case $(\inf A, \infty) \cap S \neq \emptyset$ and has no largest element. Suppose there is a sequence (s_n) in S converging to $x \in (\inf A, \sup B) - S$. Now there exist $a \in A, b \in B$ with $a < x < b$. Without loss of generality (dropping to a subsequence, if it is necessary), we will assume (s_n) is a strictly monotone sequence in (a, b). If (s_n) is strictly decreasing, $\{(-\infty, x)\} \cup \{(s_n, \infty) : n \in \mathbb{N}\}$ is an open cover with no finite subcover. The dual construction applies if (s_n) is strictly increasing. Thus, compactness of S implies the conditions listed in the theorem.

Now suppose the conditions listed are satisfied and $\mathcal{C}$ is an open cover of S. Recall that $\inf A \leq \sup B$. Now either

(1) $(-\infty, \sup B) \cap S = \emptyset$,

(2) $\min((-\infty, \sup B) \cap S) = s' \leq \inf A$, or

(3) $\min((-\infty, \sup B) \cap S) = s' \in (\inf A, \sup B)$

and either

(a) $(\inf A, \infty) \cap S = \emptyset$,

(b) $\max((\inf A, \infty) \cap S) = s'' \geq \sup B$, or

(c) $\max((\inf A, \infty) \cap S) = s'' \in (\inf A, \sup B)$.

In case (1), we have $S \subseteq [\sup B, \infty)$, and it follows that case (c) cannot occur. Cases (1) and (a) occur if and only if $S = \{\sup B\} = \{\inf A\}$, in which case S is finite and thus compact. If cases (1) and (b) occur, any $C_1 \in \mathcal{C}$ which covers s'' also covers $[\sup B, s'']$ and thus covers S.

If cases (2) and (a) occur, $S \subseteq [s', \inf A]$ and any $C_1 \in \mathcal{C}$ covering s' covers $[s', \inf A]$ and thus covers S. If cases (2) and (b) occur, any $C_1 \in \mathcal{C}$ covering s' covers $[s', \inf A + \varepsilon')$ for some $\varepsilon' > 0$ and any $C_2 \in \mathcal{C}$ covering s'' covers $(\sup B - \varepsilon'', s'']$ for some $\varepsilon'' > 0$. Consider $S' = S \cap [\inf A' + \varepsilon', \sup B' - \varepsilon'']$. If $S' = \emptyset$, then $\{C_1, C_2\}$ covers S. Otherwise, the hypothesis about no sequences in S converging to $x \in (\inf A, \sup B) - S$ implies S' is Euclidean closed. Since $S' \subseteq (\inf A, \sup B)$, the $L(A)$ topology on S' is Euclidean. As a closed and bounded Euclidean set covered by the Euclidean open sets of $\mathcal{C}$, there must be a finite subcover $\{C_k\}_{k=3}^n$ of S'. Now $\{C_k\}_{k=1}^n$ is a finite subcover of S. In case (2) and (c) hold, any $C_1 \in \mathcal{C}$ covering s' covers $[s', \inf A + \varepsilon')$ for some $\varepsilon' > 0$, and $S' = [\inf A + \varepsilon', s'']$

is a Euclidean closed and bounded set in $(\inf A, \sup B)$, and since $L(A)$ agrees with the Euclidean topology in $(\inf A, \sup B)$, there is a finite subcover $\{C_k\}_{k=2}^n$ of S'. Now $\{C_k\}_{k=1}^n$ is a finite subcover of S.

The cases involving (3) are dual to those with (2). □

For $S \subseteq \mathbb{R}$, divide S into its left, middle, and right parts $S_l = S \cap (-\infty, \inf A)$, $S_m = S \cap [\inf A, \sup B]$, and $S_r = S \cap (\sup B, \infty)$. Theorem 2.4 almost characterizes compact sets in $L(A)$ as the sets S for which S_m is closed, S_l is empty or has a least element, and S_r is empty or has a greatest element. However, this potential characterization fails due to some issues around the points $\inf A$ and $\sup B$. For example, with $A = (0, 1)$ and $S = \{-1\} \cup (0, 2]$, S is compact even though S_m is not closed. Note that Theorem 2.4 partly avoids these issues by considering $S \cap (\inf A, \infty)$ instead of $S_r = S \cap (\sup B, \infty)$ and $S \cap (-\infty, \sup B)$ instead of $S_l = (-\infty, \inf A)$.

3. Lattice Properties

Let $\mathcal{L}(\mathbb{R}) = \{L(A) : A \subseteq \mathbb{R}\}$ be the subset of the lattice $TOP(\mathbb{R})$ of all toplogies on $\mathbb{R}$, ordered by $\subseteq$. It is well-known that $TOP(X)$ is a lattice with $\tau \wedge \tau' = \tau \cap \tau'$ and $\tau \vee \tau'$ having $\tau \cup \tau'$ as a subbase (that is, $\tau \vee \tau' = [\tau \cup \tau']$).

Figures 1 and 2 suggest that $L(A)$ depends more on the pair $(i, j) = (\inf A, \sup B)$ than on the set A itself. The proof of the theorem below is straightforward.

Theorem 3.1. *Suppose A, A' are subsets of $\mathbb{R}$ with complements B, B', respectively. Let $(i, j) = (\inf A, \sup B)$ and $(i', j') = (\inf A', \sup B')$.*

(a) *$L(A) = L(A')$ if and only if $(i, j) = (i', j')$.*
(b) *If A, A' are non-empty and proper subsets of $\mathbb{R}$, $L(A) \subseteq L(A')$ if and only if $[i, j] \subseteq [i', j']$.*
(c) *If A, A' are non-empty and proper subsets of $\mathbb{R}$, $L(A) \subset L(A')$ if and only if $[i, j] \subset [i', j']$.*

We will now consider infima in $\mathcal{L}(\mathbb{R})$ and compare them to infima in $TOP(\mathbb{R})$. We start with an instructive example.

Example 3.2. $L(A) \cap L(A') \neq L(A \cap A')$. Let $A = \{0\}$ and $A' = \{3\}$. Now $L(A)$ has basis $\{(y, z) : y < 0 < z\} \cup \{(z, w) : 0 < z < w\}$, $L(A')$ has basis $\{(y, z) : y < 3 < z\} \cup \{(z, w) : 3 < z < w\}$, and $L(A \cap A') = L(\emptyset)$ has basis $\{(b, \infty) : b \in \mathbb{R}\}$. Now $L(A) \cap L(A')$ has basis $\{(y, z) : y < 3 < z\} \cup \{(z, w) : 3 < z < w\}$. In particular, $(-1, 4) \in L(A) \cap L(A')$ but $(-1, 4) \notin L(A \cap A')$.

Indeed observe that $L(\{0\}) = L(\{0, 3\})$ since $\inf\{0, 3\} = \inf\{0\}$ and $\sup(\mathbb{R} - \{0, 3\}) = \sup(\mathbb{R} - \{0\})$. Now $L(\{0\}) \cap L(\{3\}) = L(\{0, 3\}) \cap L(\{3\}) = L(\{3\})$ by Theorem 3.1.

For another example, if $A = \mathbb{Q} \cap [0, 1]$ and $A' = [0, 1] - \mathbb{Q}$, then by Thoerem 3.1, $L(A) = L(A') = L(A) \cap L(A') \neq L(A \cap A') = L(\emptyset)$.

Example 3.2 suggests that the intervals $[i, j]$, $[i', j']$ are more significant in determining $L(A) \cap L(A')$ than the actual sets A, A'. Our next result confirms this and shows that $\mathcal{L}(\mathbb{R})$ is not a lattice.

Theorem 3.3. *Suppose A, A' are non-empty, proper subsets of $\mathbb{R}$, $i = \inf A, j = \sup B$, $i' = \inf A'$, and $j' = \sup B'$.*

(a) *If $[i'', j''] = [i, j] \cap [i', j'] \neq \emptyset$, then in $\mathcal{L}(\mathbb{R})$, $L(A) \wedge L(A') = L(A) \cap L(A') = L(A'')$ where A'' arises from $[i'', j'']$ (that is, A'' is any subset of $\mathbb{R}$ with $\inf A'' = i''$ and $\sup B'' = \sup(\mathbb{R} - A'') = j''$).*
(b) *If $[i, j] \cap [i', j'] = \emptyset$, then there is no topology $\tau \in \mathcal{L}(\mathbb{R})$ with $\tau \subseteq L(A) \cap L(A')$, so in $\mathcal{L}(\mathbb{R})$, $L(A) \wedge L(A')$ fails to exists.*

Proof. (a) By Theorem 3.1, $L(A'') = L(A) \wedge L(A')$ in $\mathcal{L}(\mathbb{R})$. It is easy to verify that $L(A'') = L(A) \cap L(A')$, which is $L(A) \wedge L(A')$ in $TOP(\mathbb{R})$.

(b) Suppose $[i,j] \cap [i',j'] = \emptyset$. Without loss of generality, $i \leq j < i' \leq j'$. We will first describe $L(A) \cap L(A')$.

Suppose $U \in L(A) \cap L(A')$, $x \in U$, and $x \leq i' = \inf A'$. Now every $L(A')$ neighborhood of x must include $[x, i']$, and every $L(A)$-open set containing i' must contain $[j, i']$. Thus, $[j, i'] \subseteq U$. Indeed, every interval $(r, s) \supseteq [j, i'] \cup [x, i']$ is a $L(A) \cap L(A')$ neighborhood of $x \leq i'$.

Suppose $U \in L(A) \cap L(A')$, $x \in U$, and $x > i' = \inf A'$. Now every $L(A)$ neighborhood of x must include $[j, x]$, so $[j, i'] \subseteq [j, x] \subseteq U$. Indeed, every interval $(r, s) \supseteq [j, x]$ is a $L(A) \cap L(A')$ neighborhood of $x > i'$.

The last two paragraphs show that if $i \leq j < i' \leq j'$, the non-empty elements of $L(A) \cap L(A')$ are the intervals (r, s) with $[j, i'] \subseteq (r, s)$ $(r, s \in \mathbb{R} \cup \{\pm\infty\})$. Observe that this topology models the situation of Example 2.1 if no energy is required to move horizontally. In particular, for $j < x < y < i'$, every neighborhood of x contains y and every neighborhood of y contains x, so $L(A) \cap L(A')$ is not T_0 and thus not of form $L(A'')$ for any set A''. Indeed, there is no topology $\tau \subseteq L(A) \cap L(A')$ which is T_0, so in the poset $\mathcal{L}(\mathbb{R})$, $L(A) \wedge L(A')$ does not exist if $i \leq j < i' \leq j'$. □

Now we turn to suprema in $\mathcal{L}(\mathbb{R})$. Recall that in $TOP(\mathbb{R})$, $L(A) \vee L(A') = [L(A) \cup L(A')]$, the topology generated by the basis $L(A) \cup L(A')$.

Example 3.4. $L(A \cup A') \neq [L(A) \cup L(A')]$. Let $A = \mathbb{Q}$ and $A' = \mathbb{R} - \mathbb{Q}$. By Theorem 2.2, $L(A)$ and $L(A')$ are the Euclidean topology, so $L(A) \vee L(A')$ is the Euclidean topology. However, $L(A \cup A') = L(\mathbb{R})$ is the left-ray topology.

Notation: $convS$ is the convex hull of S. If $i = -\infty$ or $j = \infty$, by $[i, j]$ we mean $[i, j] \cap \mathbb{R}$. Infima and suprema are taken in $\mathbb{R} \cup \{\pm\infty\}$.

Theorem 3.5. *$\mathcal{L}(\mathbb{R})$ is an upper sub-semi-lattice of $Top(\mathbb{R})$. If $L(A)$ and $L(A')$ correspond to $[i, j]$ and $[i', j']$, then in $\mathcal{L}(\mathbb{R})$, $L(A) \vee L(A') = [L(A) \cup L(A')] = L(A'')$ where A'' corresponds to $[i'', j''] = [\inf\{i, i'\}, \sup\{j, j'\}] = conv([i, j] \cup [i', j'])$.*

Proof. Given $L(A), L(A')$ corresponding to $[i, j]$ and $[i', j']$, let $[i'', j''] = [\inf\{i, i'\}, \sup\{j, j'\}] = conv([i, j] \cup [i', j'])$ correspond to $L(A'')$. In $L(A'')$, $x < i''$ has a neighborhood base of form $(x - \varepsilon, i'' + \varepsilon)$, which is open in $L(A)$ or $L(A')$, depending on whether $i'' = i$ or $i'' = i'$. In $L(A'')$, $x > j''$ has a neighborhood base of form $(j'' - \varepsilon, x + \varepsilon)$, which is open in $L(A)$ or $L(A')$, depending on whether $j'' = j$ or $j'' = j'$. In $L(A'')$, $x \in [i'', j'']$ has a Euclidean neighborhood base. Now either $[i'', j''] - ([i, j] \cup [i', j'])$ is empty (if $[i, j] \cap [i', j'] \neq \emptyset$) or $[i'', j''] - ([i, j] \cup [i', j'])$ is an interval I. If $x \in [i, j] \cup [i', j']$, x has a Euclidean neighborhood base in $L(A)$ or $L(A')$. If x is in the interval I between $[i, j]$ and $[i', j']$, say $i \leq j < x < i' \leq j'$, then x has $L(A)$ neighborhoods $(j - \varepsilon, x + \varepsilon)$ and $L(A')$ neighborhoods $(x - \varepsilon, i' + \varepsilon)$, and thus has $[L(A) \cup L(A')]$ neighborhoods $(x - \varepsilon, x + \varepsilon)$. This shows that $L(A'') \subseteq [L(A) \cup L(A')]$. Since $[i, j], [i', j'] \subseteq [i'', j'']$, Theorem 3.1 shows that $L(A), L(A') \subseteq L(A'')$, so $[L(A) \cup L(A')] \subseteq L(A'')$. □

4. A quasi-metric for $L(A)$.

Theorem 4.1. *For $a \in A$, $b \in B = \mathbb{R} - A$, and $y \in \mathbb{R}$, let*

$$q(a, y) = \begin{cases} |a - y| & \text{if } a < \sup B & (1) \\ y - a & \text{if } \sup B \leq a \leq y & (2) \\ 0 & \text{if } \sup B \leq y \leq a & (3) \\ \sup B - y & \text{if } y < \sup B \leq a & (4) \end{cases}$$

$$q(b,y)=\begin{cases} |b-y| & \text{if } \inf A<b \quad (5)\\ b-y & \text{if } y<b\le \inf A \quad (6)\\ 0 & \text{if } b<y\le \inf A \quad (7)\\ y-\inf A & \text{if } b\le \inf A<y. \quad (8)\end{cases}$$

Then $q(x,y)$ is a quasi-metric which generates the topology $L(A)$.

Proof. It is easy to see that $q(x,y)\geq 0$ and $x=y$ if and only if $q(x,y)=0=q(y,x)$, and the q-balls around $x\in\mathbb{R}$ match the base of $L(A)$ neighborhoods given in Table 1. Thus, it only remains to show that $q(x,y)+q(y,z)\geq q(x,z)$ for distinct $x,y,z\in\mathbb{R}$.

If line (n) of the definition of q is used to find the distance $q(x,y)$, we will say (x,y) are "in position (n)". Observe that for $(n)=(1)$ or (5), we could more accurately say x (rather than (x,y)) is in position (n), but for consistency, we may still say (x,y) is in position (n).

Case $(x,y)=(a,a')\in A^2$: Suppose $(x,y)=(a,a')$ is in position (1). If (a',z) is in position (1) or (2), then all distances are Euclidean and the triangle inequality holds. If (a',z) is in position (3), we have $a<\sup B\leq z\leq a'$, and $q(a,a')\geq q(a,z)$ since both of these are Euclidean distances. If (a',z) are in position (4), then either (i) $a\leq z<\sup B\leq a'$ or (ii) $z\leq a\leq \sup B\leq a'$. In case (i), then $q(a,a')\geq q(a,z)$ since both of these distances are Euclidean. In case (ii), $q(a',z)=\sup B-z\geq a-z=q(a,z)$.

Suppose $(x,y)=(a,a')$ is in position (2). Note that (a',z) cannot be in position (1). If (a',z) is in position (2), then all distances are Euclidean. If (a',z) is in position (3), either (i) $\sup B\leq z\leq a\leq a'$ or (ii) $\sup B\leq z\leq a'$. In case (i), $q(a,z)=0$ and in case (ii), $q(a,a')=a'-a\geq z-a=q(a,z)$. If (a',z) is in position (4), then $z\leq \sup B\leq a\leq a'$, so $q(a,z)=\sup B-z=q(a',z)$.

Suppose $(x,y)=(a,a')$ are in positioned (3). Now (a',z) cannot be in position (1). If (a',z) is in position (2), then we have (i) $\sup B\leq a'\leq z\leq a$ or (ii) $\sup B\leq a'\leq a\leq z$. In the case (i), (a,z) are in position (3), so $q(a,z)=0$. In case (ii), $q(a',z)$ and $q(a,z)$ are computed by (2), so $q(a',z)\geq q(a,z)$. If (a',z) is in position (3) then $\sup B\leq z\leq a'\leq a$, so $q(a,z)=0$. If (a',z) is in position (4) then $z\leq \sup B\leq a'\leq a$, so $q(a,z)=\sup B-z=q(a',z)$.

Suppose $(x,y)=(a,a')$ is in position (4). Then we have $a'<\sup B\leq a$, so (a',z) is necessarily in position (1). If $z\leq a'$, then $q(a,a')+q(a',z)=\sup B-a'+a'-z=q(a,z)$. If $a'\leq z\leq \sup B\leq a$, then $q(a,a')=\sup B-a'\geq \sup B-z=q(a,z)$. If $a'\leq \sup B\leq z\leq a$, then $q(a,z)=0$. If $a'\leq \sup B\leq a\leq z$ then $q(a',z)=z-a'\geq z-a=q(a,z)$.

Case $(x,y)=(a,b)\in A\times B$: Suppose $(x,y)=(a,b)$ is in position (1). If (b,z) is in position (5) or (6), then all distances are Euclidean. If (b,z) is in position (7), then $b<z\leq \inf A\leq a\leq \sup B$, and $q(a,b)=a-b\geq a-z=q(a,z)$. If (b,z) is in position (8), either (i) $b\leq \inf A<z\leq a$ or (ii) $b\leq \inf A\leq a\leq z$. In case (i), $q(a,b)=a-b\geq a-z=q(a,z)$. In case (ii), $q(b,z)=z-\inf A\geq z-a=q(a,z)$.

Observe that $(x,y)=(a,b)\in A\times B$ cannot be in position (2), since this would imply $\sup B<b\in B$.

Suppose $(x,y)=(a,b)$ is in position (3), so $\sup B=b<a$. Suppose (b,z) is in position (5), so $\inf A<b=\sup B<a$ and $q(b,z)$ is the Euclidean distance. Either (i) $z\leq b<a$ and $q(a,z)=\sup B-z=b-z=q(b,z)$, (ii) $b<z<a$ and $q(a,z)=0$, or (iii) $b<a<z$ and $q(a,z)=z-a\leq z-b=q(b,z)$. Suppose (b,z) is in position (6) so $z<b=\sup B\leq \inf A\leq a$. Then $q(a,z)=b-z=q(b,z)$. Suppose (b,z) is in position (7) so $b=\sup B<z\leq \inf A\leq a$ and thus $q(a,z)=0$. Suppose (b,z) is in position (8) so either (i) $\sup B=b\leq \inf A<z<a$ or (ii) $\sup B=b\leq \inf A\leq a<z$. In case (i), $q(a,z)=0$ and in case (ii) $q(a,z)=z-a\leq y-\inf A=q(b,z)$.

Suppose $(x,y)=(a,b)$ is in position (4) so $b<\sup B\leq a$. These three points give four possible positions for z. If $z<b<\sup B\leq a$, then $q(a,b)=\sup B-b$ and $q(b,z)=b-z$, determined by either (5) or (6). Thus, $q(a,z)=\sup B-z=q(a,b)+q(b,z)$.

If $b < z < \sup B \leq a$, then $q(a,z) = \sup B - z \leq \sup B - b = q(b,z)$. If $b < \sup B \leq z < a$, then $q(a,z) = 0$. If $b < \sup B \leq a < z$, then (b,z) cannot be in positions (6) or (7). If (b,z) is in position (5), then $q(a,z) = z - a \leq z - b = q(b,z)$. If (b,z) is in position (8), $q(a,z) = z - a \leq z - \inf A = q(b,z)$.

The cases $\boldsymbol{(x,y) = (b,b') \in B^2}$ and $\boldsymbol{(x,y) = (b,a) \in B \times A}$ are dual. □

5. Variations

5.1. Bounded neighborhoods: $L^*(A)$

A different topology arises if we replace $\varepsilon > 0$ in the definition of the subbasis $\mathcal{S}$ for $L(A)$ by $\varepsilon \in (0,1]$. Let $L^*(A)$ be the topology on $\mathbb{R}$ having a subbasis

$$\mathcal{S}^* = \{(-\infty, a+\varepsilon) : a \in A, \varepsilon \in (0,1]\} \cup \{(b-\varepsilon, \infty) : b \in \mathbb{R} - A, \varepsilon \in (0,1]\}.$$

For example, if $A = \{0\}$, $(0.9, 1.2) = (-\infty, 0+1.2) \cap (1-0.1, \infty)$ is a $L(A)$-neighborhood of 1 (using $\varepsilon = 1.2$) which is not a $L^*(A)$-neighborhood of 1.

With $A = (-\infty, 0) \cup \{2\}$, for every $a \in A$ ($b \in B$) there exists $b \in B$ ($a \in A$) with $a < b$, so $L(A)$ is the Euclidean topology. Now consider the topology $L^*(A)$. For $a \leq -1$, a has a neighborhood base $(-\infty, a+\varepsilon)$ for $\varepsilon \in (0,1]$. For $x \in (-1,1) \cup [2,3)$, x has a Euclidean neighborhood. For $b \in [1,2)$, b has a neighborhood base $(b-\varepsilon, 2+\varepsilon)$ for $\varepsilon \in (0,1]$. For $b \geq 3$, b has a neighborhood base $(b-\varepsilon, \infty)$ for $\varepsilon \in (0,1]$.

Theorem 5.1. *The following are equivalent.*

(a) $L^*(A)$ *is Euclidean.*

(b) $\forall a \in A, (a, a+1) \cap B \neq \emptyset$ *and* $\forall b \in B, (b-1, b) \cap A \neq \emptyset$.

(c) *For every* $x \in \mathbb{R}$, $[x, x+1) \cap B \neq \emptyset$ *and* $(x-1, x] \cap A \neq \emptyset$.

Proof. (a) $\iff$ (b): If (b) holds, for any $a \in A$, there exists $b \in (a, a+1)$. For all $\varepsilon \in (0, a+1-b)$, $(a-\varepsilon, a+\varepsilon)$ is a neighborhood of a, so a has a Euclidean neighborhood. The dual argument shows that every $b \in B$ has a Euclidean neighborhood. Suppose (b) fails. Suppose there exists $a \in A$ with $[a, a+1) \subseteq A$. Thus, there is no $b \in B$ such that $a \in (b-\varepsilon, \infty)$ for $\varepsilon \in (0,1)$, so every neighborhood of a has form $(-\infty, a+\varepsilon)$, and $L^*(A)$ is not Euclidean. The dual argument covers the case of $(b-1, b] \subseteq B$.

(b) $\iff$ (c): Suppose (c) fails. Then there exists $x \in \mathbb{R}$ such that (i) $[x, x+1) \cap B = \emptyset$ or (ii) $(x-1, x] \cap A = \emptyset$. If $x \in A$, (i) contradicts (b) and if $x \in B$, (ii) contradicts (b). Conversely, suppose (b) fails. Then either there exists $x = a \in A$ with $[a, a-1) \cap B = \emptyset$ or there exists $x = b \in B$ with $(b-1, b] \cap A = \emptyset$, which shows that (c) fails. □

If B and S are subsets of $\mathbb{R}$, let B_S represent B restricted to S. That is, $B_S = B \cap S$. Define A_S analogously. The neighborhoods of a point in $L^*(A)$ are described below.

$x = a \in A$ or $x = b \in B$	form of the neighborhood base at x ($\varepsilon > 0$)
$x = a, B_{(a,a+1)} = \emptyset$	$(\sup B_{(-\infty,a)} - \varepsilon, a+\varepsilon')$, $\varepsilon, \varepsilon' \in (0,1]$
$x = a, B_{(a,a+1)} \neq \emptyset$	$(a-\varepsilon, a+\varepsilon')$, $\varepsilon \in (0, a+1-\inf B_{(a,a+1)}]$ $\varepsilon' \in (0,1]$
$x = b, A_{(b-1,b)} = \emptyset$	$(b-\varepsilon', b+\inf A_{(b,\infty)} + \varepsilon)$, $\varepsilon, \varepsilon' \in (0,1]$
$x = b, A_{(b-1,b)} \neq \emptyset$	$(b-\varepsilon', \sup A_{(b-1,b)} + \varepsilon)$, $\varepsilon, \varepsilon' \in (0,1]$ $= (b-\varepsilon', b+\varepsilon)$, $\varepsilon \in (0, \sup A_{(b-1,b)} + 1 - b]$, $\varepsilon' \in (0,1]$

5.2. Hybrids of Closed Left-Ray Right-Ray Topologies

If $\leq$ is the usual order on $\mathbb{R}$, in the specialization topology, the smallest neighborhood of x is $N(x) = \uparrow x = [x, \infty)$. This is the closed right-ray topology. The closed left-ray topology is defined dually. Note that the closed right-ray topology is finer than the (open) right ray topology, since $(a,\infty) = \bigcup_{x>a}[x,\infty)$. Let $L^c(A)$ be the topology generated by the subbase

$$\mathcal{S} = \{(-\infty, a] : a \in A\} \cup \{[b, \infty) : b \in B = \mathbb{R} - A\}.$$

A base of $L^c(A)$ neighborhoods of $a \in A$ is $\{(-\infty, a]\} \cup \{[b, a], b \in B, b < a\}$, and a base of $L^c(A)$ neighborhoods of $b \in B$ is $\{[b, \infty)\} \cup \{[b, a]], a \in A, b < a\}$.

This is modeled if each point can emit a particle in one direction only, and may block oppositely directed particles. The points of A on the real line are the points which emit particles to the left and may absorb particles arriving from the right.

Theorem 5.2. (a) *$L^c(A)$ is never Euclidean. $L^c(A)$ is finer than the Euclidean topology if and only if A and B are both order dense in $\mathbb{R}$ (i.e., dense in the Euclidean topology).*

(b) *For any $A \subseteq \mathbb{R}$, $L^c(A)$ is T_0.*

(c) *$L^c(A)$ is T_1 if and only if A and B are both order dense.*

(d) *$L^c(A)$ is connected if and only if there exists no pair $(a, b) \in A \times B$ with $a < b$. That is, if and only if A is a ray to the right (including $\emptyset$ and $\mathbb{R}$).*

(e) *$L^c(A)$ is never compact.*

Proof. (a) $L^c(A)$ must contain an open set of form $[b, \infty)$ or $(-\infty, a]$, which is not Euclidean open.

Suppose A and B are order dense. Given $x \in \mathbb{R}$ and $\varepsilon > 0$, there exist $a \in A, b \in B$ with $x - \varepsilon < b < x < a < x + \varepsilon$. This shows every Euclidean neighborhood $(x - \varepsilon, x + \varepsilon)$ of x contains a $L^c(A)$ neighborhood $[b, a]$ of x. Suppose every Euclidean neighborhood $(x-\varepsilon, x+\varepsilon)$ of x contains a $L^c(A)$ neighborhood U of x. As a bounded $L^c(A)$-neighborhood, U has form $[b, a]$ for some $b \in B, a \in A$. Thus, $\forall x \in \mathbb{R}, \forall \varepsilon > 0$, there exists $a \in A, b \in B$ with $x - \varepsilon < b \leq x \leq a < x + \varepsilon$, which shows A and B are order dense in $\mathbb{R}$.

(b) Suppose $x < y$ in $\mathbb{R}$. If $x \in A$, then $(-\infty, x]$ is a neighborhood of x excluding y. If $y \in B$, then $[y, \infty)$ separates y from x. If $x \in B, y \in A$, consider $z \in (x, y) = (b, a)$. If $z \in A$, then $(-\infty, z]$ is a neighborhood of $x = b$ which excludes $y = a$. If $z \in B$, then $[z, \infty)$ separates a from b.

(c) Note that B fails to be order dense if and only if A contains an interval of positive length. If A contains an interval $[a, a']$ with $a < a'$, then every neighborhood of a' includes a, so $L^c(A)$ is not T_1. Dually, if A is not order dense, then $L^c(A)$ is not T_1. If A and B are both order dense, (a) shows that $L^c(A)$ is finer than the Euclidean topology and thus is T_2.

(d) Suppose A is a ray to the right. If $A = \emptyset$ or $A = \mathbb{R}$, then there are no two disjoint open sets, so $L^c(A)$ cannot be disconnected. If $A = (b, \infty)$, then every open set contains b. If $A = [a, \infty)$, then every open set contains a. Thus, $L^c(A)$ has no separation. Conversely, suppose there exists $a \in A, b \in B$ with $a < b$. Let $a_+ = \limsup A \cap [a, b)$ and $b_- = \liminf B \cap (a, b]$. Now either $a_+ > a$ or $b_- < b$. The cases are dual, so suppose $a_+ > a$. Then there exists a strictly increasing sequence (a_n) in $[a, a_+) \subseteq A$ converging to a_+. If $a_+ \in B$, then $(-\infty, a_+) = \bigcup\{(-\infty, a_n] : n \in \mathbb{N}\}$ and $[a_+, \infty)$ forms a separation of $L^c(A)$. If $a_+ \in A$, then $a_+ < b$ and there exists a strictly decreasing sequence (b_n) in B converging to a_+, so $(-\infty, a_+]$ and $(a_+, \infty) = \bigcup\{[b_n, \infty) : n \in \mathbb{N}\}$ forms a separation of $L^c(A)$.

(e) If $\sup A = \infty$, then $\{(-\infty, a] : a \in A\}$ is an open cover with no finite subcover. If $\sup A = m < \infty$, then $(m, \infty) \subseteq B$. If $m \in A$, then $\{(-\infty, m]\} \cup \{[b, \infty) : b > m\}$ is an open cover with no finite subcover. If $m \in B$, then there exists a strictly increasing sequence (a_n) in A converging to m, and $\{(-\infty, a_n] : n \in \mathbb{N}\} \cup \{[m, \infty)\}$ is an open cover with no finite subcover. □

6. Hybrids of the Euclidean Topology and the Right-Ray Topology

Consider $P(A)$ having subbase

$$\mathcal{S} = \{(a-\varepsilon, a+\varepsilon) : a \in A, \varepsilon > 0\} \cup \{(b-\varepsilon, \infty) : b \in B, \varepsilon > 0\}.$$

Note that $\mathcal{S}$ is not a basis: If $A = \{0, 2\}$ and $\varepsilon = 1.5$, $(0-\varepsilon, 0+\varepsilon) \cap (2-\varepsilon, 2+\varepsilon) = (.5, 1.5)$ contains no element of $\mathcal{S}$.

$x = a \in A$ or $x = b \in B$	form of the neighborhood base at x ($\varepsilon > 0$)
a	$(a-\varepsilon, a+\varepsilon)$
$\inf A < b$	$(b-\varepsilon, b+\varepsilon)$
$b \leq \inf A$	$(b-\varepsilon, \inf A + \varepsilon)$

Table 2. Neighborhoods of points in $P(A)$.

Example 6.1. A model for $P(A)$. For a real numbers i, consider the set $X = \{(i, y) \in \mathbb{R}^2 : y \geq 0\} \cup \{(x, 0) \in \mathbb{R}^2 : i \leq x\}$ as suggested in Figure 3. Think of the x-axis as being on the ground and the y-axis extending vertically above the ground. It takes energy to travel on the ground (horizontally) and upward, but no added energy to travel downward. This suggests a quasi-metric on X defined by taking the ε-ball around $(r, s) \in X$ to be the set of points accessible from (r, s) if you have enough energy to move a distance of ε units.

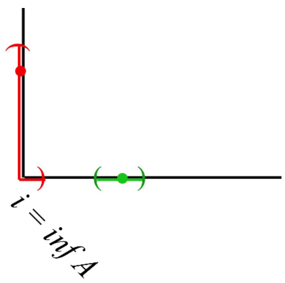

Figure 3. $P(A)$ models distances required to move overland or upward.

Comparing Figures 3 and 2 suggest that the results on $L(A)$ should carry over to $P(A)$ with minor modifications, essentially assuming $\infty \in B$ in the statements of results but not in the defintion of the basis for the topology. In this manner, we see that the results of Theorem 2.2 hold for $P(A)$, interpreting $\sup B = \infty$ in the statements.

Theorem 4.1 is also easily adapted to this situation to give a quasi-metric q for $P(A)$. There is only one case for $q(a, y)$ when $a \in A$, so we may simplify the statement as below. The proof remains valid.

Theorem 6.2. *For $a \in A$, $b \in B = \mathbb{R} - A$, and $y \in \mathbb{R}$, let*

$$q(a, y) = |a - y| \quad \text{if } y \in \mathbb{R} \qquad (1)$$

$$q(b, y) = \begin{cases} |b - y| & \text{if } \inf A < b & (5) \\ b - y & \text{if } y < b \leq \inf A & (6) \\ 0 & \text{if } b < y \leq \inf A & (7) \\ y - \inf A & \text{if } b \leq \inf A < y. & (8) \end{cases}$$

Then $q(x, y)$ is a quasi-metric which generates the topology $P(A)$.

The characterization of compact sets given in Theorem 2.4 does not carry over directly from $L(A)$ to $P(A)$. Due to the lack of symmetry, a "dual argument" given in the proof there is no longer valid. Below is a characterization of compact sets in $P(A)$.

Theorem 6.3. *Suppose $A, B \neq \emptyset$. Then a non-empty set $S \subseteq \mathbb{R}$ is not $P(A)$-compact if and only if*

(a) *S does not have a smallest element, or*
(b) $\inf A \leq \min S$ *and S is not Euclidean closed and bounded, or*
(c) $\min S < \inf A$ *and there exists a sequence (s_n) in $S \cap (2\inf A - \min S, \infty)$ converging (in the Euclidean topology) to $x \in (2\inf A - \min S, \infty] - S$.*

Proof. If S does not have a minimum element $\min S \in S$, let (s_n) be a strictly decreasing sequence in S with $\bigcup\{(s_n, \infty) : n \in \mathbb{N}\} = S$. Now $\{(s_n, \infty) : n \in \mathbb{N}\}$ is an open cover of S with no finite subcover.

If $\inf A \leq \min S$, then S inherits the Euclidean topology from $P(A)$, so S is $P(A)$ compact if and only if S is Euclidean closed and bounded.

If $\min S < \inf A$ and there exists a sequence (s_n) in $S \cap (2\inf A - \min S, \infty)$ converging to $x \in (2\inf A - \min S, \infty) - S$, without loss of generality, we may assume (s_n) is strictly monotone.

If (s_n) is strictly decreasing converging to $x \geq 2\inf A - \min S$, we consider two cases: (i) $\inf A \in A$ or (ii) $\inf A \notin A$. In case (i), set $\varepsilon = x - \inf A$ and note that $C_0 = (\inf A - \varepsilon, \inf A + \varepsilon) = (2\inf A - x, x)$ and $x > 2\inf A - \min S$ implies C_0 covers $[\min S, x)$. Now $\{C_0\} \cup \{(s_n, \infty) : n \in \mathbb{N}\}$ is an open cover of S with no finite subcover. In case (ii), let $D = \inf A - \min S$, so $\inf A + D = 2\inf A - \min S$. Now $\delta = x - (2\inf A - \min S) = x - \inf A + D > 0$. Let (a_n) be a strictly decreasing sequence in $A \cap (\inf A, \inf A + \delta)$ converging to $\inf A$. With $\varepsilon = x - a_1$, $(a_1 - \varepsilon, a_1 + \varepsilon) = (a_1 - \varepsilon, x)$ is open. Since $x = \inf A + D + \delta$ and $a_1 < \inf A + \delta$, it follows that $\varepsilon = x - a_1 > D$. Say $\varepsilon = D + \beta$. For $a_n \in A \cap (\inf A, \inf A + \beta)$, we have $a_n - \varepsilon < \inf A - D = \min S < a_n + D < a_1 + D < x$. Thus, $C_0 = (a_n - \varepsilon, a_n + \varepsilon) \cup (a_1 - \varepsilon, a_1 + \varepsilon) = (a_n - \varepsilon, x)$ covers $[\min S, x)$. Now $\{C_0\} \cup \{(s_n, \infty) : n \in \mathbb{N}\}$ is an open cover of S with no finite subcover.

If (s_n) is strictly increasing and $\inf A \in A$, then with $\varepsilon_n = s_n - \inf A$, $\{(\inf A - \varepsilon_n, \inf A + \varepsilon_n) = (\inf A - \varepsilon_n, s_n) : n \in \mathbb{N}\} \cup \{(x, \infty)\}$ is an open cover of S with no finite subcover. If $\inf A \notin A$, pick (a_n) as in the previous paragraph and let $\varepsilon_n = s_n - a_n$. Then $\{(a_n - \varepsilon_n, a_n + \varepsilon_n) = (a_n - \varepsilon_n, s_n) : n \in \mathbb{N}\} \cup \{(x, \infty)\}$ is an open cover of S with no finite subcover.

To show the converse, we must show that S is compact if

(a′) $\min S$ exists, and
(b1′) $\min S < \inf A$ or (b2′) S is Euclidean compact, and
(c1′) $\inf A \leq \min S$ or (c2′) $\min S < \inf A$ and there is no sequence in $S \cap (2\inf A - \min S, \infty)$ converging to $x \in (2\inf A - \min S, \infty] - S$.

Note that (a′, b1′, c1′) cannot occur. If (b2′) occurs, since every $P(A)$-open set is Euclidean open, every $P(A)$-open cover of S has a finite subcover, so S is $P(A)$-compact. In the remaining case (a′, b1′, c2′), suppose $\mathcal{C}$ is an open cover of S by $P(A)$-basic open sets. For $C \in \mathcal{C}$ with $\min S \in C$, if C contains $[\min S, \infty)$, then $\{C\}$ is a finite subcover of S. Otherwise, $C = (\min S - \varepsilon', \infty) \cap (a - \varepsilon, a + \varepsilon)$ for some $\varepsilon', \varepsilon > 0$ and $a \in A$. Since $\min S \in (a - \varepsilon, a + \varepsilon)$, we have $\varepsilon > \inf A - \min S$, so $a + \varepsilon > \inf A + (\inf A - \min S)$ and C covers $[\min S, 2\inf A - \min S + \delta]$ for some $\delta > 0$. Now it only remains to show that $S' = S \cap [2\inf A - \min S + \delta, \infty)$ has a finite subcover from $\mathcal{C}$. The condition (c2′) implies that S' is Euclidean closed and bounded, and since $2\inf A - \min S > \inf A$, $P(A)$ restricted to S' is the Euclidean topology, so S' and thus S are $P(A)$-compact. □

Acknowledgment. The authors would like to thank the referees, whose comments have improved the paper.

The second author acknowledges support from the Brno University of Technology (BUT) under the project MeMoV II no. CZ.02.2.69/0.0/0.0/18-053/0016962.

References

[1] A. Bouziad and E. Sukhacheva, *On Hattori spaces*, Comment. Math. Univ. Carolin. **58** (2), 213–223, 2017.

[2] A. Calderón-Villalobos and I. Sánches, *Hattori topologies on almost topological groups*, Topology Appl. **326**, 108411, 15pp., 2023.

[3] V. A. Chatyrko and Y. Hattori, *A poset of topologies on the set of real numbers*, Comment. Math. Univ. Carolin. **54** (2), 189–196, 2013.

[4] V. Chatyrko and A. Karassev, *Countable dense homogeneity and Hattori Spaces*, Q&A in Gen. Top. **39**, 73–87, 2021.

[5] P. Fletcher and W. F. Lindgren, *Quasi-uniformities with a transitive base*, Pacific J. Math. **43** (3), 619–631, 1972.

[6] R. Fox, *A short proof of the Junnila quasi-metrization theorem*, Proc. AMS **83** (3), 663–664, 1981.

[7] Y. Hattori, *Order and topological structures of posets of the formal balls on metric spaces*, Mem. Fac. Sci. Eng. Shimane Univ. Series B: Mathematical Science **43**, 13–26, 2010.

[8] H. H. Hung, *Quasi-metrizability*, Topology and its Applications **83**, 39–43, 1998.

[9] T. Khmyleva and E. Sukhacheva, *On linear hoemomorphisms of spaces of continuous functions on Hattori spaces*, Topology Appl. **281**, 107209, 8pp., 2020.

[10] R. D. Kopperman, *Which topologies are quasimetrizable?*, Topology and its Applications **52**, 99–107, 1993.

[11] H.-P. A. Künzi and F. Yildiz, *Extensions of T_0-quasi-metrics*, Acta Math. Hungar. **153** (1), 196–215, 2017.

[12] T. Richmond, *Hybrid topologies on the real line*, Applied General Topology **24** (1) 157–168, 2023.

[13] T. Richmond, *General Topology: An Introduction*, De Gruyter, 2020.

[14] L. A. Steen and J. A. Seeback, Jr., *Counterexamples in Topology*, (2nd edition) Springer-Verlag, 1970.