Hacettepe Journal of Mathematics & Statistics

Hacet. J. Math. Stat.
Volume XX (x) (XXXX), 1 – 19
DOI : 10.15672/hujms.xx

Research Article

# Wiener index of local rings

Eman S. Almotairi*[1,2], Ahmad M. Alghamdi[1]

[1] *Mathematics Department, Faculty of Sciences, Umm Al-Qura University, Makkah, Saudi Arabia*
[2] *Department of Mathematics, College of Sciences, Qassim University, Buraydah, Saudi Arabia*

## Abstract

Let $R$ be a finite commutative ring with nonzero identity. Let $Z^*(R)$ be the set of nonzero zero-divisors of $R$. We are dealing with the zero-divisor graph of $R$ which is denoted by $\Gamma(R)$ with vertex set $Z^*(R)$, where two distinct vertices $x$ and $y$ are adjacent if and only if $xy = 0$. The motivation of this study is to compute Wiener index in algebraic graph theory for special type of graph called zero-divisor graph. Wiener index is defined as the sum of all distance between all pairs of vertices in $\Gamma(R)$. In addition, we generalize the Wiener index of the zero-divisor graph in $\mathbb{Z}_p[x]/(x^2)$ for any prime number $p$. We obtain our results and methods by tables and figures.

**Mathematics Subject Classification (2020).** 05C12, 05C25, 05C76, 13M99

**Keywords.** local Rings, zero-divisor graph, compressed zero-divisor graph, Wiener index

## 1. Introduction

When Arthur Cayley [9] introduced the Cayley graph for finite groups in 1878, the study of graphs in connection with algebraic structures began. Beck defined the zero-divisor graph of a finite commutative ring $R$, [7] in his definition in 1988. Anderson and Livingston [3] revised the definition of a zero-divisor graph in 1999 by restricting the vertices to the nonzero zero-divisors of the ring $R$. This graph is denoted by $\Gamma(R)$. In 2021, Anderson, Asir, Badawi, and Chelvam [2] discussed some properties of the zero-divisor graph of a commutative ring. Many of the graphs which are related to algebraic structures were defined as a result of the relationship between ring theory and graph theory.

In this paper, $R$ will denote a finite commutative ring with identity denoted by 1. The set $Z^*(R)$ will denote all of nonzero zero-divisors of $R$. The zero-divisor graph of $R$ whose vertex set is $Z^*(R)$, denoted by $\Gamma(R)$, is a simple graph, where two distinct vertices $x, y \in Z^*(R)$ are adjacent if and only if $xy = 0$. Topological indices are considered as one of the most important in mathematical chemistry. These invariants have a good connected by physical and chemical properties of the corresponding molecules. Moreover, there are two topological indices which are, distance-based and degree-based of graph. The topological index that is distance-based is called Wiener index of graph. The paper focuses on Wiener

*Corresponding Author.
Email addresses: 3581@qu.edu.sa (E. S. Almotairi), amghamdi@uqu.edu.sa (A. M. Alghamdi)
Received: 30.05.2023; Accepted: 21.09.2024

index of graphs which is one of the important graph indices. In 1947, Harold Wiener [29], is wrote an article with about three pages, introduced the concept of Wiener index, is traditionally denoted by $W(G)$. The name was "Structural Determination of Paraffin Boiling Points", which is related to molecular branching in which Wiener discovered that the sum of distances between all pairs of vertices in the molecular graph of an alkane have a close connection. The indices of the graph are useful to understand the properties of various chemical compounds and there is a lot of applications about Wiener matrix, was introduced for acyclic graphs in chemistry can be seen in [18]. In 1971, the formal definition of the Wiener index was introduced by Hosaya [14]. In a graph $G$ the Wiener index as the sum of the lengths of the shortest paths between any pair of vertices of $G$. Then in 1993, Randit̀c [25] was introduced the hyper-Wiener index, as a generalization of the Wiener index, is denoted by $WW(G)$ of acyclic graphs and extended to all connected graphs by Klein et al [19]. In 1994, Yeh and Gutman in [33] computed the Wiener index of graphs that are obtained by specific binary operations (like product, join, and composition) on pairs of graphs. In 2001, Dobrynin, Entringer and Gutman [10] introduced the Wiener index of trees, which are connected graphs without any cycles (acyclic graph), denoted by $T$ with sets of vertices $V(T)$ and set of edges $E(T)$ of a tree. A connected graph is called "tree" with $n$ vertices and $n-1$ edges which are $|V(T)| = n$ and $|E(T)| = n-1$. That paper displayed the results for $W$ of trees, processes for computation of $W$ and combinatorial expressions for several types of trees for $W$. Moreover, there are applications of $W$ in chemistry. The molecular of tree graphs is considered as the vast majority of the Wiener index of the chemical applications with acyclic molecules. Figure 1 displays the molecular graph for tree $T$. In 2022, a survey of results consisting of the Wiener index of graphs associated with commutative rings was provided by Asir et al [5].

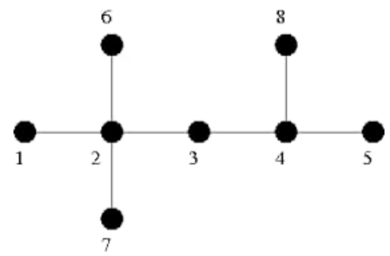



**Figure 1.** Acyclic molecular graph.

In this paper, we study Wiener index of the zero-divisor graph of finite commutative local rings $\Gamma(R)$. We denote this number by $W(\Gamma(R))$ and is defined as the sum of the length of the shortest path between all pairs of vertices in $\Gamma(R)$, i.e. $W(\Gamma(R)) = \sum_{(u,v)\subseteq V} d(u,v)$. If we denote $D_i = \sum_{u,v} d(u_i, v_j)$, then $W(\Gamma(R)) = \frac{1}{2}\sum_{i=1}^{n} D_i$.

We organize the paper as follows: Section 2 is a continuation of this introduction by more literature review. Then Section 3 contains a description of our methodology to tackle this problem. In Section 4, we review some basic notions of local rings. By Proposition 4.6, we find that any ring with identity which satisfies any condition in this proposition is called local ring. In Section 5, we give the definition of zero-divisor graph, compressed zero-divisor graph and Wiener index and some examples of them. In Section 6, we determine the implementation classification of finite commutative local rings with characteristic of the ring. In Section 7, we compute the Wiener index of zero-divisor graph for finite commutative local rings of order 8 and 27 and we generalize the Wiener index of $\Gamma(R)$ in $\mathbb{Z}_p[x]/(x^2)$, for any prime number. In Section 8, we first construct graphs of local rings and we calculate their Wiener index. Second, we focus on the Wiener index for the local matrix algebra.

## 2. Literature review

Many of the graphs which are related to algebraic structures were defined as a result of the relationship between ring theory and graph theory, such as the commuting graph and the zero-divisor graph [3, 7, 22]. In 1988, Beck introduced the zero-divisor graph of commutative ring in [7]. He let all of a commutative ring's elements to be used as the graph's vertices such that two distinct vertices $x$ and $y$ are joined by an edge if $xy = 0$. In 1999, the definition of vertex set was refined by Anderson and Livingston [3] by restricting the vertices to the nonzero zero-divisors of the ring. let $R$ be a commutative ring with identity. We denote by $Z(R)$ the set of all zero-divisors of $R$, and by $Z^*(R)$ the set of all nonzero zero-divisors of $R$. The zero-divisor graph, denoted by $\Gamma(R)$, is a simple graph with vertex set $Z^*(R)$ such that $x \neq y$, are adjacent if and only if $xy = 0$. In [3, 7, 28], the zero-divisor graphs have been studied extensively which investigated graph properties like diameter and girth. One of the most popular topological indices in chemistry for describing molecular form is the Wiener index. H. Wiener [29] used it for the first time in 1947. Hosaya [14] afterwards in 1971 provided the official description of Wiener index. The main aim of using Wiener index was to invent a lot of applications in quantitative structure relationship. The Wiener index of a graph $G$, denoted by $W(G)$, is defined as the total of the lengths of the shortest paths that connect all pairs of vertices in $G$. Graovac [12] in 1991, proposed a relationship between Wiener index and symmetry of a graph. In 1993, Randitc [25] introduced the hyper-Wiener index, as a generalization of the Wiener index, which is denoted by $WW(G)$ of acyclic graphs. In 1994, Yeh and Gutman in [33] calculated the Wiener index of graphs which are obtained by specific binary operations such as product, join, and composition on pairs of graphs. The Wiener index of trees was introduced by Dobrynin, Entringer and Gutman [10], in 2001, which are connected graphs having no cycles, denoted by $T$. In 2002, the Wiener index is used in crystallography, communication theory, facility locating, cryptography, and other fields (see [8]). In 2011, Ahmadi and Jahni-nezhad [1] introduced the concept of the Wiener index of zero-divisor graph of integers modulo n, for the case $n = p^2$ and the case $n = pq$. Other examples of the applications of the Wiener index can be found in [18, 31] and reference therein. In 2022, the Wiener index of the zero-divisor graph of a finite commutative ring with unity was introduced by Selvakumar, Gangaeswari and Arunkumar [27]. In this work, we investigate the Wiener index of the zero-divisor graph $\Gamma(R)$ of finite commutative local rings. More general analyses remain open to many of researchers.

## 3. Methodology

Graph theory research is a significantly active and vast topic of investigation. In fact, it is a field of study with a direct connection to discrete and pure mathematics.

Our methods and strategies in this study are standard. Our method is to apply the prior literature in this area to develop and investigate new issues. We focus on a certain type of graphs, which are known as zero-divisor graph and compressed zero-divisor graph.

Local ring and graph theory are very important tools for this work. Topological indices are one of the most significant graph indices. Specifically, Wiener index.

In classifying finite commutative local rings, we deal with finite local rings of order $p^n$. At the end, we provide tables that explain our methods and results.

## 4. Local rings

In this section, we study and explore the basic notions of local rings. Let $R$ be finite commutative ring with identity. Let $J(R)$ be the Jacobson radical, let $N(R)$ be the set of all nilpotent elements of $R$, let $U(R)$ be the set of all unit elements of $R$. Other undefined notation and terminology for local ring can be found in [6, 11, 16, 17].

**Definition 4.1.** Let $(R, +, \cdot)$ be a commutative ring with identity. A nonzero element $a \in R$ is called a zero-divisor if there exists nonzero element $b \in R$ such that $a \cdot b = 0$.

**Definition 4.2.** An element $a \in R$ is called a unit or an invertible element if a has a multiplicative inverse: $b \in R$ with $a \cdot b = b \cdot a = 1$.

**Lemma 4.3.** *[17] In a finite commutative ring $R$ with identity, every element of $R$ is either a unit or a zero-divisor.*

**Lemma 4.4.** *[11] Let $n \in \mathbb{Z}^+$. Then any nonzero element $a \in \mathbb{Z}_n$, the ring of integers modulo $n$, is a zero-divisor if and only if $gcd(a, n) \neq 1$.*

**Definition 4.5.** 1) The nilradical $N(R)$ of a ring $R$ is the intersection of all prime ideals of the ring.
2) The Jacobson radical $J(R)$ of a ring $R$ is the intersection of all maximal ideals of $R$.

The following proposition is a type of Fitting Lemma [20].

**Proposition 4.6.** *Let $(R, +, \cdot)$ be an associative ring with $1_R$. The following conditions are equivalent*

*a) $J(R)$ is the unique maximal right ideal in $R$;*
*b) The set of all non-unit elements of $R$ forms a proper ideal;*
*c) $J(R)$ is the set of all non-unit elements of $R$;*
*d) The factor ring $R/J(R)$ is a divison ring;*
*e) $R = J(R) \dot{\cup} U(R)$, where $R$ means the disjoint union of $J(R)$ and $U(R)$;*
*f) If $a \in R$, then either $a$ or $1 - a$ is unit;*
*g) $R$ contains exactly two idempotents namely $0_R$ and $1_R$.*

***Proof.*** (a) $\Rightarrow$ (b): Let $J(R)$ be the unique maximal right ideal of the ring $R$. Let $S$ be the set of all non-unit elements of $R$. We show that $S$ forms a proper ideal. Let $x \in S$. Since any proper right ideal of a ring $R$ with $1_R$ is contained in a maximal right ideal of the ring $R$. Then the right ideal $xR \neq R$ is contained in a maximal right ideal of the ring $R$. Hence, it is contained in $J(R)$. Therefore, $S \subseteq J(R)$, as $x$ is an arbitrary element in $S$.
If $a, b \in S$, then $a, b \in J(R)$, whence $a + b \in J(R)$. So $a + b$ is a non-unit element, that is, $a + b \in S$. If $a \in S$, $r \in R$, then $ar \in J(R)$ and $ra \in J(R)$. Hence, $ar, ra \in S$. Then, $S$ is a two-sided ideal of $R$.

(b) $\Rightarrow$ (c): Let J(R) be the unique maximal right ideal in $R$. Let $S$ be the set of all non-unit elements of $R$. We need to show that $J(R) = S$. Let $x \in J(R)$ (maximal ideal of $R$). Then $x$ is non-unit element. We know that an ideal of a ring $R$ is proper if and only if it has no unit. Hence, $x \in S$. Therefore, $J(R) \subseteq S$. By above argument, we get that $S \subseteq J(R)$. Hence, we obtain that the $J(R)$ is the set of all non-unit elements of the ring $R$.

(c) $\Rightarrow$ (d): We have $J(R)$ is the set of all non-unit elements of $R$. Then every element of $R$ which is contained in $J(R)$ is non-unit element. Hence, any element of the quotient ring $R/J(R)$ is unit element. Then, the quotient ring $R/J(R)$, is a division ring.

(d) $\Rightarrow$ (e): Let $R/J(R)$ be a division ring. We show that $R = J(R) \dot{\cup} U(R)$, where $R$ is the disjoint union of $J(R)$ and $U(R)$. It is clear that, $J(R) \dot{\cup} U(R) \subset R$. Conversely, let $x \in R$, we have two cases either $x \in J(R)$ or $x \notin J(R)$. If $x \in J(R)$, then we have the result. If $x \notin J(R)$, then every element of $R$ which is not contained in $J(R)$ is a unit. Hence, any element of the quotient ring $R/J(R)$ is a unit. Then the element $x$ has inverse in the ring $R$. Hence, $x \in U(R)$. So, $R \subset J(R) \dot{\cup} U(R)$. Then, we have $R = J(R) \dot{\cup} U(R)$. It is clear that $J(R) \dot{\cup} U(R)$ is the empty set and the union is disjoint.

(e) $\Rightarrow$ (f): We have $J(R)$ is the set of all non-unit elements. Let $a$ be a non-unit element of $R$. Then by (c), $a \in J(R)$. By Proposition [1.9,[6]], which says $x \in J(R) \Leftrightarrow$ $1 - x$ *is unit in* $R$. Then $1 - a$ is unit in $R$.

(f) $\Rightarrow$ (g): Let J(R) be the unique maximal ideal of $R$. Let $e$ be an arbitrary idempotent in the ring $R$. Then $e(1-e) = e-e^2 = 0 \in J(R)$. By hypothesis, the unique maximal ideal implies that $J(R)$ is prime ideal in $R$. Then $e \in J(R)$ or $1 - e \in J(R)$. We observe that $e$ and $1-e$ cannot both be elements of J(R) because this would imply $1 = e+(1-e) \in J(R)$, which is a contradiction. Then, if $e \in J(R)$, then $1 - e \notin J(R)$ and so $1 - e$ is a unit in $R$, then $e = 0$ in this case. Similarly, if $1 - e \in J(R)$, then $e$ is unit in $R$. Hence, $1 - e = 0$. Then, $e = 1$. Hence, $R$ contains exactly two idempotents namely $0_R$ and $1_R$.

(g) $\Rightarrow$ (a): Suppose that $J(R)$ is not unique maximal right ideal in $R$. Assume that $R$ is an Artinian ring. It follows that $R$ has many finitely maximal ideals. Let $M_1, M_2$ be two maximal ideals. Then $J(R) \subset M_1$ and $J(R) \subset M_2$. Let $x \in R$, $x \notin J(R)$ and $x \notin M$. Then we get chains of right ideals of $R$

$$R \neq xR \supseteq x^2R \supseteq x^3R \supseteq ... \neq 0.$$

and

$$\{a \in R : xa = 0\} \subseteq \{a \in R : x^2a = 0\} \subseteq \{a \in R : x^3a = 0\} \subseteq ...$$

As our assumption ($R$ is Artinian ring ), there is a positive integer $n$ such that

$$x^nR = x^{n+1}R = ... = x^{2n}R.$$

and

$$\{a \in R : x^na = 0\} = \{a \in R : x^{n+1}a = 0\} = ... = \{a \in R : x^{2n}a = 0\}.$$

In particular, there is an element $b \in R$ such that $x^n = x^{2n}b$. Then

$$x^{2n}b(1 - x^nb) = x^n(1 - x^nb) = x^n - x^{2n}b = 0.$$

which implies that $x^nb(1 - x^nb) = 0$, thus $(x^nb)^2 = x^nb$ and

$$x^nbR \subseteq x^nR = x^{2n}bR.$$

Since $rank(x^{2n}bR) \leq rank(x^nbR)$, we conclude that $x^nbR = x^nR \notin \{0, R\}$. In particular, $x^nb \notin \{0, 1\}$. Thus, $x^nb$ is a third idempotent in $R$ which is contradiction our assumption. Then $J(R)$ is the unique maximal right ideal in $R$.

□

**Definition 4.7.** Any ring with identity which satisfies any condition (and hence all) in the proposition above is called local ring. (See the book [16])

**Remark 4.8.** In the above item (d) we have:

i. If $R/J(R)$ is a commutative division ring, then $R/J(R)$ is a field.
ii. If $R/J(R)$ is a finite division ring, then $R/J(R)$ is a field according to (Wedderburns Theorem)[16].

**Corollary 4.9.** *If $R$ is a local ring, then every element in $R$ is either unit or nilpotent.*

**Corollary 4.10.** *If $R$ is a local ring, then $N(R) = J(R)$.*

***Proof.*** Let $0 \neq a \in J(R)$. Then $a$ is non-unit element. Since $R$ is a local ring, then $a$ is nilpotent element that is , $a \in N(R)$. Then $J(R) \subset N(R)$. It is clear that $N(R) \subset J(R)$ from definition the nilradical of a ring $R$ is the intersection of all of the prime ideals of the ring and define the Jacobson radical of a ring $R$ to be the intersection of all of the maximal ideals of $R$. Since all maximal ideals are prime ideals, we obtain immediately that nilradical $\subset$ Jacobson radical. Therefore, $N(R) = J(R)$. □

## 5. Graph theory

A graph theory is the study of the relationship between edges and vertices which deals with graphs. In this section, we introduce basic concepts of graph theory with properties of them. Other undefined notation and terminology for graph theory can be found in [21, 23, 30].

**Definition 5.1.** A simple graph $G$ is an ordered pair $(V, E)$, where $V$ is a nonempty set of vertices and $E$ is a nonempty set of edges. Any two distinct vertices $u, v$ in $G$ are called adjacent if $\{u, v\} \in E$. If $e = \{u, v\}$ is an edge of $G$, the vertex $u$ and the edge $e$(as well as $v$ and $e$ ) are said to be incident with each other.

**Definition 5.2.** A simple graph $G$ is called a complete graph if each pair of distinct vertices are adjacent. We denote the complete graph on $n$ vertices by $K_n$ some example of complete graphs are shown in Fig 2. This graph of order $n$ has $\frac{n(n-1)}{2}$ edges.

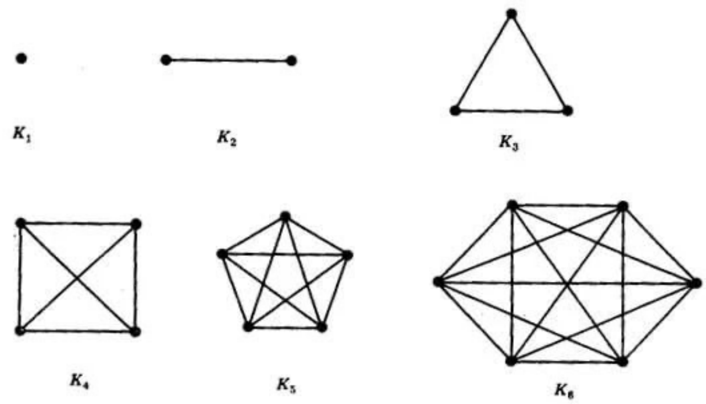


**Figure 2.** Complete graphs

**Definition 5.3.** A bipartite graph is a graph $G$ whose vertices can be partitioned into two disjoint sets $V_1$ and $V_2$ with $V = V_1 \cup V_2$, such that $uv \in E$ if $u \in V_1$ and $v \in V_2$. The bipartite graph is said to be complete in which vertex in $V_1$ is joined to every vertex in $V_2$. When $\mid V_1 \mid = n_1$ , $\mid V_2 \mid = n_2$, we denote the complete bipartite graph by $K_{n_1,n_2}$. For example, $K_{2,3}$ and $K_{3,3}$ are shown in Figure below.

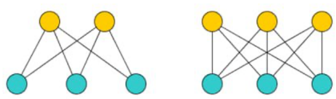


**Figure 3.** $K_{2,3}$ and $K_{3,3}$

**Definition 5.4.** A proper vertex coloring or proper coloring of a graph $G$ is the assignment of colors to the vertices such that all adjacent vertices have different colors. The smallest number of colors needed to get a proper vertex coloring is called the chromatic number of the graph, denoted $\chi(G)$.

**Example 5.5.** In Figure 3, $\chi(K_{2,3}) = 2$ and $\chi(K_{3,3}) = 2$.

**Definition 5.6.** In a graph $G = (V, E)$, a clique is defined as an induced subgraph of $G$ which is complete. The number of vertices in the largest clique of $G$ is called the clique number, denoted by $\omega(G)$.

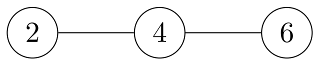
**Definition 5.7.** A path on $n$ vertices, denoted $P_n$, is a sequence of edges such that each consecutive pair has a vertex in common.

**Definition 5.8.** Let $R$ be a commutative ring with nonzero identity. A simple graph with vertex set being the set of nonzero zero-divisors of $R$ with $(x, y)$ an edge if $x \neq y$ and $xy = 0$ is called the zero-divisor graph of $R$, denoted by $\Gamma(R)$.

**Example 5.9.** (1) Let $R = \mathbb{Z}_4$, then $\Gamma(R)$ is a singleton graph.

(2) Let $R = \mathbb{Z}_2 \times \mathbb{Z}_2$, then $\Gamma(R)$ consists of two vertices which are connected by an edge.

(3) Let $R = \mathbb{Z}_8$ or $\mathbb{Z}_2(x)/(x^3)$ then $\Gamma(R)$ is a path of length 2 since $\mathbb{Z}_8 = \{0, 1, 2, 3, 4, 5, 6, 7\}$. We get $Z^*(\mathbb{Z}_8) = \{2, 4, 6\}$ is the vertex set since $2 \cdot_8 4 = 0$, $4 \cdot_8 6 = 0$, $\{2, 4\}$ and $\{4, 6\}$ are edges in $\Gamma(\mathbb{Z}_8)$. But, the set $\{2, 6\}$ is not an edge in $\Gamma(\mathbb{Z}_8)$ as $2 \cdot_8 6 \neq 0$ in $\mathbb{Z}_8$. The graph $\Gamma(\mathbb{Z}_8)$ is given below.




**Definition 5.10.** The compressed zero-divisor graph, denoted by $\Gamma_c(R)$, is the compression of a zero-divisor graph, with vertices the equivalence classes induced by $\sim$ other than $[0]$ and $[1]$, defined by

$$Z(R_c) = \{[r] \mid r \in R\}$$

where $[r] = \{s \in R \mid ann(r) = ann(s)\}$ and two distinct equivalence classes $[r]$ and $[s]$ are adjacent if and only if $rs = 0$.

**Definition 5.11.** The distance between two distinct vertices $u$ and $v$ of $\Gamma(R)$, denoted by $d(u, v)$, is the length of the shortest path from $u$ to $v$.
If there is no such path, then we write $d(u, v) = \infty$.

The following definition was introduced by Wiener [29].

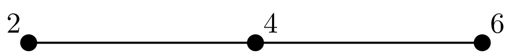
**Definition 5.12.** Let $\Gamma = (V, E)$ be a simple graph with vertex set $V(\Gamma(R))$. The Wiener index of $G$ is defined as

$$W(\Gamma(R)) = \sum_{\{u,v\} \subseteq V(\Gamma(R))} d(u, v),$$

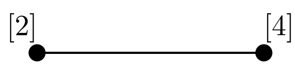
where $d(u, v)$ is the length of the shortest path between all pairs of vertices in $\Gamma(R)$.

Now, we will study the Wiener index of the some of local rings.

**Example 5.13.** (1) Consider $R = \mathbb{Z}_8 = \{0, 1, 2, 3, 4, 5, 6, 7\}$. Here $Z^*(R) = \{2, 4, 6\}$ is the vertex set of $\Gamma(\mathbb{Z}_8)$. The zero-divisor graph of $\mathbb{Z}_8$ is complete bipartite graph $K_{1,2}$, which is given below.




Now, we compute the Wiener index of $\Gamma(\mathbb{Z}_8)$. We obtain that $d(2, 4) = 1, d(2, 6) = 2, d(4, 6) = 1$, then, $W(\Gamma(\mathbb{Z}_8)) = d(2, 4) + d(2, 6) + d(4, 6) = 1 + 2 + 1 = 4$. For the vertex set of $\Gamma_c(R)$ we have, $ann(2) = \{4\}, ann(4) = \{2, 6\}, ann(6) = \{4\}$. Then $Z(R_c) = \{[2], [4]\}$ is the vertex set of $\Gamma_c(R)$.




(2) Let $R = \mathbb{Z}_{27} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, ..., 26\}$.
Here $Z^*(R) = \{3, 6, 9, 12, 15, 18, 21, 24\}$ be the vertex set of $\Gamma(R)$. For the vertex set of $\Gamma_c(R)$ we have, $ann(3) = \{9, 18\}, ann(6) = \{9, 18\}, ann(9) = \{3, 6, 9, 12, 15, 18, 21, 24\}$, $ann(12) = \{9, 18\}, ann(15) = \{9, 18\}, ann(18) = \{3, 6, 9, 12, 15, 18, 21, 24\}, ann(21) =$

$\{9, 18\}, ann(24) = \{9, 18\}$. Then $Z(R_c) = \{[3], [9]\}$ is the vertex set of $\Gamma_c(R)$ which is given below.

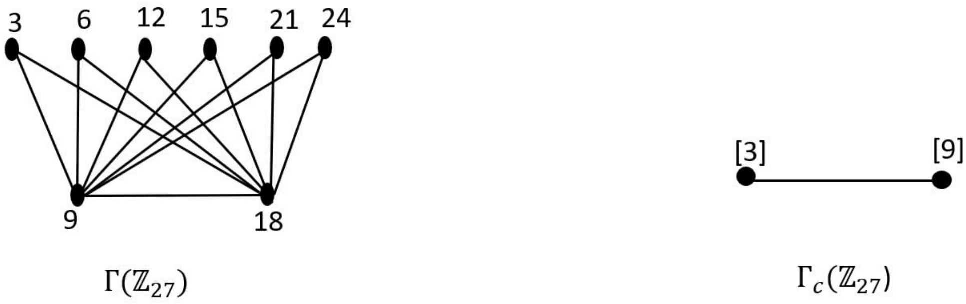
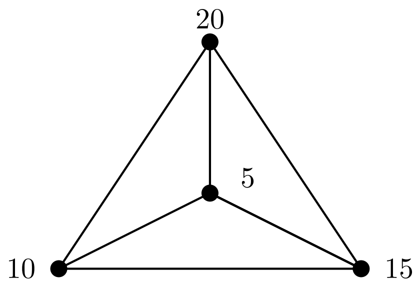



**Figure 4.** $\Gamma(\mathbb{Z}_{27})$ and $\Gamma_c(\mathbb{Z}_{27})$

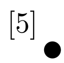
We obtain that $d(3, 6) = 2, d(3, 9) = 1, d(3, 12) = 2, d(3, 15) = 2, d(3, 18) = 1, d(3, 21) = 2, d(3, 24) = 2, d(6, 9) = 1, d(6, 12) = 2, d(6, 15) = 2, d(6, 18) = 1, d(6, 21) = 2, d(6, 24) = 2, d(9, 12) = 1, d(9, 15) = 1, d(9, 18) = 1, d(9, 21) = 1, d(9, 24) = 1, d(12, 15) = 2, d(12, 18) = 1, d(12, 21) = 2, d(12, 24) = 2, d(15, 18) = 1, d(15, 21) = 2, d(15, 24) = 2, d(18, 21) = 1, d(18, 24) = 1, d(21, 24) = 2$, then $W(\Gamma(\mathbb{Z}_{27})) = 43$.

**Remark 5.14.** For Figure 4, which is related to the zero-divisor graph $\Gamma(\mathbb{Z}_{27})$, we obtain the complete bipartite graph $K_{2,6}$ if we delete the specific edge between 9 and 18.

(3) Consider $R = \mathbb{Z}_{25} = \{0, 1, 2, 3, 4, 5, 6, ..., 24\}$. Here $Z^*(R) = \{5, 10, 15, 20\}$ be the vertex set of $\Gamma(R)$. The zero-divisor graph of $\mathbb{Z}_{25}$ is given below.




For the vertex set of $\Gamma_c(R)$ we have, $ann(5) = \{5, 10, 15, 20\}, ann(10) = \{5, 10, 15, 20\}, ann(15) = \{5, 10, 15, 20\}, ann(20) = \{5, 10, 15, 20\}$. Hence $Z(R_c) = \{[5]\}$ is the vertex set of $\Gamma_c(R)$, which is a singleton graph.




Since $\Gamma(\mathbb{Z}_{25})$ is the zero-divisor graph, then Wiener index is

$$W(\Gamma(\mathbb{Z}_{25})) = d(5, 10) + d(5, 15) + d(5, 20) + d(10, 15) + d(10, 20) + d(15, 20)$$
$$= 1 + 1 + 1 + 1 + 1 + 1 = 6.$$

## 6. Implementation of classification of finite commutative local rings

Every finite commutative ring is well known to be a product of finite commutative local rings. Moreover, we assume in this section that our finite commutative local rings have $p^n$ elements, for $p \in \{2, 3, 5, 7\}$ and $n \in \{1, 2, 3, 4, 5\}$.

Our main concern is to use the characteristic of the ring as the following definition.

**Definition 6.1.** [11] If for a ring $R$ a positive integer $n$ exists such that $n \cdot a = 0$ for all $a \in R$, then the least such positive integer is the characteristic of the ring $R$, denoted by $Ch(R)$. If no such positive integer exists, then $R$ is of characteristic 0.

**Remark 6.2.** The following computations are well known in the literature for finite commutative local rings. We have four cases for $|R| = p^4$ with $p \in \{2, 3, 5, 7\}$.

1. There are, up to isomorphism, exactly 7, 11, 2 and a unique finite commutative local rings of order $2^4$ with characteristic 2, $2^2$, $2^3$ and $2^4$, respectively. Hence, the overall of finite commutative local rings of order $2^4$ is exactly 21.
2. There are, up to isomorphism, exactly 7, 11, 3 and a unique finite commutative local rings of order $3^4$ with characteristic 3, $3^2$, $3^3$ and $3^4$, respectively. Then, the overall of finite commutative local rings of order $3^4$ is exactly 22.
3. There are, up to isomorphism, exactly 7, 11, 3 and a unique finite commutative local rings of order $5^4$ with characteristic 5, $5^2$, $5^3$ and $5^4$ respectively. Then, the overall of finite commutative local rings of order $5^4$ is exactly 22.

4. There are, up to isomorphism, exactly 7, 13, 3 and a unique finite commutative local rings of order $7^4$ with characteristic 7, $7^2$, $7^3$ and $7^4$, respectively. So, the overall of finite commutative local rings of order $7^4$ is exactly 24.

**Table 1.** Characteristic of finite commutative local rings of order $p^n$ $(n \leq 5)$

| | Char | | | | |
|---|---|---|---|---|---|
| Order | $p$ | $p^2$ | $p^3$ | $p^4$ | $p^5$ |
| $p$ | 1 | | | | |
| $p^2$ | 2 | 1 | | | |
| $p^3$ | 3 | 3<br>2 | 1 | | |
| $p^4$ | 7 | 11<br>13 | 2<br>3 | 1 | |
| $p^5$ | 12 | 27<br>38<br>40<br>44 | 13<br>7<br>12<br>14 | 2<br>8<br>3 | 1 |

## 7. Main Results

Now, we show that the zero-divisor graphs and compressed zero-divisor graphs for finite commutative local rings. Moreover, we will investigate the Wiener index for some finite commutative local rings of order $p^n$, where $p \in \{2, 3\}$ and $n = 3$.

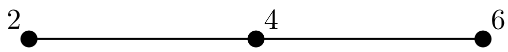
### 7.1. Wiener Index of $\Gamma(R)$ of finite commutative local rings of order $p^n$, for $p \in \{2, 3\}$ and $n = 3$

. Firstly, we compute Wiener index of zero-divisor graph for finite commutative local rings when $p = 2$ and $n = 3$ such that $|R| = 2^3$

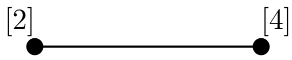
1) $R_1 = \mathbb{Z}_8$ with characteristic 8.
2) $R_2 = \mathbb{Z}_2[x]/(x^3)$ with characteristic 2.

3) $R_3 = \mathbb{Z}_2[x,y]/(x^2, xy, y^2)$ with characteristic 2.
4) $R_4 = \mathbb{Z}_4[x]/(2x, x^2)$ with characteristic 4.
5) $R_5 = \mathbb{Z}_4[x]/(2x, x^2-2)$ with characteristic 4.

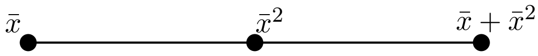
**Example 7.1.** Let $R_1 = \mathbb{Z}_8$ then $\Gamma(R)$ is a path of length 2 since $\mathbb{Z}_8 = \{0,1,2,3,4,5,6,7\}$. Then $Z^*(\mathbb{Z}_8) = \{2,4,6\}$ is the vertex set since $2 \cdot_8 4 = 0$, $4 \cdot_8 6 = 0$, $\{2,4\}$ and $\{4,6\}$ are edges in $\Gamma(\mathbb{Z}_8)$. But, the set, $\{2,6\}$ is not an edge in $\Gamma(\mathbb{Z}_8)$ as $2 \cdot_8 6 \neq 0$ in $\mathbb{Z}_8$. The graph $\Gamma(\mathbb{Z}_8)$ is given below.



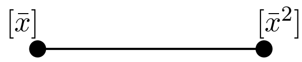

which is complete bipartite graph $K_{1,2}$. For the vertex set of $\Gamma_c(R)$ we have, $ann(2) = \{4\}, ann(4) = \{2,6\}, ann(6) = \{4\}$. Then $Z(R_c) = \{[2],[4]\}$ is the vertex set of $\Gamma_c(R)$, and we get that $W(\Gamma(\mathbb{Z}_8)) = 4$.



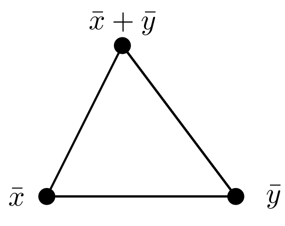

**Example 7.2.** Consider $R_2 = \mathbb{Z}_2[x]/(x^3) = \{f(x) + (x^3) \mid f(x) \in \mathbb{Z}_2[x]\} = \{ax^2 + bx + c + (x^3) \mid a,b,c \in \mathbb{Z}_2\} = \{\bar{0}, \bar{1}, \bar{x}, \bar{x}^2, \bar{1}+\bar{x}, 1+\bar{x}^2, \bar{x}+\bar{x}^2, \bar{1}+\bar{x}+\bar{x}^2\}$.
Now, we want to show that $R_2$ is local ring. We have, $J(R_2) = \{\bar{0}, \bar{x}, \bar{x}^2, \bar{x}+\bar{x}^2\}$. We know $x \in J(R) \Leftrightarrow 1-x$ *is unit in* $R$. Then $U(R_2) = \{\bar{1}, \bar{1}+\bar{x}, \bar{1}+\bar{x}^2, \bar{1}+\bar{x}+\bar{x}^2\}$, which is unit. Hence every element of $R_2$ is either unit or nilpotent. Hence $R_2$ is local ring. Therefore, $Z^*(R) = \{\bar{x}, \bar{x}^2, \bar{x}+\bar{x}^2\}$ is the vertex set of $\Gamma(R)$ and we obtain that $W(\Gamma(\mathbb{Z}_2[x]/(x^3))) = 4$.




For the vertex set of $\Gamma_c(R)$ we have, $ann(\bar{x}) = \{\bar{x}^2\}, ann(\bar{x}^2) = \{\bar{x}, \bar{x}+\bar{x}^2\}, ann(\bar{x}+\bar{x}^2) = \{\bar{x}^2\}$ . Then, $Z(R_c) = \{[\bar{x}], [\bar{x}^2]\}$ is the vertex set of $\Gamma_c(R)$, Therefore, $\Gamma_c(R)$ is a single edge.

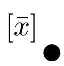



**Example 7.3.** Let $R_3 = \mathbb{Z}_2[x,y]/(x^2,xy,y^2) = \{\bar{0}, \bar{1}, \bar{x}, \bar{y}, \bar{1}+\bar{x}, \bar{1}+\bar{y}, \bar{x}+\bar{y}, \bar{1}+\bar{x}+\bar{y}\}$
Now, we want to show that $R_3$ is local ring. We have, $J(R_3) = \{\bar{0}, \bar{x}, \bar{y}, \bar{x}+\bar{y}\}$. We know $x \in J(R) \Leftrightarrow 1-x$ *is unit in* $R$. Then, $U(R_3) = \{\bar{1}, \bar{1}+\bar{x}, \bar{1}+\bar{y}, \bar{1}+\bar{x}+\bar{y}\}$. Hence, every element of $R_3$ is either unit or nilpotent. Therefore, $R_3$ is local ring. Then, $Z^*(R) = \{\bar{x}, \bar{y}, \bar{x}+\bar{y}\}$ is the vertex set of $\Gamma(R)$.



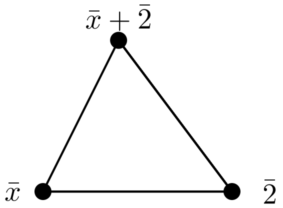

$$W(\Gamma(\mathbb{Z}_2[x,y]/(x^2,xy,y^2)) = d(\bar{x},\bar{y}) + d(\bar{x},\bar{x}+\bar{y}) + d(\bar{y},\bar{x}+\bar{y}) = 1+1+1 = 3.$$

For the vertex set of $\Gamma_c(R)$ we have, $ann(\bar{x}) = \{\bar{x}, \bar{y}, \bar{x}+\bar{y}\}, ann(\bar{y}) = \{\bar{x}, \bar{y}, \bar{x}+\bar{y}\}, ann(\bar{x}+\bar{y}) = \{\bar{x}, \bar{y}, \bar{x}+\bar{y}\}$. Then $Z(R_c) = \{[\bar{x}]]\}$ is the vertex set of $\Gamma_c(R)$ which is a singleton graph.




**Example 7.4.** Consider $R_4 = \mathbb{Z}_4[x]/(2x,x^2) = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{x}, \bar{x}+\bar{1}, \bar{x}+\bar{2}, \bar{x}+\bar{3}\}$. Firstly, we will show that $R_4$ is local ring. We have $J(R_4) = \{\bar{0}, \bar{2}, \bar{x}, \bar{x}+\bar{2}\}$. We know $x \in J(R) \Leftrightarrow 1-x$ *is unit in* $R$. Then, $U(R_4) = \{\bar{1}, \bar{3}, \bar{x}+\bar{1}, \bar{x}+\bar{3}\}$. Therefore, every element of $R_4$ is

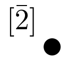
either unit or nilpotent. Hence $R_4$ is a local ring. Then, $Z^*(R) = \{\bar{2}, \bar{x}, \bar{x}+\bar{2}\}$ is the vertex set of $\Gamma(R)$.



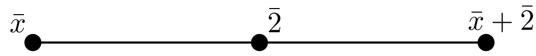

Secondly, we will show the Wiener index of $\Gamma(\mathbb{Z}_4[x]/(2x, x^2))$.

$$W(\Gamma(\mathbb{Z}_4[x]/(2x, x^2))) = d(\bar{x}, \bar{2}) + d(\bar{x}, \bar{x}+\bar{2}) + d(\bar{2}, \bar{x}+\bar{2}) = 1 + 1 + 1 = 3.$$

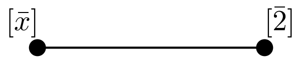
However, for the vertex set of $\Gamma_c(R)$ we have, $ann(\bar{2}) = \{\bar{2}, \bar{x}, \bar{x}+\bar{2}\}, ann(\bar{x}) = \{\bar{2}, \bar{x}, \bar{x}+\bar{2}\}, ann(\bar{x}+\bar{2}) = \{\bar{2}, \bar{x}, \bar{x}+\bar{2}\}$. Then $Z(R_c) = \{[\bar{2}]\}$ is the vertex set of $\Gamma_c(R)$ which is a singleton graph.




**Example 7.5.** Let $R_5 = \mathbb{Z}_4[x]/(2x, x^2-2) = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{x}, \bar{x}+\bar{1}, \bar{x}+\bar{2}, \bar{x}+\bar{3}\}$. Now, we will show that $R_5$ is local ring. We have, $J(R_5) = \{\bar{0}, \bar{2}, \bar{x}, \bar{x}+\bar{2}\}$. We know $x \in J(R) \Leftrightarrow 1 - x$ *is unit in R*. Then, $U(R_5) = \{\bar{1}, \bar{3}, \bar{x}+\bar{1}, \bar{x}+\bar{3}\}$. Therefore, then every element of $R_5$ is either unit or nilpotent. Hence $R_5$ is a local ring.Thus, $Z^*(R) = \{\bar{2}, \bar{x}, \bar{x}+\bar{2}\}$ is the vertex set of $\Gamma(R)$.




The Wiener index is 4. On the other hand, for the vertex set of $\Gamma_c(R)$ we have, $ann(\bar{2}) = \{\bar{2}, \bar{x}, \bar{x}+\bar{2}\}, ann(\bar{x}) = \{\bar{2}\}, ann(\bar{x}+\bar{2}) = \{\bar{2}\}$. Then $Z(R_c) = \{[\bar{2}], [\bar{x}]\}$ is the vertex set of $\Gamma_c(R)$. Therefore, $\Gamma_c(R)$ is a single edge.




**Remark 7.6.** We note that $\mathbb{Z}_8$ , $\mathbb{Z}_2[x]/(x^3)$ and $\mathbb{Z}_4[x]/(2x, x^2-2)$ have the same zero-divisor graphs and compressed zero-divisor graphs which are single edge and they have the same Wiener index. Also, $\mathbb{Z}_4[x]/(2x, x^2)$ and $\mathbb{Z}_2[x, y]/(x^2, xy, y^2)$ have the same zero-divisor graphs and compressed zero-divisor graphs which are singleton graph and they have also the same Wiener index.

**Table 2.** Classification of finite commutative local rings of order 8.

| $R$ | $\|R\|$ | $Ch(R)$ | $\|\Gamma(R)\|$ | $\Gamma(R)$ | $\Gamma_c(R)$ | $W(\Gamma(R))$ | $\chi(\Gamma(R))$ |
|---|---|---|---|---|---|---|---|
| $\mathbb{Z}_8$ | $2^3$ | $2^3$ | 3 | $K_{1,2}$ | $K_2$ | 4 | 2 |
| $\mathbb{Z}_2[x]/(x^3)$ | $2^3$ | 2 | 3 | $K_{1,2}$ | $K_2$ | 4 | 2 |
| $\mathbb{Z}_2[x, y]/(x^2, xy, y^2)$ | $2^3$ | 2 | 3 | $K_3$ | $K_1$ | 3 | 3 |
| $\mathbb{Z}_4[x]/(2x, x^2)$ | $2^3$ | $2^2$ | 3 | $K_3$ | $K_1$ | 3 | 3 |
| $\mathbb{Z}_4[x]/(2x, x^2-2)$ | $2^3$ | $2^2$ | 3 | $K_{1,2}$ | $K_2$ | 4 | 2 |

**Proposition 7.7.** *Let $R$ be a commutative local ring of order $p^3$. Then,* $|\Gamma(R)| = p^2 - 1$.

***Proof.*** Given $R$ is a commutative local ring, then every element of $R$ is either a unit or a zero-divisor. Since $|R| = p^3$, then, by multiplication table, we get that $|Z(R)| = p^2$. From Definition 5.8, the zero-divisor graph $\Gamma(R)$ has vertices as elements of $Z^*(R) = Z(R)\setminus\{0\}$, and two distinct vertices are joined by an edge if and only if $xy = 0$. Hence, $|Z^*(R)| = |\Gamma(R)| = p^2 - 1$. □

**Theorem 7.8.** *The zero-divisor graph of finite commutative local ring of order $p^3$ is either $\Gamma(R) = K_{p^2-1}$ or $\Gamma(R) = K_{p-1,p^2-p}$.*

***Proof.*** It is straightforward to check that the theorem holds if

$$R = \{\mathbb{Z}_8, \mathbb{Z}_2[x]/(x^3), \mathbb{Z}_2[x,y]/(x^2,xy,y^2), \mathbb{Z}_4[x]/(2x,x^2), \mathbb{Z}_4[x]/(2x,x^2-2)\}.$$

First, we notice that distinct vertices of $\Gamma(R)$ are adjacent if and only if $xy = 0$ for all $x, y \in V(\Gamma(R))$. Therefore, $\Gamma(R) = K_{p^2-1}$, which is complete graph. Next, we can split $V(\Gamma(R))$ into two different sets $V_1 = \{v_1\}$ and $V_2 = Z^*(R) - V_1$. Therefore, $|V_1| = p - 1$ and $|V_2| = p^2 - p$, which is complete bipartite graph $K_{p-1,p^2-p}$.

□

From the above theorem, we get the following result.

**Lemma 7.9.** *The compressed zero-divisor graph $\Gamma_c(R)$ of finite commutative local ring of order $p^3$ is either $\Gamma_c(R) = K_p$ or $K_{p-1}$.*

***Proof.*** According to the definition, the compressed zero-divisor graph $\Gamma_c(R)$ of $R$, is the graph where the vertices are the equivalence classes of the nonzero zero-divisor. Let $|\Gamma(R)| = 3$ and $x, y, x+y \in V(\Gamma(R))$. By Theorem 7.8, we get that $\Gamma(R)$ is the complete graph with 3 vertices, then $ann(x) = \{x, y, x+y\}$, $ann(y) = \{x, y, x+y\}$ and $ann(x+y) = \{x, y, x+y\}$. Hence, $ann(x) = ann(y) = ann(z)$, it follows that the vertex set of $\Gamma_c(R)$ is $\{[x]\}$. Therefore, the compressed zero-divisor graph is a singleton graph. Thus, $\Gamma_c(R) = K_{p-1}$. On the other hand, $\Gamma(R)$ is the complete bipartite graph with 3 vertices, then $ann(x) = \{y\}$, $ann(y) = \{x, x+y\}$ and $ann(x+y) = y$. Then $ann(x) = ann(x+y)$, it follows that $x \sim y$. Hence, $Z(R_c) = \{[x], [y]\}$ is the vertex set of $\Gamma_c(R)$. Then, the compressed zero-divisor graph is a single edge. Therefore, $\Gamma_c(R) = K_p$ □

**Remark 7.10.** If $R \cong \mathbb{Z}_8, \mathbb{Z}_2[x]/(x^3), \mathbb{Z}_2[x,y]/(x^2,xy,y^2), \mathbb{Z}_4[x]/(2x,x^2), \mathbb{Z}_4[x]/(2x,x^2-2)$, then $W(\Gamma(R))$ is either $p^{n-1}$ or $p^{n-1} - 1$.

. Secondly, we calculate Wiener index of zero-divisor graph for finite commutative local rings when $p = 3$ such that $|R| = 3^3$

1) $\mathbb{Z}_{27}$ with characteristic $3^3$
2) $\mathbb{Z}_3[x]/(x^3)$ with characteristic 3
3) $\mathbb{Z}_3[x,y]/(x^2,xy,y^2)$ with characteristic 3
4) $\mathbb{Z}_9[x]/(3x,x^2)$ with characteristic $3^2$
5) $\mathbb{Z}_9[x]/(3x,x^2-3)$ with characteristic $3^2$
6) $\mathbb{Z}_9[x]/(3x,x^2-6)$ with characteristic $3^2$

**Table 3.** Classification of finite commutative local rings of order 27.

| $R$ | $\|R\|$ | $Ch(R)$ | $\|\Gamma(R)\|$ | $\Gamma(R)$ | $\Gamma_c(R)$ | $W(\Gamma(R))$ | $\chi(\Gamma(R))$ |
|---|---|---|---|---|---|---|---|
| $\mathbb{Z}_{27}$ | $3^3$ | $3^3$ | 8 | Fig. 4 | $K_2$ | 43 | 3 |
| $\mathbb{Z}_3[x]/(x^3)$ | $3^3$ | 3 | 8 | Fig. 4 | $K_2$ | 43 | 3 |
| $\mathbb{Z}_3[x,y]/(x^2,xy,y^2)$ | $3^3$ | 3 | 8 | $K_8$ | $K_1$ | 28 | 8 |
| $\mathbb{Z}_9[x]/(3x,x^2)$ | $3^3$ | $3^2$ | 8 | $K_8$ | $K_1$ | 28 | 8 |
| $\mathbb{Z}_9[x]/(3x,x^2-3)$ | $3^3$ | $3^2$ | 8 | Fig. 4 | $K_2$ | 43 | 3 |
| $\mathbb{Z}_9[x]/(3x,x^2-6)$ | $3^3$ | $3^2$ | 8 | Fig. 4 | $K_2$ | 43 | 3 |

**Proposition 7.11.** *Let $R$ be a commutative local ring of order $p^3$. Then, $|\Gamma(R)| = p^2 - 1$.*

***Proof.*** The proof is the same as Proposition 7.7. □

**Theorem 7.12.** *The zero-divisor graph for finite commutative local ring of order $p^3$ is either $\Gamma(R) = K_{p^2-1}$ or Fig.4.*

***Proof.*** The proof follows from the above discussions. □

From the above theorem, we obtain the following Lemma.

**Lemma 7.13.** *Let $R$ be a commutative local ring of order $p^3$. Then, the compressed zero-divisor graph is either $\Gamma_c(R) = K_{p-2}$ or $K_{p-1}$.*

***Proof.*** The proof is the same as Lemma 7.9.

□

**Remark 7.14.** We observe that in Fig.4, which is related to the local rings $\mathbb{Z}_{27}$, $\mathbb{Z}_3[x]/(x^3)$, $\mathbb{Z}_9[x]/(3x, x^2-3)$ and $\mathbb{Z}_9[x]/(3x, x^2-6)$, the complete bipartite graph $K_{2,6}$ if we delete the specific edge between 9 and 18.

## 7.2. Wiener Index of $\Gamma(R)$ for the local ring $\mathbb{Z}_p[x]/(x^2)$

When the ring $R$ is commutative local rings with 1, there are only two such rings (up to isomorphism) of order $p^2$ which are $\mathbb{Z}_p[x]/(x^2)$ and $\mathbb{Z}_{p^2}$, for any prime number $p$.

**Table 4.** Classification for the local ring $\mathbb{Z}_p[x]/(x^2)$; $p$ is prime number.

| $R$ | $\lvert R\rvert$ | $Ch(R)$ | $\lvert\Gamma(R)\rvert$ | $\Gamma(R)$ | $\Gamma_c(R)$ | $W(\Gamma(R))$ | $\chi(\Gamma(R))$ | $\omega(\Gamma(R))$ |
|---|---|---|---|---|---|---|---|---|
| $\mathbb{Z}_2[x]/(x^2)$ | 4 | 2 | 1 | $K_1$ | $K_1$ | 0 | 1 | 1 |
| $\mathbb{Z}_3[x]/(x^2)$ | 9 | 3 | 2 | $K_2$ | $K_1$ | 1 | 2 | 2 |
| $\mathbb{Z}_5[x]/(x^2)$ | 25 | 5 | 4 | $K_4$ | $K_1$ | 6 | 4 | 4 |
| $\mathbb{Z}_7[x]/(x^2)$ | 49 | 7 | 6 | $K_6$ | $K_1$ | 15 | 6 | 6 |
| $\mathbb{Z}_{11}[x]/(x^2)$ | 121 | 11 | 10 | $K_{10}$ | $K_1$ | 45 | 10 | 10 |
| $\mathbb{Z}_{13}[x]/(x^2)$ | 169 | 13 | 12 | $K_{12}$ | $K_1$ | 66 | 12 | 12 |

**Observation 7.15.** 1. If $R = \mathbb{Z}_p[x]/(x^2)$, where $p$ is prime then $|R| = p^2$.
2. For the local ring $\mathbb{Z}_p[x]/(x^2)$, $\chi(\Gamma(R)) = \omega(\Gamma(R))$ in Table 4.

**Proposition 7.16.** *If $R = \mathbb{Z}_p[x]/(x^2)$ and $|R| = p^2$, then $|\Gamma(R)| = p - 1$.*

***Proof.*** Let $p$ be a prime number. We have $Z^*(\mathbb{Z}_p[x]/(x^2)) = \{x, 2x, ...(p-1)x\}$. For every $x, y$ in $Z^*(\mathbb{Z}_p[x]/(x^2))$, clearly $xy = 0$. So, $|\Gamma(\mathbb{Z}_p[x]/(x^2)| = p - 1$. □

**Theorem 7.17.** *The zero-divisor graph of $\mathbb{Z}_p[x]/(x^2)$ is $K_{p-1}$.*

***Proof.*** If $a \in Z(\mathbb{Z}_p[x]/(x^2))$, then for any two distinct vertices $a, b \in Z^*(\mathbb{Z}_p[x]/(x^2))$, so $ab = 0$, each vertex of $\Gamma(\mathbb{Z}_p[x]/(x^2))$ is adjacent to every other vertex. Furthermore, there are $p - 1$ nonzero elements, so the zero-divisor graph of $(\mathbb{Z}_p[x]/(x^2))$ has $p - 1$ vertices. Therefore, $\Gamma(\mathbb{Z}_p[x]/(x^2))$ is $K_{p-1}$. □

**Lemma 7.18.** *The graph $\Gamma_c(\mathbb{Z}_p[x]/(x^2))$ is a singleton graph.*

***Proof.*** It is obvious. □

**Theorem 7.19.** *If $p$ be a prime number and $R = \mathbb{Z}_p[x]/(x^2)$, then*

$$W(\Gamma(R)) = (p-1)(p-2)/2.$$

***Proof.*** Let $p$ be a prime number. If $R = \mathbb{Z}_p[x]/(x^2)$, then by Theorem 7.17, $\Gamma(R) = K_{p-1}$, i.e. each vertex is adjacent to the remaining $(p-1)$ vertices. Therefore, if every two distinct vertices $u, v \in \Gamma(R)$, then we have $d(u, v) = 1$. Hence

$$W(\Gamma(R)) = \sum d(u,v) = \sum_{u \neq v} 1 = \binom{p-1}{2} = (p-1)(p-2)/2.$$

□

**Remark 7.20.** If p be a prime number and $R = \mathbb{Z}_p[x]/(x^2)$, then

$$|\Gamma(R)| = \chi(\Gamma(R)) = \omega(\Gamma(R)).$$

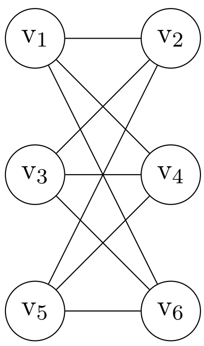
## 8. Construction graphs with Wiener index

We first explain the construction and obtain some examples regarding getting zero-divisor graphs for finite commutative rings. Second, we will find the Wiener index of zero-divisor graph for this finite commutative ring.

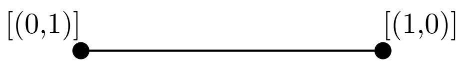
### 8.1. Wiener index with the direct product of two finite fields

Let $\mathbb{F}_q$ be a finite field of order $q$ and characteristic $p$, where $q = p^n$ for a prime number $p$ and a natural number $n$.

**Example 8.1.** (1) Let $F_4 = \mathbb{Z}_2[x]/(x^2+x+1)$ be a field of order four and characteristic 2. Then $R \cong F_4 \times F_4 = \{(0,0),(0,1),(0,x),(0,x+1),(1,0),(1,1),(1,x),(1,x+1),(x,0),(x,1),(x,x),(x,x+1),(x+1,0),(x+1,1),(x+1,x),(x+1,x+1)\}$. Hence $Z^*(R) = \{(0,1),(1,0),(0,x),(x,0),(0,x+1),(x+1,0)$. Then $\Gamma(R)$ consists of six vertices and the Wiener index, $W(\Gamma(R)) = 21$.




Now, we compute $\Gamma_c(R)$. Then $Z(R_c) = \{[(0,1)],[(1,0)]\}$ is the vertex set of $\Gamma_c(R)$. Then we obtain that $\Gamma_c(R)$ is a single edge with two vertices, which is complete bipartite graph $K_{1,1}$ and $W(\Gamma_c(R)) = 1$.




**Theorem 8.2.** *If $R \cong \mathbb{F}_q \times \mathbb{F}_q$, where $\mathbb{F}_q$ be a finite field, then $|\Gamma(R)| = 2(q-1)$.*

***Proof.*** Let $\mathbb{F}_q$ be a finite field of order $q$, where $q = p^n$ for a prime number $p$ and a natural number $n$. Let $R \cong \mathbb{F}_q \times \mathbb{F}_q$. Then there exist two maximal ideals which are either of the form $A \times \mathbb{F}_q$ where $A$ is maximal ideal in $\mathbb{F}_q$ or of the form $\mathbb{F}_q \times B$ where $B$ is maximal ideal in $\mathbb{F}_q$. Suppose $A \neq \{0\}$ and $B \neq \{0\}$, then $A$ and $B$ have inverses in $\mathbb{F}_q$, which are $A-1$ and $B-1$. Therefore, the vertex set of nonzero zero-divisors of $R$ is $2(q-1)$. Thus, $|Z^*(R)| = |\Gamma(R)| = 2(q-1); q = p^n$. □

**Theorem 8.3.** *If $R \cong \mathbb{F}_q \times \mathbb{F}_q$, where $\mathbb{F}_q$ be a finite field, then the Wiener index of $\Gamma(\mathbb{F}_q \times \mathbb{F}_q)$ is*

$$W(\Gamma((\mathbb{F}_q \times \mathbb{F}_q)) = 3q^2 - 8q + 5$$

***Proof.*** By Theorem 8.2, if $R \cong \mathbb{F}_q \times \mathbb{F}_q$, where $\mathbb{F}_q$ be a finite field then $|\Gamma(R)| = 2(q-1)$. We partition $Z^*(R)$ into two different sets $V_1$ and $V_2$ with $V = V_1 \cup V_2$ where $|V_1| = |V_2| = q-1$. Let $x, y \in V(\Gamma(R))$. It is clear that $d(x,y) = 0$ if $x = y$. Thus we may assume that $x$ and $y$ are distinct vertices. We have the following cases.

**Case 1**: $x, y \in V_1$. Then $xy \neq 0$, so $d(x,y) \neq 1$. Since every vertex in $V_1$ is adjacent to every vertex in $V_2$, we get that $d(x,y) = 2$. Therefore, $\sum_{x,y \in V_1} d(x,y) = 2(q-2)$.

**Case 2**: $x \in V_1$ and $y \in V_2$. Then $xy = 0$, so $d(x,y) = 1$ and $\sum_{x \in V_1, y \in V_2} d(x,y) = q-1$. Therefore, by above cases and the definition of the Wiener index 5.12. We obtain that

$$\begin{aligned} W(\Gamma(\mathbb{F}_q \times \mathbb{F}_q)) &= \frac{2(q-1) \quad [2(q-2)+(q-1)]}{2} \\ &= (q-1)(3q-5) \\ &= 3q^2 - 8q + 5. \end{aligned}$$

□

**Remark 8.4.** If $R \cong \mathbb{F}_q \times \mathbb{F}_q$, where $\mathbb{F}_q$ be a finite field, then the compressed zero-divisor graph of $\mathbb{F}_q \times \mathbb{F}_q$ is a single edge with the Wiener index 1.

## 8.2. Wiener index with the tensor product of two finite fields

Now, we are dealing with the tensor product of two finite fields. We shall use the following result for our purpose. The proof of this theorem is well-known and we will omit it.

**Theorem 8.5.** *Let $p$ be a prime number and let $m$, $n$ be two natural numbers. Write $gcd(m,n) = d$ to be the greatest common divisor of $m$ and $n$. Also, write $l = lcm(m,n)$ to be the least common multiple. The tensor product $\mathbb{F}_{p^m} \bigotimes \mathbb{F}_{p^n}$ is isomorphic to a direct sum of $d$ copies of $\mathbb{F}_{p^l}$. In particular, $\mathbb{F}_{p^m} \bigotimes \mathbb{F}_{p^n}$ is a field if and only if $d = 1$.*

**Example 8.6.** Let $\mathbb{F}_{2^4}$ be a field of order 16 where the elements of $GF(16)$ are

$$GF(16) = \mathbb{Z}_2[x]/ < p(x) >= \{ax^3 + bx^2 + cx + d \quad | \quad a,b,c,d \in \mathbb{Z}_2\}.$$

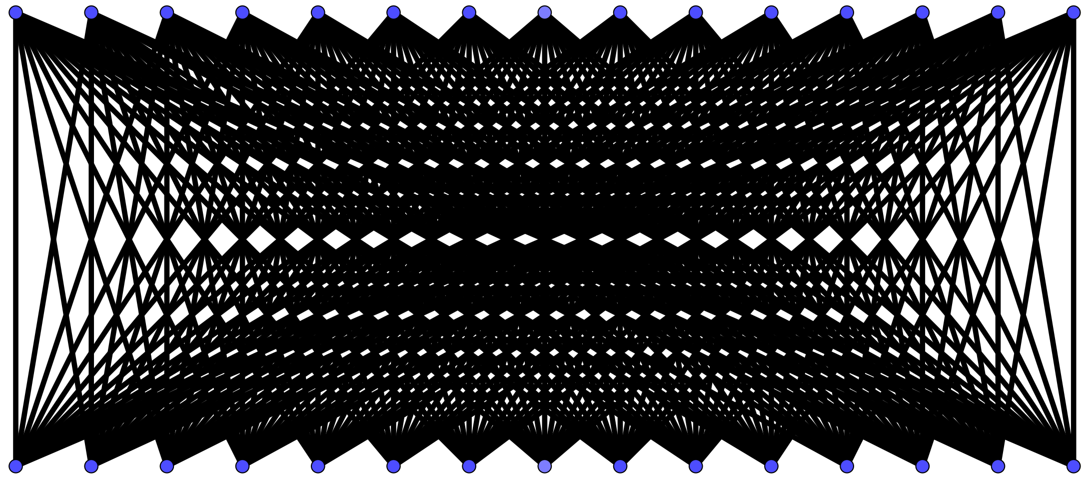
We denote these as :

$$\begin{aligned} g_0 &= 0x^3 + 0x^2 + 0x + 0 & g_8 &= 1x^3 + 0x^2 + 0x + 0 \\ g_1 &= 0x^3 + 0x^2 + 0x + 1 & g_9 &= 1x^3 + 0x^2 + 0x + 1 \\ g_2 &= 0x^3 + 0x^2 + 1x + 0 & g_{10} &= 1x^3 + 0x^2 + 1x + 0 \\ g_3 &= 0x^3 + 0x^2 + 1x + 1 & g_{11} &= 1x^3 + 0x^2 + 0x + 1 \\ g_4 &= 0x^3 + 1x^2 + 0x + 0 & g_{12} &= 1x^3 + 1x^2 + 0x + 0 \\ g_5 &= 0x^3 + 1x^2 + 0x + 1 & g_{13} &= 1x^3 + 1x^2 + 0x + 1 \\ g_6 &= 0x^3 + 1x^2 + 1x + 0 & g_{14} &= 1x^3 + 1x^2 + 1x + 0 \\ g_7 &= 0x^3 + 1x^2 + 1x + 1 & g_{15} &= 1x^3 + 1x^2 + 1x + 1 \end{aligned}$$

It follows that, $GF(16) = \{g_0, g_1, g_2, g_3, g_4, g_5, g_6, g_7, g_8, g_9, g_{10}, g_{11}, g_{12}, g_{13}, g_{14}, g_{15}\}$.

Then, we construct the ring $R = \mathbb{F}_{2^4} \bigotimes \mathbb{F}_{2^2} \cong \mathbb{F}_{2^4} \times \mathbb{F}_{2^4}$. This ring has zero-divisors which are $v_1 = (g_0, g_1), v_2 = (g_0, g_2), v_3 = (g_0, g_3), v_4 = (g_0, g_4), v_5 = (g_0, g_5), v_6 = (g_0, g_6), v_7 = (g_0, g_7), v_8 = (g_0, g_8), v_9 = (g_0, g_9), v_{10} = (g_0, g_{10}), v_{11} = (g_0, g_{11}), v_{12} = (g_0, g_{12}), v_{13} = (g_0, g_{13}), v_{14} = (g_0, g_{14}), v_{15} = (g_0, g_{15}), v_{16} = (g_1, g_0), v_{17} = (g_2, g_0), v_{18} = (g_3, g_0), v_{19} = (g_4, g_0), v_{20} = (g_5, g_0), v_{21} = (g_6, g_0), v_{22} = (g_7, g_0), v_{23} = (g_8, g_0), v_{24} = (g_9, g_0), v_{25} = (g_{10}, g_0), v_{26} = (g_{11}, g_0), v_{27} = (g_{12}, g_0), v_{28} = (g_{13}, g_0), v_{29} = (g_{14}, g_0), v_{30} = (g_{15}, g_0)$. Hence, $\Gamma(R)$ consists of 30 vertices. Therefore,

$Z^*(R) = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}, v_{11}, v_{12}, v_{13}, v_{14}, v_{15}, v_{16}, v_{17}, v_{18}, v_{19}, v_{20}, v_{21}, v_{22}, v_{23}, v_{24}, v_{25}, v_{26}, v_{27}, v_{28}, v_{29}, v_{30}\}$.



The Wiener index of this graph is $W(\Gamma(R)) = 645$.

**Remark 8.7.** We observe that the figure above in this zero-divisor graph can be partitioned into different sets $V_1 = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}, v_{11}, v_{12}, v_{13}, v_{14}, v_{15}\}$ and $V_2 = Z^*(R) - V_1$ where every $v_1 \in V_1$ and every $v_2 \in V_2$ such that $v_1v_2 \in E$, where $\mid V_1 \mid = 15$ , $\mid V_2 \mid = 15$. The bipartite graph is called complete, denoted by $K_{15,15}$.

## 8.3. Wiener Index for the local matrix algebra

We compute the Wiener index of zero-divisor graph for finite local ring. Also, we calculate the compressed zero-divisor graph for the same local ring.

**Example 8.8.** For the field $K = \mathbb{F}_2$

Consider $R = \left\{ \begin{pmatrix} a & 0 & b \\ 0 & a & c \\ 0 & 0 & a \end{pmatrix} : \ a, b, c \in \mathbb{F}_2 \right\}$

It is clear to show that $R$ is commutative local ring with identity $1_R$ which has characteristic 2. Now, we will show that $(R, +, \cdot)$ is local ring. We have $|R| = 2^3 = 8$.

Then $J(R) = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \right\}$

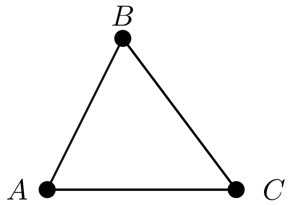
Since $x \in J(R) \Leftrightarrow 1 - x$ *is unit in* $R$. Then the units of $R$ are

$U(R) = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \right\}$

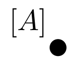
We observe that $R$ is the disjoint union of the unit and the Jacobson radical. Also, every element of $R$ is either unit or nilpotent. Therefore, $(R, +, \cdot\ )$ is local ring.

Here $Z^*(R) = \left\{ \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \right\}$ is the vertex set of $\Gamma(R)$.

Then the zero-divisor graph of $R$ is given below.




The Wiener index of this graph of the local ring, $W(\Gamma(R)) = d(A,B)+d(A,C)+d(B,C) = 1+1+1 = 3$. For the vertex set of $\Gamma_c(R)$, we have $ann(A) = \{A,B,C\}$, $ann(B) = \{A,B,C\}$, $ann(C) = \{A,B,C\}$. So, $Z(R_c) = \{[A]\}$ is the vertex set of $\Gamma_c(R)$.




**Theorem 8.9.** *Let $p$ be a prime number.*
*Let* $R = \left\{ \begin{pmatrix} a & 0 & b \\ 0 & a & c \\ 0 & 0 & a \end{pmatrix} : a,b,c \in \mathbb{F}_p \right\}$ *be a commutative local ring of order $p^3$. Then*

- $|V(\Gamma(R))| = p^2 - 1$.
- *The zero-divisor graph of $R$ is a complete graph.*
- *The compressed zero-divisor graph of $R$ is a singleton graph.*
- $W(\Gamma(R)) = \dfrac{(p^2-1)(p^2-2)}{2}$.

***Proof.*** Firstly, since $R$ is a commutative local ring, then every element of $R$ is either a unit or a zero-divisor. The Jacobson radical of $R$ is the subalgebra of $R$ with zero diagonal. That is, $J(R) = \left\{ \begin{pmatrix} 0 & 0 & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} : b,c \in \mathbb{F}_p \right\}$ and has order $p^2$. Since $R$ has order $p^3$, the number of units in $R$ is $p^3 - p^2$. Therefore, $|V(\Gamma(R))| = p^2 - 1$. Secondly, we observe that the diagonal in Jacobson matrices are all zero. Hence, any two distinct vertices $x, y \in V(\Gamma(R))$ are adjacent; $xy = 0$. Therefore, $\Gamma(R) = K_{p^2-1}$ which is a complete graph. Thirdly, by definition of the compressed zero-divisor graph, we get that all vertices collapse to a singleton graph. Lastly, it is straightforward from the definition of the Wiener index and the direct computations. □

## 9. Discussion

In the past several years, there are a lot of unsolved problems in abstract algebra and graph theory. Furthermore, many researchers have determined some connection between commutative ring theory and graph theory to solve it by associating a suitable graph. The notion of the Wiener index of zero-divisor graph was introduced by Ahmadi and Jahni-nezhad [1], who demonstrated the Wiener index of zero-divisor graph of rings $\mathbb{Z}_n$, for the case $n = p^2$ and the case $n = pq$. In this paper, we discuss the Wiener index of zero-divisor graph of finite commutative local rings of order 8 and 27. Moreover, we generalize the Wiener index of the zero-divisor graph of $\mathbb{Z}_p[x]/(x^2)$ for any prime number $p$. Many of researchers can general theses analyses. We would like to mention that the work in zero-divisor graphs of commutative rings are active and we mention the following new articles [4, 15, 26].

## 10. Conclusion

In this paper, we investigate a problem in algebraic graph theory. We give an extensive overview of the relevant literature. In Section 4, we study the basic notions of local rings.

By Proposition 4.6, we find that any ring with identity which satisfies any condition in this proposition is called local ring. In Section 5, we give the definition of zero-divisor graph, compressed zero-divisor graph and Wiener index. Furthermore, we calculate the Wiener index of zero-divisor graphs of finite commutative rings. Also, we investigate the some examples of the Wiener index of zero-divisor graph $\Gamma(R)$ for finite commutative local rings. In Section 6, we determine the classification of finite commutative local rings with characteristic of the ring. In Section 7, we first compute the Wiener index of zero-divisor graph for finite commutative local rings when $p = 2$ and $p = 3$. Second, we will focus on the Wiener index of $\Gamma(R)$ for $\mathbb{Z}_p[x]/(x^2)$. In Section 8, we first give the construction of the zero-divisor graph of the tensor product of two finite fields. This constructions helps to understand of the graph structure. Second, we focus on the Wiener index for the local matrix algebra. The researchers observe that there are substantial types of finite commutative local rings, which also motivate others to compute the Wiener index for them. The research in this topic is vast. For further theory and investegation, we refere to the new references [13, 24, 32].

## Acknowledgements

This work is a part of the first author's PhD study at Umm Al-Qura University under the supervision of the professor Ahmad M. Alghamdi. The authors are grateful to the reviewer for his/her valuable comments and suggestions that have improve the writing of this paper.

**Author contributions.** All the co-authors have contributed equally in all aspects of the preparation of this submission.

**Conflict of interest statement.** The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

**Funding.** This work received no funding.

**Data availability.** No data was used for the research described in the article.

## References

[1] MR. Ahmadi and R. Jahani-Nezhad, *Energy and Wiener index of zero divisor graphs*, J. Iranian of Math Chem. **2** (1), 45-51, 2011.

[2] D. F. Anderson, T. Asir, A. Badawi, and T. T. Chelvam, *Graphs from rings*, Springer, 2021.

[3] D. F. Anderson and P. S. Livingston, *The zero-divisor graph of a commutative ring*, J. Algebra 217, No. **2**, 434- 447, 1999.

[4] T. Asir, A. Kumar and A. Mehdi, *On the zero-divisor hypergraph of a reduced ring*, Acta Math. Hungar. 1-14, 2023.

[5] T. Asir, V. Rabikka, A. M. Anto and N. Shunmugapriya, *Wiener index of graphs over rings: a survey*, AKCE International Journal of Graphs and Combinatorics **19**(3), 316-324, 2022.

[6] M. F. Atiyah and I. G. Macdonald, *Introduction to Commutative Algebra*, Addison-Wesley Publishing, Reading, MA, 1969.

[7] I. Beck, *Coloring of commutative rings*, J. Algebra **116**, 208226, 1988.

[8] D. Bonchev, *The Wiener numbersome applications and new developments*, In Topology in Chem, 58-88, 2002.

[9] A. Cayley, *Desiderata and suggestions: The theory of groups: Graphical representation*, Amer. J. Math. **1**, 174176, 1878.

[10] A. Dobrynin, R. Entringer and I. N Gutman, *Wiener index of trees: theory and applications*, Acta Appl. Math. **66** 3, 211-249, 2001.
[11] J. B. Fraleigh, *A first course in abstract algebra*, Pearson Education India 2003.
[12] A. Graovac and T. Pisanski, *On the Wiener index of a graph*, J. math chem. **8** (1), 53-62, 1991.
[13] M. Hazewinkel, *Formal groups and applications*, Vol. **78**. Elsevier, 1978.
[14] H. Hosoya, *Topological index. a newly proposed quantity characterizing the topological nature of structural isomers of saturated hydrocarbons*, Bull. Chem. Soc. Jpn. **44** (9), 23322339, 1971.
[15] H. M. Hoque, H. K. Saikia, J. Goswami and D. Patwari, *Non-Nilpotent Graph of Commutative Rings*, J. Algebraic System **12** (1), 149-162, 2024.
[16] T. W. Hungerford, *Algebra*, Springer Science and Business Media, 2003.
[17] W. Hungerford, Thomas *Abstract algebra: an introduction*, Cengage Learning, 2012.
[18] D. Janei, A. Milievi, S. Nikoli, N. Trinajsti, *Graph-theoretical matrices in chemistry*, Second edition, CRC Press, Boca Raton. F. xiv+160 pp. ISBN: 978-1-4987-0115-0, 2015.
[19] D. J. Klein, I Lukovits, I Gutman, *On the defination of the hyper Wiener index for cycle-containing structures*, J. Chem. Inf. Comput. Sci. **35**, 50-52, 1995.
[20] B. Külshammer, *Lectures on block theory*, Cambridge University Press, 1991.
[21] J.J. Molitierno, *Applications of combinatorial matrix theory to Laplacian matrices of graphs*, CRC Press, 2016.
[22] GR. Omidi and E. Vatandoost, *On the commuting graph of rings*, J. Algebra Appl. **10** (3), 521-52, 2011.
[23] S. Pirzada, *An Introduction to Graph Theory*, Universities Press, Hyderabad, India, 2011.
[24] Q. Ma, *Extremal polygonal chains with respect to the Kirchhoff index*, Disc. Appl. Math. **342**, 218-226, 2024.
[25] M. Randiťc, *Novel molecular descriptor for structure-property studies*, Chem. Phys. Lett. **211**, 478-483, 1993.
[26] N. U. Rehman, M. Nazim and K. Selvakumar, *On the genus of extended zero-divisor graph of commutative rings*, Rend. Circ. Mat. Palermo **2**, 1-10, 2022.
[27] K. Selvakumar, P. Gangaeswari and G. Arunkumar, *The Wiener index of the zero-divisor graph of a finite commutative ring with unity*, Dis. Appl. Math. **311**, 72-84, 2022.
[28] NH. Shuker and PA. Rashed, *The zero divisor graph of the ring* $Z_{pqr}$, Int. J. Sci. **6** (2), 569-574, 2015.
[29] H. Wiener, *Structural determination of the paraffin boiling points*. J. Am. Chem. Soc. **69**, 17-20, 1947.
[30] R. Wilson. J, *Introduction to graph theory*. Pearson Education India, 1979.
[31] K. Xu, M. Liu, K. C. Das, I. Gutman, and B. Furtula, *A survey on graphs extremal with respect to distance-based topologicalindices*, MATCH Commun. Math. Comput. Chem, **71** (3), 461508, 2014.
[32] K. Yamazaki, M. Qian and R. Uehara, *Efficient enumeration of non-isomorphic distance-hereditary graphs and related graphs*, Disc. App. Math. **342**, 190-199, 2024.
[33] Y.-N. Yeh, I. Gutman, *On the sum of all distances in composite graphs*, Disc. Math, **135**, 359365, 1994.