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TWO POINT FUZZY BOUNDARY VALUE PROBLEM WITH EXTENSION PRINCIPLE USING HEAVISIDE FUNCTION

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Abstract. In this paper we deal with the fuzzy eigenfunctions of the two point fuzzy boundary value problem (FBVP) with fuzzy coefficient of the boundary conditions. The fuzzy solution is obtained from the Zadeh's extension principle using the Heaviside function. The eigenvalues and the fuzzy eigenfunctions of the boundary value problem are found using the Wronskian functions. We present an example in order to compare the proposed solution.

1. Introduction

$$\widehat{u}'' + \lambda \widehat{u} = 0, \quad t \in [a, b] \tag{1.1}$$

which satisfy the conditions

$$\widehat{a}_1 \widehat{u}(a) = \widehat{a}_2 \widehat{u}'(a) \tag{1.2}$$

$$\widehat{b}_1 \widehat{u}(b) = \widehat{b}_2 \widehat{u}'(b) \tag{1.3}$$

where $\widehat{a}_1, \widehat{a}_2, \widehat{b}_1, \widehat{b}_2$ nonnegative triangular fuzzy numbers, $\lambda > 0$, at least one of the numbers $\widehat{a}_1$ and $\widehat{a}_2$ and at least one of the numbers $\widehat{b}_1$ and $\widehat{b}_2$ are nonzero.

The topic of fuzzy differential equations (FDEs) has been rapidly growing in recent years. There were many suggestions to define the fuzzy derivative concept. The concept of the fuzzy derivative was first introduced by Chang and Zadeh [1], it was developed by Puri and Ralescu [2], and generalized and extended the concept of Hukuhara differentiability [3]. But in some cases the fuzzy solutions with Hukuhara derivative suffers certain disadvantages since the diameter of the solutions is unbounded as time increases [4]. To solve the disadvantage, Bede and Gal [4] introduced a generalized defination of fuzzy derivative which was name Strongly generalized differentiability. One can provide a fuzzy solution to the problem by extending the classical solution, via Zadeh's extension principle [5, 6].

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So far, several published results are proposed to investigate the solution of two-point fuzzy boundary value problems (FBVPs). Liu [7] investigated the solutions of two-point (FBVPs) using monotone function under the lateral type of H-derivative. Gultekin Citil [8] studied fuzzy eigenvalue of the (FBVPs) under the approach of generalized differentiability. Ceylan and Altınısık [9] examined fuzzy eigenfunction of the fuzzy problem via generalized differentiability.

In this paper we investigate the fuzzy eigenfunction of two-point (FBVPs) by using Zadeh's extension principle with help the Heaviside function. This principle allows us to find the solution of the fuzzy problem without considering the signs of the first and second derivatives of the solution and the sign of the solution itself [11].

2. Basic concepts of fuzzy sets

Before we give the solution method, let us introduce the notation which we will be used throughout the paper. We place a bar $\widehat{}$ over a letter to denote a fuzzy number. We also write, $\widehat{u}(t)$ for fuzzy valued functions defined on real numbers. The $\alpha-$level representation of fuzzy-valued function is given by $[\widehat{u}_\lambda(t)]^\alpha = [u_\alpha^-(t,\lambda), u_\alpha^+(t,\lambda)]$, for all $x \in [a,b]$ *and* $\alpha \in [0,1]$.

Definition 2.1. ([2]) Let E be a universal set. A fuzzy subset $\widehat{A}$ of E is given by its membership function $\mu_{\widehat{A}} : E \to [0,1]$, where $\mu_{\widehat{A}}(t)$ represents the degree to which $t \in E$ belongs to $\widehat{A}$. We denote the class of the fuzzy subsets of E by the sembol $F(E)$.

Definition 2.2. ([12]) The $\alpha-$ level of a fuzzy set $\widehat{A} \subseteq E$, denoted by $\left[\widehat{A}\right]^\alpha$, is defined as $\left[\widehat{A}\right]^\alpha = \left\{x \in E : \widehat{A}(t) \geq \alpha\right\}$, $\forall \alpha \in (0,1]$. If E is also topological space, then the $0-$ level is defined as the closure of the support of $\widehat{A}$,that is, $\left[\widehat{A}\right]^0 = \overline{\left\{x \in E : \widehat{A}(t) > 0\right\}}$. The $1-$ level of a fuzzy subset $\widehat{A}$ is also called as core of $\widehat{A}$ and denoted by $\left[\widehat{A}\right]^1 = core\left(\widehat{A}\right)$

Definition 2.3. ([13]) A fuzzy subset $\widehat{u}$ on $\mathbb{R}$ is called a fuzzy real number (fuzzy interval), whose $\alpha-$ cut set is denoted by $[\widehat{u}]^\alpha$, i.e., $[\widehat{u}]^\alpha = \{x : \widehat{u}(t) \geq 0\}$, if it satisfies two axioms:

i. There exists $r \in \mathbb{R}$ such that $\widehat{u}(r) = 1$,

ii. For all $0 < \alpha \leq 1$, there exist real numbers $-\infty < u_\alpha^- \leq u_\alpha^+ < +\infty$ such that $[\widehat{u}]^\alpha$ is equal to the closed interval $[u_\alpha^-, u_\alpha^+]$.

Definition 2.4. ([14]) An arbitrary fuzzy number $\widehat{u}$ in the parametric form is represented by an ordered pair of functions $[u_\alpha^-, u_\alpha^+]$, $0 \leq \alpha \leq 1$, which satisfy the following requirements

i. u_α^- is bounded non-decreasing left continuous function on $(0,1]$ and right-continuous for $\alpha = 0$,

ii. u_α^+ is bounded non- increasing left continuous function on $(0,1]$ and right-continuous for $\alpha = 0$,

iii. $u_\alpha^- \leq u_\alpha^+$, $0 < \alpha \leq 1$.

Definition 2.5. ([14]) For $\widehat{u}, \widehat{v} \in \mathbb{R}_F$, and $\lambda \in \mathbb{R}$, the sum $\widehat{u} \oplus \widehat{v}$ and the product $\lambda \odot \widehat{u}$ are defined for all $\alpha \in [0,1]$,

$$\begin{aligned} [\widehat{u} \oplus \widehat{v}]^\alpha &= [\widehat{u}]^\alpha + [\widehat{v}]^\alpha = \{x + y : x \in [\widehat{u}]^\alpha, y \in [\widehat{v}]^\alpha\}, \\ [\lambda \odot \widehat{u}]^\alpha &= \lambda \odot [\widehat{u}]^\alpha = \{\lambda x : x \in [\widehat{u}]^\alpha\}. \end{aligned}$$

Definition 2.6. ([15]) A fuzzy number $\widehat{A}$ is said to be triangular if the parametric representation of its $\alpha-$ level is of the form $\left[\widehat{A}\right]^\alpha = [(m - a_\alpha^-)\,\alpha + a_\alpha^-, (m - a_\alpha^+)\,\alpha + a_\alpha^+]$, for all $\alpha \in [0,1]$, where $\left[\widehat{A}\right]^0 = [a_\alpha^-, a_\alpha^+]$ and $core\left(\widehat{A}\right) = m$. A triangular fuzzy number is denoted by the triple $(a_\alpha^-; m; a_\alpha^+)$.

The Zadeh's extension principle is a mathematical method to extend classical functions to deal with fuzzy sets as input arguments [1]. For multiple fuzzy variables as arguments, the Zadeh's extension principle is defined as follows.

Definition 2.7. ([16]) Let $f : X_1 \times X_2 \to Z$ a classical function and let $\widehat{A}_i \in F(X_i)$, for $i = 1, 2$. The Zadeh's extension $\widehat{f}$ of f, applied to $(\widehat{A}_1, \widehat{A}_2)$, is the fuzzy set $\widehat{f}$ $(\widehat{A}_1, \widehat{A}_2)$ of Z, whose membership function is defined by

$$\widehat{f}(\widehat{A}_1, \widehat{A}_2)(z) = \begin{cases} \sup\limits_{(x_1,x_2)\in f^{-1}(z)} min\{\widehat{A}_1(x_1), \widehat{A}_2(x_2)\}, & \text{if } f^{-1}(z) \neq \varnothing, \\ 0 & , \text{if } f^{-1}(z) = \varnothing \end{cases}$$

where $f^{-1}(z) = \{(x_1, x_2) \in X_1 \times X_2 : f(x_1, x_2) = z\}$.

We can apply the Zadeh's extension principle to define the standard arithmetic operations for fuzzy numbers [1] . Let $[\widehat{u}]^\alpha = [u_\alpha^-, u_\alpha^+]$ and $[\widehat{v}]^\alpha = [v_\alpha^-, v_\alpha^+]$. For all $\alpha \in [0,1]$ and $\alpha \in \mathbb{R}$, we have

$$\begin{aligned} [\widehat{u} \oplus \widehat{v}]^\alpha &= [\widehat{u}]^\alpha + [\widehat{v}]^\alpha = [u_\alpha^- + v_\alpha^-, u_\alpha^+ + v_\alpha^+] \\ [\widehat{u} - \widehat{v}]^\alpha &= [\widehat{u}]^\alpha - [\widehat{v}]^\alpha = [u_\alpha^- - v_\alpha^+, u_\alpha^+ + v_\alpha^-] \\ [\lambda \odot \widehat{u}]^\alpha &= \lambda \odot [\widehat{u}]^\alpha = \begin{cases} [\lambda u_\alpha^-, \lambda u_\alpha^+]\,, \text{ if } \lambda \geq 0, \\ [\lambda u_\alpha^+, \lambda u_\alpha^-]\,, \text{ if } \lambda < 0. \end{cases} \end{aligned}$$

Definition 2.8. ([11]) The function

$$\theta\,(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

is called the Heaviside step function.

3. Solution Method of the FBVP

In this section, we investigate the eigenvalues and the fuzzy eigenfunctions of the problem (1.1)-(1.3). Then we shall make use of solutions of (1.1) satisfying the fuzzy initial conditions instead of fuzzy boundary conditions in a manner similar to Titchmarsh [17].

We define two fuzzy solutions $\widehat{\Phi}_\lambda\,(t)$ and $\widehat{\Psi}_\lambda\,(t)$ of the equation (1.1). Let $\widehat{\Phi}_\lambda\,(t) = \widehat{\Phi}\,(t, \lambda)$ be the solution of equation (1.1) which satisfies the initial conditions

$$\begin{pmatrix} u\,(a) \\ u'\,(a) \end{pmatrix} = \begin{pmatrix} \widehat{a}_2 \\ \widehat{a}_1 \end{pmatrix} \tag{3.1}$$

and $\widehat{\Psi}_\lambda(t) = \widehat{\Psi}(t,\lambda)$ be the solution of equation (1.1) , which satisfies the initial conditions

$$\begin{pmatrix} u(b) \\ u'(b) \end{pmatrix} = \begin{pmatrix} \widehat{b}_2 \\ \widehat{b}_1 \end{pmatrix}. \tag{3.2}$$

Let us consider the following linear and homogeneous differential equation with (3.1) and (3.2) fuzzy initial conditions,

$$\begin{cases} \Phi'' + \lambda\Phi = 0 \\ \Phi(a) = \widehat{a}_2, \ \Phi'(a) = \widehat{a}_1 \end{cases} \tag{3.3}$$

and

$$\begin{cases} \Psi'' + \lambda\Psi = 0 \\ \Psi(b) = \widehat{b}_2, \ \Psi'(b) = \widehat{b}_1. \end{cases} \tag{3.4}$$

where $\widehat{a}_1, \widehat{a}_2, \widehat{b}_1, \widehat{b}_2$ triangular fuzzy numbers, $\lambda > 0$, at least one of the numbers $\widehat{a}_1$ and $\widehat{a}_2$ and at least one of the numbers $\widehat{b}_1$ and $\widehat{b}_2$ are nonzero..

First, let's search the solution of the problem in Eq. (3.3) with the help of the algorithm created by Akin et al. [11]. Then we find a solution for the problem (3.4) by doing similar operations. We will firstly solve the following crisp initial value problem related to the fuzzy initial value problem in Eq. (3.3) and then apply Zadeh's Extension Principle to the solution [10]:

$$\begin{cases} \Phi'' + \lambda\Phi = 0 \\ \Phi(a) = a_2, \ \Phi'(a) = a_1 \end{cases} \tag{3.5}$$

where a_1, a_2 and λ are real numbers. The general solution of the differential equation (3.5) can be written as:

$$\Phi_\lambda(t) = C_1\Phi_1(t) + C_2\Phi_2(t), \tag{3.6}$$

where C_1 and C_2 are arbitrary constants; $\Phi_1(t)$ and $\Phi_1(t)$ are linearly independent functions satisfying the Eq. (3.5).

Let us next obtain the solution of the crisp initial value problem (3.5) given by Eq. (3.6). Therefore, we obtain the following system of equations:

$$\begin{cases} C_1\Phi_1(a) + C_2\Phi_2(a) = a_2 \\ C_1\Phi_1'(a) + C_2\Phi_2'(a) = a_1 \end{cases} \tag{3.7}$$

In Eq. (3.7) C_1 and C_2 unknowns. From now on, we use the following notations for the sake of shortness.

$$\begin{aligned} W &= \begin{pmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{pmatrix}; \\ w_{11} &= \Phi_1(a), \ w_{12} = \Phi_2(a), \ w_{21} = \Phi_1'(a), \ w_{22} = \Phi_2'(a); \\ \overrightarrow{C} &= \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}, \ \overrightarrow{a} = \begin{pmatrix} a_2 \\ a_1 \end{pmatrix}. \end{aligned}$$

According to these notations, we write (3.7) in the matrix form:

$$W\overrightarrow{C} = \overrightarrow{a}.$$

Using Cramer's method, we obtain C_1 and C_2 as follows:

$$C_J = \frac{|W_1|}{|W|} - \frac{|W_2|}{|W|}.$$

Here

$$\begin{aligned} |W| &= \begin{vmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{vmatrix} = w_{11}w_{22} - w_{21}w_{12}, \\ |W_1| &= \begin{vmatrix} a_2 & w_{12} \\ a_1 & w_{22} \end{vmatrix} = a_2w_{22} - a_1w_{12}, \\ |W_2| &= \begin{vmatrix} w_{11} & a_2 \\ w_{21} & a_1 \end{vmatrix} = a_1w_{11} - a_2w_{21}. \end{aligned}$$

Thus, C_1 and C_2 can be rewritten as

$$\begin{aligned} C_1 &= \frac{|W_1|}{|W|} = \frac{a_2w_{22} - a_1w_{12}}{|W|}, \\ C_2 &= \frac{|W_2|}{|W|} = \frac{a_1w_{11} - a_2w_{21}}{|W|}. \end{aligned}$$

To simplify the results above, C_1 and C_2 can be rewritten in the following form, respectively

$$\begin{aligned} C_1 &= a_2f_{22} - a_1f_{12}, \\ C_2 &= a_1f_{11} - a_2f_{21} \end{aligned}$$

where $f_{ij} = \frac{w_{ij}}{|W|}; i, j = 1, 2$.

From the results for C_1 and C_2, we can now derive the classical solution of the given crisp initial value problem as follows:

$$\begin{aligned} \Phi_\lambda(t) &= C_1\Phi_1(t) + C_2\Phi_2(t), \\ &= (a_2f_{22} - a_1f_{12})\,\Phi_1(t) + (a_1f_{11} - a_2f_{21})\,\Phi_2(t). \end{aligned}$$

This solution can also be written as:

$$\Phi_\lambda(t) = a_2\left(f_{22}\Phi_1(t) - f_{21}\Phi_2(t)\right) + a_1\left(f_{11}\Phi_2(t) - f_{12}\Phi_1(t)\right).$$

Next we use the following notations for the sake of its comprehension:

$$\begin{aligned} K_{11}(t) &= f_{22}\Phi_1(t) - f_{21}\Phi_2(t), \\ K_{12}(t) &= f_{11}\Phi_2(t) - f_{12}\Phi_1(t). \end{aligned} \tag{3.8}$$

Thus the solution of the crisp initial value problem (3.5) can be written as:

$$\Phi_\lambda(t) = a_2K_{11}(t) + a_1K_{12}(t). \tag{3.9}$$

It can be easily seen that the solution in Eq. (3.9) is linearly dependent only on the initial values. Now, we apply Zadeh's Extension Principle and write the solution of the fuzzy initial value problem as follows:

$$\widehat{\Phi}_\lambda(t) = \widehat{a}_2K_{11}(t) + \widehat{a}_1K_{12}(t) \tag{3.10}$$

where $\alpha-$levels ($\alpha \in (0;1]$) of $\widehat{a}_i$ and $\widehat{\Phi}_\lambda(t)$ are defined as follows:

$$\begin{aligned} [\widehat{a}_i]^\alpha &= \left[(a_i)_\alpha^-, (a_i)_\alpha^+\right]; \\ \left[\widehat{\Phi}_\lambda(t)\right]^\alpha &= \left[\Phi_\alpha^-(t,\lambda), \Phi_\alpha^+(t,\lambda)\right]. \end{aligned}$$

Here $(a_i)_\alpha^-$, and $\Phi_\alpha^-(t,\lambda)$ are lower bounds and $(a_i)_\alpha^+$, and $\Phi_\alpha^+(t,\lambda)$ are the upper bounds of $\alpha-$levels, respectively.

By taking these $\alpha-$levels into account in the solution (3.10), we obtain the following result:

$$\left[\Phi_{\alpha}^{-}(t,\lambda),\Phi_{\alpha}^{+}(t,\lambda)\right]=\sum_{i=1}^{2}\left[(a_i)_{\alpha}^{-},(a_i)_{\alpha}^{+}\right]K_{1i}(t)$$

where for $i=1,2$

$$\begin{aligned}\Phi_{\alpha}^{-}(t,\lambda) &= \sum_{i=1}^{2}\min\left\{\left[(a_i)_{\alpha}^{-},(a_i)_{\alpha}^{+}\right]K_{1i}(t)\right\}\\ \Phi_{\alpha}^{+}(t,\lambda) &= \sum_{i=1}^{2}\max\left\{\left[(a_i)_{\alpha}^{-},(a_i)_{\alpha}^{+}\right]K_{1i}(t)\right\}.\end{aligned}$$

Here the min and max are evaluated for each $x\geq 0$ and $\alpha\in(0;1]$. Using the Heaviside function, we can write the $\alpha-$levels of the solution of $\widehat{\Phi}_{\lambda}(t)$ as follows:

$$\Phi_{\alpha}^{-}(t,\lambda)=\sum_{i=1}^{2}\left[(a_i)_{\alpha}^{+}-\left((a_i)_{\alpha}^{+}-(a_i)_{\alpha}^{-}\right)\phi\left(K_i(t)\right)\right]K_i(t) \tag{3.11}$$

and

$$\Phi_{\alpha}^{+}(t,\lambda)=\sum_{i=1}^{2}\left[(a_i)_{\alpha}^{-}+\left((a_i)_{\alpha}^{+}-(a_i)_{\alpha}^{-}\right)\phi\left(K_i(t)\right)\right]K_i(t). \tag{3.12}$$

For $\left[\widehat{\Psi}_{\lambda}(t)\right]^{\alpha}$, we find a solution for the problem (3.4) by doing similar operations. So the solution of the crisp initial value problem $\Psi_{\lambda}(t)$ can be written as:

$$\Psi_{\lambda}(t)=b_2K_{21}(t)+b_1K_{22}(t). \tag{3.13}$$

Then we apply Zadeh's Extension Principle and write the solution of the fuzzy initial value problem as follows:

$$\widehat{\Psi}_{\lambda}(t)=\widehat{a}_2K_{21}(t)+\widehat{a}_1K_{22}(t). \tag{3.14}$$

By taking $\alpha-$levels into account in the solution (3.4) and using the Heaviside function, we can write the $\alpha-$levels of the solution of $\widehat{\Psi}_{\lambda}(t)$ as follows:

$$\Psi_{\alpha}^{-}(t,\lambda)=\sum_{i=1}^{2}\left[(b_i)_{\alpha}^{+}-\left((b_i)_{\alpha}^{+}-(b_i)_{\alpha}^{-}\right)\phi\left(K_{2i}(t)\right)\right]K_{2i}(t) \tag{3.15}$$

and

$$\Psi_{\alpha}^{+}(t,\lambda)=\sum_{i=1}^{2}\left[(b_i)_{\alpha}^{-}+\left((b_i)_{\alpha}^{+}-(b_i)_{\alpha}^{-}\right)\phi\left(K_{2i}(t)\right)\right]K_{2i}(t). \tag{3.16}$$

Since the eigenvalues of the boundary value problem (1.1)-(1.3) if and only if are consist of the zeros of function $W(\Phi,\Psi)(t,\lambda)$ in [8], we find Wronskian function from the classical solutions (3.9) and (3.13) as follows:

$$W(\Phi,\Psi)(t,\lambda)=\Phi_{\lambda}(t)\Psi_{\lambda}'(t)-\Phi_{\lambda}'(t)\Psi_{\lambda}(t). \tag{3.17}$$

The analytical forms of the $\alpha-$levels for the solutions of fuzzy initial value problems depends on the behavior of $\Phi_{\lambda}(t)$, $\Psi_{\lambda}(t)$ and derivatives of $\Phi_{\lambda}(t)$, $\Psi_{\lambda}(t)$. This new formulation gives us the $\alpha-$levels of the solution of the fuzzy initial value problem in Eq. (3.3) and (3.4) without considering the signs of the first and second derivatives of the solution and the sign of the solution itself [11].

Now let us consider the following numerical example.

Example 3.1. Consider the two point fuzzy boundary value problem

$$u'' + \lambda u = 0 \tag{3.18}$$

$$\widehat{2}u(0) = \widehat{1}u'(0) \tag{3.19}$$

$$\widehat{4}u(1) = \widehat{3}u'(1) \tag{3.20}$$

where $\widehat{1} = (0,1,2)$, $\widehat{2} = (1,2,3)$, $\widehat{3} = (2,3,4)$, $\widehat{4} = (3,4,5)$ triangular fuzzy numbers and $\lambda = p^2$, $p > 0$.

From problem (3.18)-(3.20) , we get two FIVPs as follows

$$\Phi'' + p^2\Phi = 0, \quad \Phi(0) = \widehat{1}, \quad \Phi'(0) = \widehat{2} \tag{3.21}$$

and

$$\Psi'' + p^2\Psi = 0, \quad \Psi(1) = \widehat{3}, \quad \Psi'(1) = \widehat{4}. \tag{3.22}$$

Let us first solve the crisp initial value problem:

$$\Phi'' + p^2\Phi = 0, \quad \Phi(0) = 1, \quad \Phi'(0) = 2.$$

By solving the differential equation in the crisp initial value problem, we obtain the general crisp solution as:

$$\Phi(t,\lambda) = C_1 \cos(pt) + C_2 \sin(pt).$$

The functions $K_{11}(t)$ and $K_{12}(t)$ are obtained as follows:

$$\begin{aligned} K_{11}(t) &= \cos(pt) \\ K_{12}(t) &= \frac{1}{p}\sin(pt). \end{aligned} \tag{3.23}$$

Thus the solution of the crisp initial value problem can be written as:

$$\begin{aligned} \Phi(t,\lambda) &= a_2 K_{11}(t) + a_1 K_{12}(t) \\ &= \frac{1}{p}\sin(pt) + 2\cos(pt) \end{aligned} \tag{3.24}$$

Similarly, we can write the solution $\Psi(t,\lambda)$ as:

$$\Psi(t,\lambda) = \frac{3}{p}\sin(pt - p) + 4\cos(pt - p). \tag{3.25}$$

Then, we can get Wronskian functions from Eq. (3.17) as:

$$W(\lambda) = W(\Phi,\Psi)(t,\lambda) = \left(3p + \frac{8}{p}\right)\sin(p) + (-2)\cos(p).$$

Since the clasic eigenvalues of the problem (3.18)-(3.20) if and only if are consist of the zeros of functions $W(\lambda)$, computing the values p with Matlab Program we can get infinitely many eigenvalues satisfying the equation $W(\lambda) = 0$ in Table 1.

For the purposes of this example we found the first five numerically and then we will use the approximation of the remaining eigenvalues.

	p_n	λ_n
$n=1$	3.3024	10.9058
$n=2$	6.3809	40.7158
$n=3$	9.4929	90.1151
$n=4$	12.6183	159.2215
$n=5$	15.7498	248.0562
$n \approx$	$n\pi$	$(n\pi)^2$

Table 1. Eigenvalues of the fuzzy problem

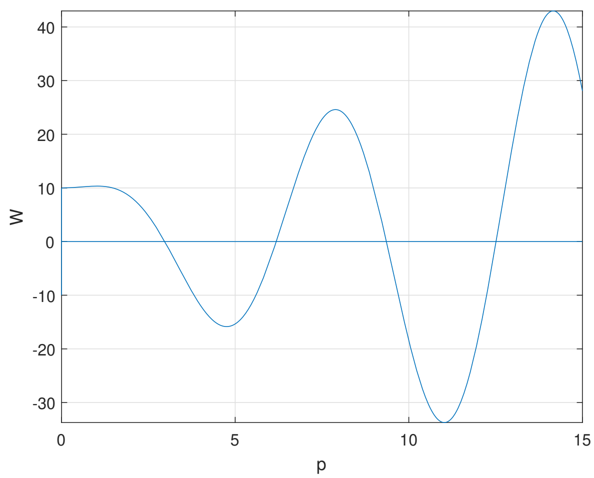

FIGURE 1. The function $W(\lambda) = \left(3p + \frac{8}{p}\right)\sin(p) + (4-6)\cos(p)$

According to (3.23), the $\alpha-$levels of the solution $\Phi(t,\lambda)$ can be found as follows:

(3.26)
$$\Phi_\alpha^-(t,\lambda) = \left[3-\alpha-2(1-\alpha)\theta(K_{11}(t))\right]K_{11}(t)+\left[2-\alpha-2(1-\alpha)\theta(K_{12}(t))\right]K_{12}(t)$$

and

(3.27)
$$\Phi_\alpha^+(t,\lambda) = \left[\alpha+1+2(1-\alpha)\theta(K_{11}(t))\right]K_{11}(t)+\left[\alpha+2(1-\alpha)\theta(K_{12}(t))\right]K_{12}(t).$$

Similarly we can get the $\alpha-$levels of the solution $\Psi(t,\lambda)$ as follows:

(3.28)
$$\Psi_\alpha^-(t,\lambda) = \left[5-\alpha-2(1-\alpha)\theta(K_{11}(t))\right]K_{11}(t)+\left[4-\alpha-2(1-\alpha)\theta(K_{12}(t))\right]K_{12}(t)$$

and

(3.29)
$$\Psi_\alpha^+(t,\lambda) = \left[\alpha+3+2(1-\alpha)\theta(K_{11}(t))\right]K_{11}(t)+\left[\alpha+2+2(1-\alpha)\theta(K_{12}(t))\right]K_{12}(t).$$

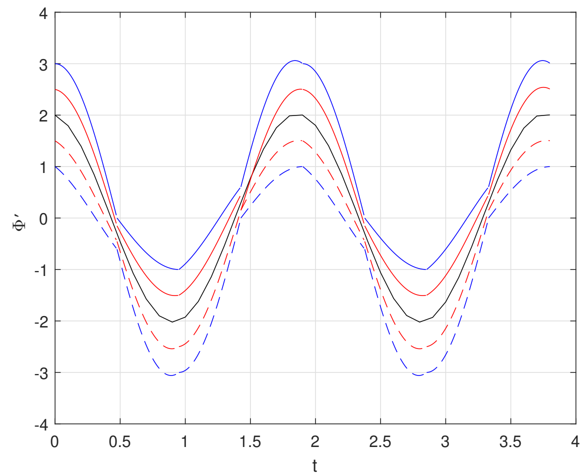

FIGURE 2. The fuzzy solution $\widehat{\Phi}_{\lambda_1}(x)$ for Example 1. The black line represents the crisp solution. The blue and red lines represent upper solution for $\alpha = 0$ and $\alpha = 0.5$, respectively and the dashed blue and red lines represent lower solution for $\alpha = 0$ and $\alpha = 0.5$, respectively

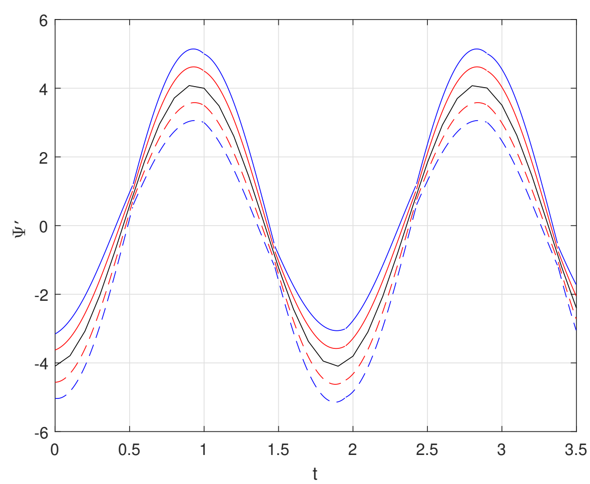

FIGURE 3. The fuzzy solution $\widehat{\Psi}_{\lambda_1}(t)$ for Example 1. The black line represents the crisp solution. The blue and red lines represent upper solution for $\alpha = 0$ and $\alpha = 0.5$, respectively and the dashed blue and red lines represent lower solution for $\alpha = 0$ and $\alpha = 0.5$, respectively

In particular, we select $p_2 = 3.3024$ in Table 1. The $\alpha-$levels of the solution for this example, obtained for $\left[\widehat{\Phi}\left(t, (3.3024)\right)\right]^{\alpha}$ by using Eq. (3.26) and Eq. (3.27) and for $\left[\widehat{\Psi}\left(t,(3.3024)\right)\right]^{\alpha}$ by using Eq. (3.28) and Eq. (3.29) different values of α, are given in Figure 2 and Figure 3

4. Conclusion

In this study we researched the fuzzy eigenfunctions of FBVP by using Zadeh extension principle with the Heaviside function. We solved an example. We examined the fuzzy eigenfunctions on graphs for a selected value of eigenvalues λ in Table 1. Using the Zadeh extension principle we got the $\alpha-$levels of the fuzzy solutions without considering the signs of the first and second derivatives of the solution and the sign of the solution itself.

This new formulation gives us the $\alpha-$levels of the solution of the fuzzy initial value problem in Eq. (3.13) and (3.14) without considering the signs of the first and second derivatives of the solution and the sign of the solution itself.

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The Declaration of Research and Publication Ethics
The authors declared that they comply with the scientific, ethical, and citation rules of Journal of Universal Mathematics in all processes of the study and that they do not make any falsification on the data collected. Besides, the authors declared that Journal of Universal Mathematics and its editorial board have no responsibility for any ethical violations that may be encountered and this study has not been evaluated in any academic publication environment other than Journal of Universal Mathematics.

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