Necmettin Erbakan Üniversitesi
Fen ve Mühendislik Bilimleri Dergisi
Journal of Science and Engineering (NEJSE)
DOI: 10.47112/neufmbd.2023.5

Cilt: 5 Sayı:1 Yıl: 2023 | Araştırma Makelesi / Research Article | E-ISSN: 2667- 7989

Cebirsel Bir Grafın Harary İndeksi

Nihat AKGÜNEŞ [1] **Büşra AYDIN** [2]

[1] Necmettin Erbakan University, Faculty of Science, Department of Mathematics and Computer Science, Konya, Türkiye, nakgunes@erbakan.edu.tr *(Sorumlu Yazar/ Corresponding Author)*
[2] Necmettin Erbakan University, Institute of Science, Department of Mathematics, Konya, Türkiye,

Makale Bilgileri

Makale Geçmişi
Geliş: 31.05.2023
Kabul: 21.06.2023
Yayın: 30.06.2023

Anahtar Kelimeler:
Graf teori, Harary indeks, Nokta çarpım grafı, Topolojik indeks.

ÖZ

Topolojik indekslerin matematiksel kimyada kulanım alanı bulunmaktadır. Uzaklık-bazlı topolojik indekslerin ise moleküler graf teoride oldukça önemi vardır. Harary indeksi uzaklık-bazlı bir graf değişmezidir. Yakın zamanda cebirsel bir yapı üzerinde nokta çarpım grafı çalışıldı. Bu çalışmada da bu grafın Harary indeksi verilecektir.

Harary Index for an Algebraic Graph

Article Info

Article History
Received: 31.05.2023
Accepted: 21.06.2023
Published: 30.06.2023

Keywords:
Dot product graph,
Graph theory,
Harary index,
Topological index.

ABSTRACT

Topological indices are used in mathematical chemistry. Distance-based topological indices have a great interest in molecular graph theory. Harary index is one of the distance-based graph invariant. Recently, a dot product graph for an algebraic structure has been studied. In this study, the Harary index of this graph will be given.

Atıf/Citation: Akgüneş, N. & Aydın, B. (2023). Cebirsel bir grafın harary indeksi, *Necmettin Erbakan Üniversitesi Fen ve Mühendislik Bilimleri Dergisi*, *5*(1), 9-13.

INTRODUCTION AND PRELIMINARIES

Graph and Ring Theory connection was constituted by Beck, in 1988 [1]. Graphs for zero-divisors of algebraic structures have been studied in [2] and [3]. Also dot product graphs for rings, commutative, has been studied by Badawi in [4]. Then this graph for monogenic semigroups has been defined in [5] and [6]. Also, some products of this graphs have been studied in [7] and [8].

The basic definitions are given in (Harary [9], Bondy and Murty [10]). For any simple-connected graph *G*, the vertex set is symbolized by $V(G) = \{v_1, v_2, \dots v_n\}$. Then elements of edge set *E(G)* are unordered pairs of adjacent vertices. The order of a graph is the number of vertices $|V| = n$. Then the size of a graph is the number of vertices $|E| = m$. Then the distance of two vertices *u* and *v* in *V(G)* is defined as length of shortest path between vertices *u* and *v*. The distance the vertices *u* and *v* is denoted by *d(u,v)*. The degree of any vertex *v* in *V(G)* is the number of edges incident of *v*. The degree is symbolized by *deg(v)*.

Definition 1. [5] The finite multiplicative semigroup $S_M^n = \{0, x, x^2, \dots, x^n\}$ and its undirected zero-divisor graph $\Gamma(S_M^n)$. The vertex set of $\Gamma(S_M^n)$ is the non-zero elements of monogenic semigroup S_M^n and the edge set is pair of vertices which are provide the rule $x^i . x^j = 0_{S_M^n}$ for $i, j \in \{0,1,2,\dots,n\}$ where $x^0 = 0_{S_M^n}$.

Also the dot product graph over monogenic semigroups for finite times is given in [6].

Definition 2. A cartesian product of monogenic semigroup for k times $S = S_M^n \times S_M^n \times \dots \times S_M^n$. The undirected graph of S is $\Gamma(S)$ and the vertex set of $\Gamma(S)$ is consist of non-zero elements of S.

Some parameters of these graphs have been obtained in a lot of research. Further some topological indices over these graphs have been obtained in [11] and [12].

The topological index is a graph invariant-used to characterize the properties of molecular graphs. Molecules are represented by graphs with atoms by vertices and bonds by edges. Topological indices have wide application in chemical graph theory (for example [13]). The Harary index is a distance-based topological index. Harary index has been introduced to characterize the molecular graphs ([14]).

Definition 3. For a graph G, Harary index is obtained as the sum of the reciprocals of distances between all pairs of vertices.

$$H(G) = \sum_{\{u,v\} \in V(G)} \frac{1}{d(u,v)}$$

Harary index, one of the popular distance-based topological index, has been extensively researched [15-17].

Bounds for Harary index depend on the order of graph.

Lemma 4. [18] Let *G* be a graph with order $n \geq 2$ then

$$1 + n\sum_{k=2}^{n-1} \frac{1}{k} \leq H(G) \leq \frac{n(n-1)}{2}$$

With lower inequality holds iff $G \cong P_n$ and upper inequality holds iff $G \cong K_n$.

MAIN RESULTS

There is a good relations with many physico-chemical properties of molecules and their graphs. Topological indices of graphs are divided into three as distance, degree and eccentric. The Harary index is a distance-based topological index.

Let give the diameter of any dot product graph over monogenic semigroup.

Lemma 5. [5] Let S be cartesian product as Definition 2. Then diameter of $\Gamma(S)$ is equal to

$$diam\big(\Gamma(S)\big) = 2$$

We defined dot product of monogenic semigroup finitely k times, above. Now, it will be continued for $k = 2$.

Let $S_M^n = \{x, x^2, \dots, x^n\}$ be a monogenic semigroup for $n \in \mathrm{N}^+$. Then $\Gamma(S)$ be the dot product graph for *2* times of S_M^n monogenic semigroup. It is known that vertices of $\Gamma(S)$ are non-zero elements of S.

The Harary index for dot product graph, it is known that the distances of any two vertices in the vertex set of the dot product graph. Topological index are very important for graph theory and topology [19]. The Wiener index is a distnce-based topological index, too.

Theorem 6. Let $\Gamma(S)$ be an algebraic structure graph then Harary index is

$$H\big(\Gamma(S)\big) = \frac{1}{4}\left[\left((n+1)^2 - 1 - \left\lfloor\frac{n}{2}\right\rfloor^2 - 2\left\lfloor\frac{n}{2}\right\rfloor\right) + \sum_{t=0}^{n-1}[n + (n+1)(n-t)] + \sum_{k=0}^{n-1}[n + (n+1)(n-k)] + \sum_{k,t=0}^{n-1}[2n - k - t + (n-k)(n-t)]\right]$$

Where $S = S_M^n \times S_M^n$ and S_M^n be a monogenic semigroup of order n.

Proof: Let u and v be any two vertices of $\Gamma(S)$ then $d(u,v) \leq 2$ since Lemma 5. So for any two vertices distance is equal to *1* or *2*. Then the vertex set of $\Gamma(S)$ can be written by seperating into two disjoint sets as

$$V_1 = \{u_1 : d(u_1, u_2) = 1, u_1, u_2 \in \Gamma(S)\}$$

And

$$V_2 = \{u_1 : d(u_1, u_2) = 2, u_1, u_2 \in \Gamma(S)\}$$

So, the Harary index of the dot product graph $\Gamma(S)$ be

$$H\big(\Gamma(S)\big) = \frac{1}{2}\sum_{u \in V(\Gamma(S))}\left[1.|V_1| + \frac{1}{2}.|V_2|\right]$$

Since all pairs of vertices are counted twice and $d(u,v) = d(v,u)$.

It is known that $|V| = |V_1| + |V_2|$ then the Harary index is in the form

$$H\big(\Gamma(S)\big) = \frac{1}{2}\sum_{u \in V(\Gamma(S))}\left[|V_1| + \frac{1}{2}.(|V| - |V_1|)\right] = \frac{1}{4}\sum_{u \in V(\Gamma(S))}[|V| + |V_1|]$$

Let determine the set of the adjacent vertices since the elements of set V_1becomes vertices of distances *1n*.

Let any two no-zero elements of S $X = (x^{i_1}, x^{i_2})$ and $Y = (x^{j_1}, x^{j_2})$ for $i_1, i_2, j_1, j_2 \in \{0,1,2,\dots,n\}$ $where\ x^{i_t} = 0_{S_M^n}\ if\ and\ only\ if\ i_t = 0.$

Then $\Gamma(S)$ is defined as the vertices are $X, Y \in S^* = S\backslash\{0_S\}$ such that they are adjacent vertices if and only if $X.Y = \big(x^{i_1}, x^{i_2}\big).\big(x^{j_1}, x^{j_2}\big) = x^{i_1}.x^{j_1} + x^{i_2}.x^{j_2} = x^{i_1+j_1} + x^{i_2+j_2} = 0_{S_M^n}$ then $X \sim Y$ shorthand for two adjacent vertices.

Any vertex in the vertex set of $V(\Gamma(S))$ can be written $(x^{n-k}, x^{n-t}) \in \Gamma(S)$ for $k, t \in \{0,1,2,\dots,n\}$, here we assumed that $x^0 = 0_{S_M^n}$. Then we can seperate three cases for k is equal to n or t is equal to n or $k, t \in \{0,1,2,\dots,n-1\}$.

First case: If k is equal to n then the vertex $(x^{n-k}, x^{n-t}) = (0, x^{n-t})$. If any $(x^a, x^b) \in V(\Gamma(S))$ is adjacent to $(0, x^{n-t})$ then $n-t+b=0$. Hence $n-t+b>n$ or $b=0_{S_M^n}$. Consequently there are $(n-t)(n+1)+n$ vertices are adjacent to the vertex $(0, x^{n-t})$.

Second case: If t is equal to n then the vertex $(x^{n-k}, x^{n-t}) = (x^{n-k}, 0)$. If any $(x^c, x^d) \in V(\Gamma(S))$ is adjacent to $(x^{n-k}, 0)$ then $n-k+c>n$ or $d=0_{S_M^n}$. Similar way to first case there are $(n-k)(n+1)+n$ vertices are adjacent to the vertex $(x^{n-k}, 0)$.

Last case: If $k, t \in \{0,1,2,\dots,n\}$ then the vertex (x^{n-k}, x^{n-t}) is adjacent to any vertex $(x^e, x^f) \in V(\Gamma(S))$ then $n-k+e>n$ or $e=0_{S_M^n}$ and $n-t+f>n$ or f= $0_{S_M^n}$. So there are $(n-k)(n-t)+2n-k-t$ vertices are adjacent to the vertex (x^{n-k}, x^{n-t}) for $k, t \in \{0,1,2,\dots,n\}$.

However; in above cases, the vertices which are adjacent to themselves are counted. It should be subtracted from Harary index. So $\left(n-\left\lfloor\frac{n}{2}\right\rfloor\right)^2 + 2\left(n-\left\lfloor\frac{n}{2}\right\rfloor\right)$ vertices adjacent to each other.

Consequently

$$H(\Gamma(S)) = \frac{1}{4}\sum_{u\in V(\Gamma(S))}[|V|+|V_1|]$$
$$= \frac{1}{4}\left[\left((n+1)^2 - 1 - \left\lfloor\frac{n}{2}\right\rfloor^2 - 2\left\lfloor\frac{n}{2}\right\rfloor\right) + \sum_{t=0}^{n-1}[(n-t)(n+1)+n] + \sum_{k=0}^{n-1}[(n-k)(n+1)+n] + \sum_{k,t=0}^{n-1}[(n-k)(n-t)+2n-k-t]\right]$$

is obtained.

DISCUSSION AND CONCLUSIONS

By applying a distance-based topological index on a product graph, we obtained the Harary index of the dot product graph over monogenic semigroup. Also other topological indices can be studied on some product graphs.

REFERENCES

[1] I. Beck, Coloring of Commutating Ring, *Journal of Algebra*, 116 (1988), 208-226.

[2] D. F. Anderson, P. S. Livingston, The Zero-Divisor Graph of a Commutative Ring, *Journal of Algebra*, 217 (1999), 434-447.

[3] F. R. De Meyer, L. De Meyer, Zero-Divisor Graphs of Semigroups, *Journal of Algebra*, 283 (2005), 190- 198.

[4] A. Badawi, On the Dot Product Graph of a Commutative Ring, *Communications in Algebra*, 43 (2015), 43-50.

[5] K. C. Das, N. Akgüneş, A. S. Çevik, On a Graph of Monogenic Semigroups, *Journal of Inequalities and Applications*, 2013:44 (2013).

[6] N. Akgüneş, B. Çağan, On the Dot Product of Graphs Over Monogenic Semigroups, *Applied Mathematics and Computation*, 322 (2018), 1-5.

[7] Y. Nacaroğlu, On the corona product of monogenic semigroup graphs, *Adv. and Appl. in Discrete Math.*, 19(2018) 409-420.

[8] Y. Nacaroğlu, On Join Operation of Graphs by Obtained Monogenic Semigroups, *Turkish Journal of Mathematics and Computer Science*, 13(1)(2021) 57-62.

[9] F. Harary, Graph Theory, *Addison Wesley, Reading, Mass.*, 1969.

[10] G. A. Bondy, U. S. R. Murty, Graph Teory with Applications, *Elsevier Science, New York, NY, USA*,

(1982).

[11] N. Akgüneş, K. C. Das, A. S. Çevik, Topological indices on a graph of monogenic semigroups, *Topics in Chemical Graph Theory*, 16 (2014), 3-20.

[12] B. Aydın, N. Akgüneş, İ. N. Cangül, On the Wiener index of the dot product graph over monogenic semigroups, *European Journal of Pure and Applied Mathematics*, Vol.13 No.5 (2020), 1231-1240.

[13] I. Gutman, O. Polansky, Mathematical Concepts in Organic Chemistry, *Springer-Verlag, Berlin, Germany*, (1986).

[14] D. Plavši´c, S. Nikoli´c, N. Trinajsti´c, Z. Mihali´c, On the Harary index for the characterization of chemical graphs, *Journal of Mathematical Chemistry,* 12 (1993), 235–250.

[15] K. C. Das, K. Xu, I. N. Cangül, A. S. Çevik, A. Graovac, On the Harary index of graph operations, *Journal of Inequalities and Applications*, vol. 339 (2013).

[16] K. Xu, K. C. Das, On harary index of graphs, *Discrete Applied Mathematics*, vol. 159 no. 15 (2011), 1631–1640.

[17] B. H. Xing, G. D. Yu, L. X. Wang, J. Cao, The Harary index of all unicyclic graphs with given diameter, *Discrete Dynamics in Nature and Society*, (2018).

[18] B. Zhou, X. Cai, N. Trinajstic, On Harary Index, *Journal of Mathematical Chemistry*, 44 (2008), 611-618.

[19] S. Pak , Ö. Gürmen Alansal ve U. Cesur , "Pseudo 2- Çaprazlanmış Modüller ve Pseudo 3-Çaprazlanmış Modüller", Necmettin Erbakan Üniversitesi Fen ve Mühendislik Bilimleri Dergisi, 2(2), (2020), 22-37.