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Research Article

The direct product of a star and a path is antimagic

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Abstract

A graph G is antimagic if there exists a bijection f from $E(G)$ to $\{1, 2, \ldots, |E(G)|\}$ such that the vertex sums for all vertices of G are distinct, where the vertex sum is defined as the sum of the labels of all incident edges. Hartsfield and Ringel conjectured that every connected graph other than K_2 admits an antimagic labeling. It is still a challenging problem to address antimagicness in the case of disconnected graphs. In this paper, we study antimagicness for the disconnected graph that is constructed as the direct product of a star and a path.

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Keywords. direct product, star, path, disconnected graph, antimagic labeling

1. Introduction

In 1990, the concept of antimagic labeling was first introduced by Hartsfield and Ringel [7]. They called a graph G *antimagic* if there exists a bijection $f : E(G) \to \{1, 2, \ldots, |E(G)|\}$ such that for all vertices their weights, defined as the sum of all the incident edge labels, are distinct. In the same paper [7] some simple graphs such as paths, cycles, complete graphs and wheels are proven to be antimagic and, Hartsfield and Ringel posed the strong conjecture that every connected graph except K_2 is antimagic.

Although several researchers have attempted to settle the above conjecture, still the conjecture remains open. For instance, Alon et al., [1] validated the conjecture for graphs with sufficiently large minimum degree. They proved that there exists an absolute constant c such that every graph with n vertices and minimum degree at least $c \log n$ is antimagic. They also proved that every graph with at least four vertices and the maximum degree $\Delta(G) \geq n-2$ is antimagic. A similar result was improved by Yilma [20] for $\Delta(G) \geq n-3$.

Only some authors studied antimagicness for disconnected graphs. Not all disconnected graphs admit an antimagic labeling. Since, we know K_2 is not an antimagic, so we only

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need to consider the graphs with no component isomorphic to K_2. Exploring the set of all disconnected antimagic graphs seems to be another interesting open problem. In this context, Wang et al. [17] studied antimagicness for the unions of some graphs. That is, $mK_{1,n}$ for $n \geq 2$, $2P_n$ for $n \geq 2$, $K_{1,n} \cup P_n$ for $n \geq 3$, $K_{1,n} \cup P_{n+1}$ for $n \geq 3$ and $C_n \cup K_{1,m}$ if $m \geq 2\sqrt{n}+2$. Shang et al. [15] considered a star forest with no component isomorphic to K_2 and at most one component isomorphic to $K_{1,2}$ is antimagic. Also, they shown that if a star forest mK_2 is antimagic then $m = 1$. Moreover, they investigated that the star forest $mK_{1,2} \cup K_{1,n}$ is antimagic if and only if $m \leq \min\left\{2n+1, (2n-5+\sqrt{8n^2-24n+17})/2\right\}$. Shang [14] showed that a linear forest with no component isomorphic to P_2, P_3 and P_4 is antimagic. Chen et al. [3] proved that $mK_{1,2} \cup K_{1,n}$, $n \geq 3$ is antimagic if and only if $n \geq \max\left\{(m-1)/2, (1-2m+\sqrt{8m^2+16m+9})/2\right\}$. They also gave a necessary condition and a sufficient condition for a star forest $mK_{1,2} \cup mK_{1,n_1} \cup mK_{1,n_2} \cup \cdots \cup mK_{1,n_k}$, $n_1, n_2, \ldots, n_k \geq 3$ to be antimagic. In addition, they proved that a star forest with an extra disjoint path is antimagic. Many works related to the antimagicness for product of graphs was discussed by various authors for connected graphs. The readers can refer to the following references: for the Cartesian product [4, 5, 8, 12, 16], for the lexicographic product [9–11], for the corona product [6], for the join of graphs [2, 18]. However, none of the researchers focused on the antimagicness of the direct product of some graph.

The direct product of graphs was first considered by Weichsel [19] in 1962, which was originally derived from Kronecker product of matrices. There are several names used for the direct product of graphs that are used by different authors. Those are cardinal product, Kronecker product, tensor product, categorical product and graph conjunction. The direct product of graphs G and H, denoted by $G \times H$, is the graph with the vertex sets same as the Cartesian product of these graphs, i.e., $V(G \times H) = V(G) \times V(H)$ such that the vertex pairs (x, y) and (x', y') are adjacent in $G \times H$ if and only if x is adjacent to x' in G and y is adjacent to y' in H.

The connectedness of the direct product of two graphs is characterized in the following theorem.

Theorem 1.1 ([19])**.** *Let G and H be connected graphs. The direct product $G \times H$ is connected if and only if either G or H contains an odd cycle.*

Corollary 1.2 ([19])**.** *If G and H are connected graphs with no odd cycles then the direct product $G \times H$ has exactly two connected components.*

In this paper we study the antimagicness of the direct product of a star and a path. Our main result is the following.

Theorem 1.3. *The graph $K_{1,s} \times P_n$ is antimagic for all positive integers $s \geq 1$, $n \geq 2$ except three cases when $(s, n) \in \{(1, 2), (1, 3), (2, 2)\}$.*

According to Corollary 1.2 the direct product $K_{1,n} \times P_m$ has exactly two connected components. Evidently, when $s = 1$ the graph $K_{1,s} \times P_n$ is isomorphic to two copies of the path P_n. Trivially, $2P_2$ is not antimagic. In [17] Wang, Lui and Li proved that mP_3 is not antimagic for $m \geq 2$. Moreover, they proved that the union of two copies of a path on at least four vertices is an antimagic graph. This immediately implies that $K_{1,1} \times P_n$ is antimagic if and only if $n \geq 4$.

To complete the proof of Theorem 1.3 for $s \geq 2$ we distinguish two cases according to the parity on n. These cases are discussed in the following two sections.

2. A path on even number of vertices

First consider a graph $K_{1,s} \times P_{2m+2}$, $s \geq 2$, $m \geq 0$. This graph is disconnected and consists of two isomorphic copies. Let us denote the vertices and edges of $K_{1,s} \times P_{2m+2}$

in the following way.

$$\begin{aligned} V(K_{1,s} \times P_{2m+2}) = & \{a_i^j, b_i^j, v_i, u_i : i = 0, 1, \dots, m, j = 1, 2, \dots, s\}, \\ E(K_{1,s} \times P_{2m+2}) = & \{a_i^j v_i, b_i^j u_i : i = 0, 1, \dots, m, j = 1, 2, \dots, s\} \\ & \cup \{a_i^j v_{i-1}, b_i^j u_{i-1} : i = 1, 2 \dots, m, j = 1, 2, \dots, s\}. \end{aligned}$$

The structure of graph is shown in Figure 1.

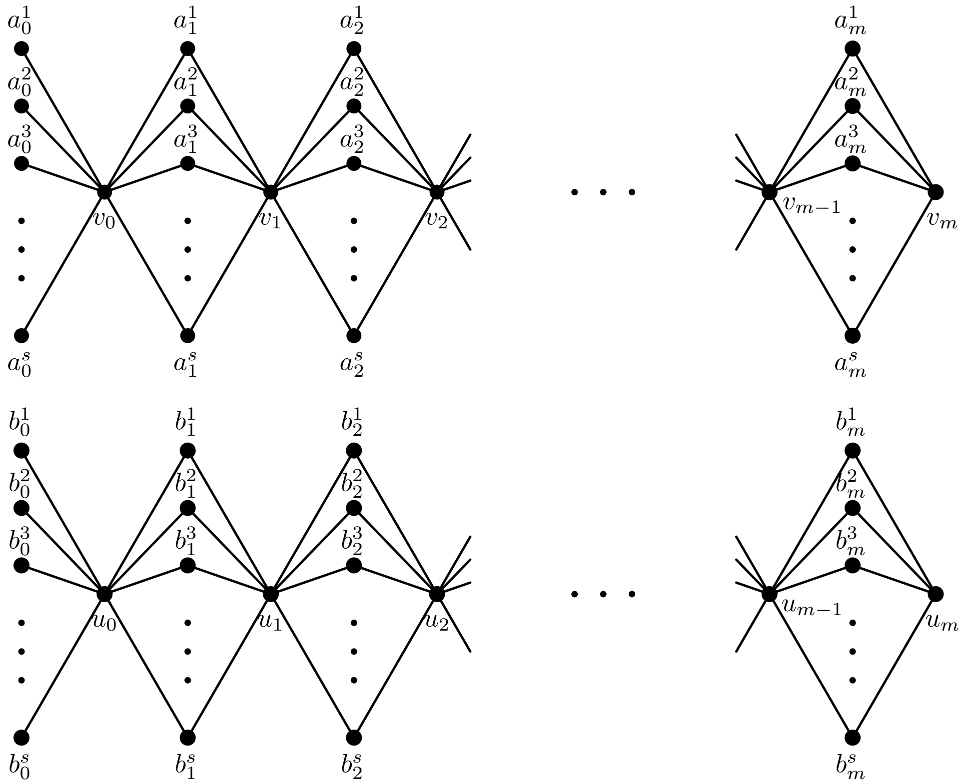

Figure 1. The general representation of graph $K_{1,s} \times P_{2m+2}$.

Before all else we solve two small cases. More precisely, we consider two special cases when $m = 0$, i.e., the direct product $K_{1,s} \times P_2$ for $s \geq 2$ and the case when $s = 2$, i.e., the direct product $K_{1,2} \times P_{2m+2}$ for $m \geq 1$.

Lemma 2.1. *The graph $K_{1,s} \times P_2$ is antimagic for $s \geq 3$.*

Proof. Sampathkumar [13] proved that if a connected graph G contains no odd cycle then $G \times K_2$ is isomorphic to $2G$. Thus the graph $K_{1,s} \times P_2$ is isomorphic to two copies of the star $K_{1,s}$. Evidently, the graphs $2K_{1,1}$ and $2K_{1,2} \cong 2P_3$ are not antimagic. In [17], it is proved that $2K_{1,s}$ is antimagic for $s \geq 3$. $\square$

Lemma 2.2. *The graph $K_{1,2} \times P_{2m+2}$ is antimagic for $m \geq 1$.*

Proof. Let us define an edge labeling f of $K_{1,2} \times P_{2m+2}$ in the following way:

$$\begin{aligned} f(a_0^j v_0) = & 2 + j, && \text{for } j = 1, 2, \\ f(b_0^j u_0) = & j, && \text{for } j = 1, 2, \\ f(b_i^j u_{i-1}) = & 3 + 2i + 4m(j-1), && \text{for } i = 1, 2, \dots, m \text{ and } j = 1, 2, \\ f(b_i^j u_i) = & 4 + 2i + 4m(j-1), && \text{for } i = 1, 2, \dots, m \text{ and } j = 1, 2, \end{aligned}$$

$$f(a_i^j v_{i-1}) = 3 + 2i + 2m(2j-1), \qquad \text{for } i = 1,2,\dots,m \text{ and } j = 1,2,$$
$$f(a_i^j v_i) = 4 + 2i + 2m(2j-1), \qquad \text{for } i = 1,2,\dots,m \text{ and } j = 1,2.$$

Evidently, the edges are labeled with distinct numbers from 1 to $8m+4$. Now, we evaluate the induced vertex labels under the function f. The weights of vertices of degree one are

$$wt_f(a_0^j) = f(a_0^j v_0) = 2 + j,$$
$$wt_f(b_0^j) = f(u_0 b_0^j) = j$$

for $j = 1, 2$, thus the weights are $1, 2, 3$ and 4.
Now we evaluate the weights of vertices of degree 2. For $i = 1, 2, \dots, m$ and $j = 1, 2$ we obtain

$$wt_f(a_i^j) = f(v_{i-1} a_i^j) + f(a_i^j v_i) = 4m + 7 + 4i + 8m(j-1),$$
$$wt_f(b_i^j) = f(v_{i-1} b_i^j) + f(b_i^j v_i) = 7 + 4i + 8m(j-1).$$

Thus the weights of vertices of degree 2 are the following odd numbers $11, 15, \dots, 16m+7$. More precisely, all of them are congruent 3 modulo 4. The weights of vertices v_m and u_m are divisible by 4, as

$$wt_f(v_m) = f(a_m^1 v_m) + f(a_m^2 v_m) = 12m + 8,$$
$$wt_f(u_m) = f(b_m^1 u_m) + f(b_m^2 u_m) = 8m + 8.$$

Now, we evaluate the weights of vertices of degree 4. For $i = 1, 2, \dots, m-1$, we get

$$wt_f(v_i) = \sum_{j=1}^{2} f(a_i^j v_i) + \sum_{j=1}^{2} f(a_{i+1}^j v_i) = 16m + 18 + 8i,$$
$$wt_f(u_i) = \sum_{j=1}^{2} f(b_i^j u_i) + \sum_{j=1}^{2} f(b_{i+1}^j u_i) = 8m + 18 + 8i.$$

Which means that they are distinct numbers and all of them are congruent 2 modulo 4. Finally,

$$wt_f(v_0) = \sum_{j=1}^{2} f(a_0^j v_0) + \sum_{j=1}^{2} f(a_1^j v_0) = 8m + 17,$$
$$wt_f(u_0) = \sum_{j=1}^{2} f(b_0^j u_0) + \sum_{j=1}^{2} f(b_1^j u_0) = 4m + 13.$$

Thus they are congruent 1 modulo 4. Evidently, all vertex weight induced by the labeling f are distinct numbers thus f is an antimagic labeling of $K_{1,2} \times P_{2m+2}$. $\square$

Now consider the case when $s \geq 3$.

Theorem 2.1. *The graph $K_{1,s} \times P_{2m+2}$ is antimagic for $s \geq 3$, $m \geq 1$.*

Proof. Let us define an edge labeling f_ε, $\varepsilon \in \{0, 1, \dots, s^2\}$, of $K_{1,s} \times P_{2m+2}$, $s \geq 3$, $m \geq 1$, such that the pendant edges are labeled as follows:

$$\{f_\varepsilon(a_0^j v_0), f_\varepsilon(b_0^j u_0) : j = 1, 2, \dots, s\} = \{1, 2, \dots, 2s\} \tag{2.1}$$

and

$$\sum_{j=1}^{s} f_\varepsilon(a_0^j v_0) = \frac{s(s+1)}{2} + \varepsilon, \tag{2.2}$$

thus

$$\sum_{j=1}^{s} f_\varepsilon(b_0^j u_0) = s^2 + \frac{s(s+1)}{2} - \varepsilon. \tag{2.3}$$

The labels of the remaining edges are

$$f_\varepsilon(v_{i-1}a_i^j) = \begin{cases} 2j-1+2is, & \text{for } i \equiv 1 \pmod 2,\ 1 \le i \le m \text{ and } j=1,2,\dots,s,\\ 2j+2is, & \text{for } i \equiv 0 \pmod 2,\ 2 \le i \le m \text{ and } j=1,2,\dots,s,\end{cases}$$

$$f_\varepsilon(a_i^j v_i) = \begin{cases} 2j+2is, & \text{for } i \equiv 1 \pmod 2,\ 1 \le i \le m \text{ and } j=1,2,\dots,s,\\ 2j-1+2is, & \text{for } i \equiv 0 \pmod 2,\ 2 \le i \le m \text{ and } j=1,2,\dots,s,\end{cases}$$

$$f_\varepsilon(u_{i-1}b_i^j) = \begin{cases} 2j-1+2is+2ms, & \text{for } i \equiv 1 \pmod 2,\ 1 \le i \le m \text{ and } j=1,2,\dots,s,\\ 2j+2is+2ms, & \text{for } i \equiv 0 \pmod 2,\ 2 \le i \le m \text{ and } j=1,2,\dots,s,\end{cases}$$

$$f_\varepsilon(b_i^j u_i) = \begin{cases} 2j+2is+2ms, & \text{for } i \equiv 1 \pmod 2,\ 1 \le i \le m \text{ and } j=1,2,\dots,s,\\ 2j-1+2is+2ms, & \text{for } i \equiv 0 \pmod 2,\ 2 \le i \le m \text{ and } j=1,2,\dots,s.\end{cases}$$

Evidently, the edges are labeled with distinct numbers from 1 to $4ms+2s$.

Now we evaluate the induced vertex labels under the labeling f_ε. The weights of vertices of degree one are

$$\begin{aligned} wt_{f_\varepsilon}(a_0^j) &= f_\varepsilon(a_0^j v_0),\\ wt_{f_\varepsilon}(b_0^j) &= f_\varepsilon(b_0^j u_0) \end{aligned}$$

for $j=1,2,\dots,s$. According to (2.1) we get that they are distinct numbers from 1 to $2s$.

For the weights of vertices of a_i^j, $i=1,2,\dots,m$, $j=1,2,\dots,s$, we get

$$wt_{f_\varepsilon}(a_i^j) = f_\varepsilon(v_{i-1}a_i^j) + f_\varepsilon(a_i^j v_i) = 4j+4is-1$$

thus they are numbers from the set $\{4s+3, 4s+7, \dots, 4ms+4s-1\}$.

The weights of vertices of b_i^j, $i=1,2,\dots,m$, $j=1,2,\dots,s$ are

$$wt_{f_\varepsilon}(b_i^j) = f_\varepsilon(u_{i-1}b_i^j) + f_\varepsilon(b_i^j u_i) = 4ms+4j+4is-1,$$

i.e., they form the set $\{4ms+4s+3, 4ms+4s+7, \dots, 8ms+4s-1\}$. Thus the weights of vertices of degree two are all distinct numbers of the form $4s+4k-1$, $k=1,2,\dots,2ms$, thus they are all odd, moreover all are congruent 3 modulo 4.

Now we check the weights of vertices of degree s. We get

$$wt_{f_\varepsilon}(v_m) = \sum_{j=1}^{s} f_\varepsilon(a_m^j v_m) = \begin{cases} 2ms^2+s^2+s, & \text{when } m \text{ is odd},\\ 2ms^2+s^2, & \text{when } m \text{ is even},\end{cases}$$

$$wt_{f_\varepsilon}(u_m) = \sum_{j=1}^{s} f_\varepsilon(b_m^j u_m) = \begin{cases} 4ms^2+s^2+s, & \text{when } m \text{ is odd},\\ 4ms^2+s^2, & \text{when } m \text{ is even}.\end{cases}$$

Now we evaluate the weights of vertices of degree $2s$. For v_i, u_i, $i=1,2,\dots,m-1$ we get

$$wt_{f_\varepsilon}(v_i) = \sum_{j=1}^{s} f_\varepsilon(a_i^j v_i) + \sum_{j=1}^{s} f_\varepsilon(v_i a_{i+1}^j) = \begin{cases} 4s^2+4is^2+2s, & \text{when } i \text{ is odd},\ 1 \le i \le m-1,\\ 4s^2+4is^2, & \text{when } i \text{ is even},\ 2 \le i \le m-1,\end{cases}$$

$$wt_{f_\varepsilon}(u_i) = \sum_{j=1}^{s} f_\varepsilon(b_i^j u_i) + \sum_{j=1}^{s} f_\varepsilon(u_i b_{i+1}^j) = \begin{cases} 4ms^2+4s^2+4is^2+2s, & \\ \qquad \text{when } i \text{ is odd},\ 1 \le i \le m-1, & \\ 4ms^2+4s^2+4is^2, & \\ \qquad \text{when } i \text{ is even},\ 2 \le i \le m-1, & \end{cases}$$

thus all these weights are even numbers. Finally, according to (2.2) and (2.3) we get

$$wt_{f_\varepsilon}(v_0) = \sum_{j=1}^{s} f_\varepsilon(a_0^j v_0) + \sum_{j=1}^{s} f_\varepsilon(v_0 a_1^j) = 3s^2 + \frac{s(s+1)}{2} + \varepsilon,$$

$$wt_{f_\varepsilon}(u_0) = \sum_{j=1}^{s} f_\varepsilon(b_0^j u_0) + \sum_{j=1}^{s} f_\varepsilon(u_0 b_1^j) = 4s^2 + \frac{s(s+1)}{2} + 2ms^2 - \varepsilon.$$

Note that only the weights of vertices u_0, v_0 and a_0^j, b_0^j, $j = 1, 2, \dots, s$, depend on the value of ε.

Evidently, the weights of vertices of degree 1 are distinct and they are different (smaller) from all the other vertex weights.

Moreover, as the weights of the vertices of degree 2 are odd, they are different from the weights of the vertices v_i, u_i, $i = 1, 2, \dots, m-1$, as they are all even numbers.

It is easy to see that

$$wt_{f_\varepsilon}(v_0) < wt_{f_\varepsilon}(v_1) < wt_{f_\varepsilon}(v_2) < \dots < wt_{f_\varepsilon}(v_{m-1}) < wt_{f_\varepsilon}(u_1) < wt_{f_\varepsilon}(u_2) < \dots < wt_{f_\varepsilon}(u_{m-1}),$$
$$wt_{f_\varepsilon}(v_m) < wt_{f_\varepsilon}(u_0) < wt_{f_\varepsilon}(u_1),$$
$$wt_{f_\varepsilon}(v_0) < wt_{f_\varepsilon}(v_m) < wt_{f_\varepsilon}(u_m).$$

Now we prove that also the other vertex weights are distinct. To prove it we have to show the following.

(1) $wt_{f_\varepsilon}(u_m) \neq wt_{f_\varepsilon}(u_i)$ for every $i = 1, 2, \dots, m-1$.

This follows from the fact that

$$wt_{f_\varepsilon}(u_m) \leq 4ms^2 + s^2 + s < 4ms^2 + 8s^2 + 2s = wt_{f_\varepsilon}(u_1) < wt_{f_\varepsilon}(u_2) < \dots < wt_{f_\varepsilon}(u_{m-1}).$$

(2) $wt_{f_\varepsilon}(u_m) \neq wt_{f_\varepsilon}(v_i)$ for every $i = 1, 2, \dots, m-1$.
By contradiction. Consider that for some $t \in \{1, 2, \dots, m-1\}$ holds the equality $wt_{f_\varepsilon}(u_m) = wt_{f_\varepsilon}(v_t)$. We distinguish two subcases.

When m is odd then

$$4ms^2 + s^2 + s = wt_{f_\varepsilon}(u_m) = wt_{f_\varepsilon}(v_t) = \begin{cases} 4s^2 + 4ts^2 + 2s, & \text{when } t \text{ is odd,} \\ 4s^2 + 4ts^2, & \text{when } t \text{ is even,} \end{cases}$$

$$s(4m - 4t - 3) = \begin{cases} 1, & \text{when } t \text{ is odd,} \\ -1, & \text{when } t \text{ is even.} \end{cases}$$

However, this is not possible when $s \geq 2$.

When m is even then

$$4ms^2 + s^2 = wt_{f_\varepsilon}(u_m) = wt_{f_\varepsilon}(v_t) = \begin{cases} 4s^2 + 4ts^2 + 2s, & \text{when } t \text{ is odd,} \\ 4s^2 + 4ts^2, & \text{when } t \text{ is even,} \end{cases}$$

$$s(4m - 4t - 3) = \begin{cases} 2, & \text{when } t \text{ is odd,} \\ 0, & \text{when } t \text{ is even.} \end{cases}$$

This is not possible when $s \geq 3$.

(3) $wt_{f_\varepsilon}(u_m)$ is distinct from the weights of vertices of degree 2.

When m is odd then $wt_{f_\varepsilon}(u_m) = 4ms^2 + s(s+1)$ is even and thus it is different from the weights of vertices of degree 2 as these weights are odd.

When m and s are both even then $wt_{f_\varepsilon}(u_m) = 4ms^2 + s^2$ is even. Thus it is different from the weights of vertices of degree 2.

When m is even and s is odd then $wt_{f_\varepsilon}(u_m) = 4ms^2 + s^2 \equiv 1 \pmod 4$. As the weights of the vertices of degree 2 are congruent 3 modulo 4 we have that are distinct.

(4) $wt_{f_\varepsilon}(v_m) \neq wt_{f_\varepsilon}(v_i)$ for every $i = 1, 2, \dots, m-1$.

By contradiction. Consider that for some $t \in \{1, 2, \dots, m-1\}$ holds the equality $wt_{f_\varepsilon}(v_m) = wt_{f_\varepsilon}(v_t)$. We distinguish two subcases.

When m is odd then

$$2ms^2+s^2+s = wt_{f_\varepsilon}(v_m) = wt_{f_\varepsilon}(v_t) = \begin{cases} 4s^2+4ts^2+2s, & \text{when } t \text{ is odd,}\\ 4s^2+4ts^2, & \text{when } t \text{ is even,}\end{cases}$$

$$s(2m-4t-3) = \begin{cases} 1, & \text{when } t \text{ is odd,}\\ -1, & \text{when } t \text{ is even.}\end{cases}$$

However, this is not possible when $s \geq 2$.

When m is even then

$$2ms^2+s^2 = wt_{f_\varepsilon}(v_m) = wt_{f_\varepsilon}(v_t) = \begin{cases} 4s^2+4ts^2+2s, & \text{when } t \text{ is odd,}\\ 4s^2+4ts^2, & \text{when } t \text{ is even,}\end{cases}$$

$$s(2m-4t-3) = \begin{cases} 2, & \text{when } t \text{ is odd,}\\ 0, & \text{when } t \text{ is even.}\end{cases}$$

This is not possible when $s \geq 3$.

(5) $wt_{f_\varepsilon}(v_m)$ is distinct from the weights of vertices of degree 2.

When m is odd then $wt_{f_\varepsilon}(v_m) = 2ms^2 + s(s+1)$ is even and thus it is different from the weights of vertices of degree 2 as these weights are odd.

When m and s are both even then $wt_{f_\varepsilon}(v_m) = 2ms^2 + s^2$ is even. Thus it is different from the weights of vertices of degree 2.

When m is even and s is odd then $wt_{f_\varepsilon}(v_m) = 2ms^2 + s^2 \equiv 1 \pmod 4$. As the weights of vertices are congruent 3 modulo 4 we have that they are distinct.

(6) Now we prove that for at least one integer ε^* from the set $\{0,1,2,3\}$ under the labeling f_{ε^*} the weight of the vertex v_0 is different from the weights of vertices of degree 2, the weight of the vertex u_0 is different from the weights of vertices of degree 2 and also $wt_{f_{\varepsilon^*}}(u_0) \neq wt_{f_{\varepsilon^*}}(v_i)$ for every $i = 1,2,\dots,m-1$.

But this follows from the fact that the difference between two weights of vertices of degree 2 is four and the difference between the weights of vertices v_i, $i = 1,2,\dots,m-1$, is at least $4s^2-2s$.

(7) Finally we show that $wt_{f_\varepsilon}(u_0) \neq wt_{f_\varepsilon}(u_m)$.

When $m \geq 2$ then

$$\begin{aligned} wt_{f_\varepsilon}(u_0) &= 4s^2 + \frac{s(s+1)}{2} + 2ms^2 - \varepsilon \leq 4s^2 + \frac{s(s+1)}{2} + 2ms^2 < 4ms^2+s^2 \\ &\leq wt_{f_\varepsilon}(u_m). \end{aligned}$$

When $m = 1$ then the weights of all non pendant vertices are the following

$$\begin{aligned} wt_{f_\varepsilon}(v_0) &= 3s^2 + \tfrac{s(s+1)}{2},\\ wt_{f_\varepsilon}(v_1) &= 3s^2 + s,\\ wt_{f_\varepsilon}(u_0) &= 6s^2 + \tfrac{s(s+1)}{2},\\ wt_{f_\varepsilon}(u_1) &= 5s^2 + s. \end{aligned}$$

Since $s \geq 3$, all vertices have distinct weights.

This concludes the proof. □

Figure 2 illustrates an antimagic labeling of $K_{1,3} \times P_8$.

Combing Lemmas 2.1, 2.2 and Theorem 2.1 we get the following result for the direct product of a star with a path on even number of vertices.

Theorem 2.2. *The graph $K_{1,s} \times P_{2m+2}$ is antimagic for $s \geq 2$, $m \geq 0$ except the case when $(s,n) = (2,2)$.*

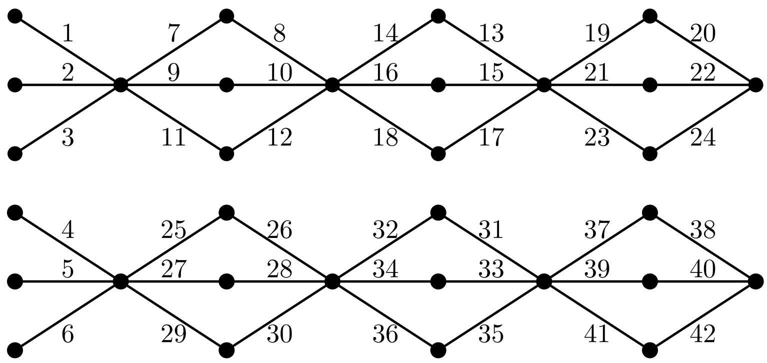

Figure 2. An antimagic labeling of of $K_{1,3} \times P_8$.

3. A path on odd number of vertices

In this section we consider the direct product of a star with a path on odd number of vertices. Also in this case the graph $K_{1,s} \times P_{2m+1}$ consist of two connected components however, these components are not isomorphic. Let us denote the vertices and edges of $K_{1,s} \times P_{2m+1}$, $s \geq 2$, $m \geq 1$, in the following way

$$\begin{aligned} V(K_{1,s} \times P_{2m+1}) = &\left\{a_i^j, u_i : i = 0,1,\dots,m, j = 1,2,\dots,s\right\} \\ &\cup \left\{b_i^j : i = 1,2,\dots,m, j = 1,2,\dots,s\right\} \cup \{v_i : i = 0,1,\dots,m-1\}, \\ E(K_{1,s} \times P_{2m+1}) = &\left\{a_i^j v_i, a_{i+1}^j v_i : i = 0,1,\dots,m-1, j = 1,2,\dots,s\right\} \\ &\cup \left\{b_i^j u_i, b_i^j u_{i-1} : i = 1,2,\dots,m, j = 1,2,\dots,s\right\}. \end{aligned}$$

The structure of graph is shown in Figure 3.

First we solve the small cases $K_{1,s} \times P_3$ and $K_{1,s} \times P_5$.

Lemma 3.1. *The graph $K_{1,s} \times P_3$ is antimagic for $s \geq 2$.*

Proof. Let us define an edge labeling f of $K_{1,s} \times P_3$ such that:

$$\begin{aligned} f(a_0^j v_0) =& j, && \text{for } j = 1,2,\dots,s, \\ f(v_0 a_1^j) =& s + j, && \text{for } j = 1,2,\dots,s, \\ f(u_1 b_1^j) =& 2s + 2j - 1, && \text{for } j = 1,2,\dots,s, \\ f(u_0 b_1^j) =& 2s + 2j, && \text{for } j = 1,2,\dots,s. \end{aligned}$$

The induced vertex weights are

$$\begin{aligned} wt_f(a_0^j) =& f(a_0^j v_0) = j, && \text{for } j = 1,2,\dots,s, \\ wt_f(a_1^j) =& f(v_0 a_1^j) = s + j, && \text{for } j = 1,2,\dots,s, \\ wt_f(b_1^j) =& \sum_{j=1}^{s} f(u_0 b_1^j) + \sum_{j=1}^{s} f(b_1^j u_1) = 4s + 4j - 1, && \text{for } j = 1,2,\dots,s, \\ wt_f(v_0) =& \sum_{j=1}^{s} f(a_0^j v_0) + \sum_{j=1}^{s} f(a_1^j v_0) = 2s^2 + s, \\ wt_f(u_1) =& \sum_{j=1}^{s} f(u_1 b_1^j) = 3s^2, \end{aligned}$$

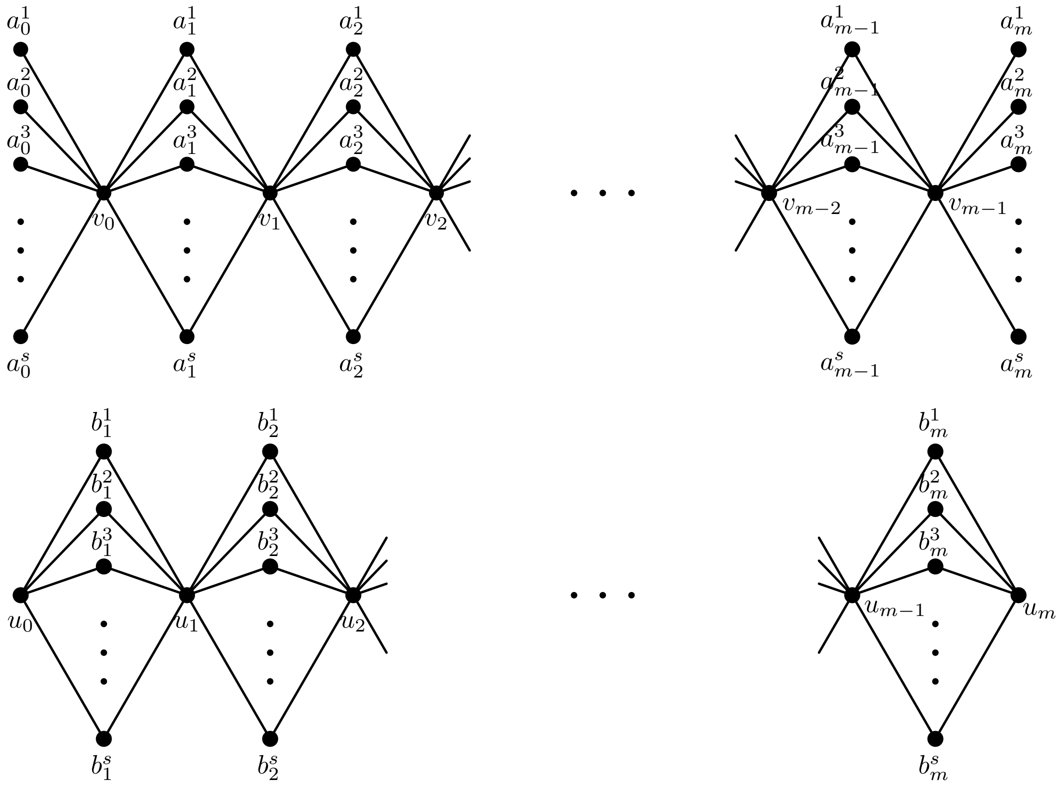

Figure 3. The general representation of graph $K_{1,s} \times P_{2m+1}$.

$$wt_f(u_0) = \sum_{j=1}^{s} f(u_0 b_1^j) = 3s^2 + s.$$

Clearly, for $s \geq 2$ all the weights are distinct. □

Lemma 3.2. *The graph $K_{1,s} \times P_5$ is antimagic for $s \geq 2$.*

Proof. In this case consider an edge labeling f of $K_{1,s} \times P_5$ in the following way

$$\begin{aligned}
f(a_0^j v_0) &= j, && \text{for } j = 1, 2, \ldots, s,\\
f(v_1 a_2^j) &= s + j, && \text{for } j = 1, 2, \ldots, s,\\
f(v_1 a_1^j) &= 2s + 2j - 1, && \text{for } j = 1, 2, \ldots, s,\\
f(v_0 a_1^j) &= 2s + 2j, && \text{for } j = 1, 2, \ldots, s,\\
f(u_1 b_1^j) &= 4s + 2j - 1, && \text{for } j = 1, 2, \ldots, s,\\
f(u_0 b_1^j) &= 4s + 2j, && \text{for } j = 1, 2, \ldots, s,\\
f(u_1 b_2^j) &= 6s + 2j - 1, && \text{for } j = 1, 2, \ldots, s,\\
f(u_2 b_2^j) &= 6s + 2j, && \text{for } j = 1, 2, \ldots, s.
\end{aligned}$$

Evidently f is a bijection. The vertex weights are

$$\begin{aligned}
wt_f(a_0^j) &= f(a_0^j v_0) = j, && \text{for } j = 1, 2, \ldots, s,\\
wt_f(a_2^j) &= f(a_2^j v_1) = s + j, && \text{for } j = 1, 2, \ldots, s,\\
wt_f(a_1^j) &= \sum_{j=1}^{s} f(v_0 a_1^j) + \sum_{j=1}^{s} f(v_1 a_1^j) = 4s + 4j - 1, && \text{for } j = 1, 2, \ldots, s,
\end{aligned}$$

$$wt_f(b_1^j) = \sum_{j=1}^{s} f(u_0b_1^j) + \sum_{j=1}^{s} f(u_1b_1^j) = 8s + 4j - 1, \qquad \text{for } j = 1, 2, \dots, s,$$
$$wt_f(b_2^j) = \sum_{j=1}^{s} f(u_1b_2^j) + \sum_{j=1}^{s} f(u_2b_2^j) = 12s + 4j - 1, \qquad \text{for } j = 1, 2, \dots, s,$$
$$wt_f(v_0) = \sum_{j=1}^{s} f(v_0a_0^j) + \sum_{j=1}^{s} f(v_0a_1^j) = \frac{7s^2 + 3s}{2},$$
$$wt_f(v_1) = \sum_{j=1}^{s} f(v_1a_1^j) + \sum_{j=1}^{s} f(v_1a_2^j) = \frac{9s^2 + s}{2},$$
$$wt_f(u_0) = \sum_{j=1}^{s} f(u_0b_1^j) = 5s^2 + s,$$
$$wt_f(u_2) = \sum_{j=1}^{s} f(u_2b_2^j) = 7s^2 + s,$$
$$wt_f(u_1) = \sum_{j=1}^{s} f(u_1b_1^j) + \sum_{j=1}^{s} f(u_1b_2^j) = 12s^2.$$

It is clear from the above vertex sums that for $s \geq 2$ the weights of the vertices are distinct. $\square$

Theorem 3.1. *The graph $K_{1,s} \times P_{2m+1}$ is antimagic for $s \geq 2$, $m \geq 3$.*

Proof. Let us define an edge labeling f_ε, $\varepsilon \in \{0, 1 \dots, s^2\}$, of $K_{1,s} \times P_{2m+1}$, $s \geq 2$, $m \geq 3$, such that the pendant edges are labeled as follows:

$$\{f_\varepsilon(a_0^jv_0), f_\varepsilon(v_{m-1}a_m^j) : j = 1, 2, \dots, s\} = \{1, 2, \dots, 2s\} \tag{3.1}$$

and

$$\sum_{j=1}^{s} f_\varepsilon(a_0^jv_0) = \frac{s(s+1)}{2} + \varepsilon, \tag{3.2}$$

thus

$$\sum_{j=1}^{s} f_\varepsilon(a_m^jv_{m-1}) = s^2 + \frac{s(s+1)}{2} - \varepsilon. \tag{3.3}$$

The labels of the remaining edges are

$$f_\varepsilon(v_{i-1}a_i^j) = \begin{cases} 2j + 2is, & \text{for } i \equiv 1 \pmod 2,\ 1 \leq i \leq m-1 \text{ and } j = 1, 2, \dots, s, \\ 2j + 2is - 1, & \text{for } i \equiv 0 \pmod 2,\ 2 \leq i \leq m-1 \text{ and } j = 1, 2, \dots, s, \end{cases}$$
$$f_\varepsilon(a_i^jv_i) = \begin{cases} 2j + 2is - 1, & \text{for } i \equiv 1 \pmod 2,\ 1 \leq i \leq m-1 \text{ and } j = 1, 2, \dots, s, \\ 2j + 2is, & \text{for } i \equiv 0 \pmod 2,\ 2 \leq i \leq m-1 \text{ and } j = 1, 2, \dots, s, \end{cases}$$
$$f_\varepsilon(u_{i-1}b_i^j) = \begin{cases} 2j + 2is + 2(m-1)s, & \text{for } i \equiv 1 \pmod 2,\ 1 \leq i \leq m \text{ and } j = 1, 2, \dots, s, \\ 2j + 2is + 2(m-1)s - 1, & \text{for } i \equiv 0 \pmod 2,\ 2 \leq i \leq m \text{ and } j = 1, 2, \dots, s, \end{cases}$$
$$f_\varepsilon(b_i^ju_i) = \begin{cases} 2j + 2is + 2(m-1)s - 1, & \text{for } i \equiv 1 \pmod 2,\ 1 \leq i \leq m \text{ and } j = 1, 2, \dots, s, \\ 2j + 2is + 2(m-1)s & \text{for } i \equiv 0 \pmod 2,\ 2 \leq i \leq m \text{ and } j = 1, 2, \dots, s. \end{cases}$$

Evidently, the edge labels are distinct numbers from 1 to $4ms$. Now, we evaluate the induced vertex labels under the function f_ε. The weights of vertices of degree one is

$$wt_{f_\varepsilon}(a_0^j) = f_\varepsilon(a_0^jv_0),$$
$$wt_{f_\varepsilon}(a_0^m) = f_\varepsilon(v_{m-1}a_0^j)$$

for $j = 1, 2, \dots, s$. According to (3.1) we get that they are distinct numbers from 1 to $2s$.

For the weights of vertices of a_i^j, $i = 1, 2, \dots, m-1$, $j = 1, 2, \dots, s$, we obtain

$$wt_{f_\varepsilon}(a_i^j) = f_\varepsilon(v_{i-1}a_i^j) + f_\varepsilon(a_i^j v_i) = 4j + 4is - 1$$

thus they are odd numbers from the set $\{4s+3, 4s+7, \dots, 4ms-1\}$, i.e., they are congruent 3 modulo 4.

For the weights of vertices of b_i^j, $i = 1, 2, \dots, m$, $j = 1, 2, \dots, s$, we have

$$wt_{f_\varepsilon}(b_i^j) = f_\varepsilon(u_{i-1}b_i^j) + f_\varepsilon(b_i^j u_i) = 4ms + 4is + 4j - 4s - 1$$

which are odd numbers from the set $\{4ms+3, 4ms+7, \dots, 8ms-1\}$. Again these numbers are congruent 3 modulo 4.

Now we check the weights of vertices of degree $2s$. For v_i, $i = 1, 2, \dots, m-2$, and u_i, $i = 1, 2, \dots, m-1$, we get

$$wt_{f_\varepsilon}(v_i) = \sum_{j=1}^{s} f_\varepsilon(a_i^j v_i) + \sum_{j=1}^{s} f_\varepsilon(v_i a_{i+1}^j) = \begin{cases} 4is^2 + 4s^2, & \text{when } i \text{ is odd, } 1 \le i \le m-2, \\ 4is^2 + 4s^2 + 2s, & \text{when } i \text{ is even, } 2 \le i \le m-2, \end{cases}$$

$$wt_{f_\varepsilon}(u_i) = \sum_{j=1}^{s} f_\varepsilon(b_i^j u_i) + \sum_{j=1}^{s} f_\varepsilon(u_i b_{i+1}^j) = \begin{cases} 4ms^2 + 4s^2 i, & \text{when } i \text{ is odd}, 1 \le i \le m-1 \\ 4ms^2 + 4s^2 i + 2s, & \text{when } i \text{ is even}, 2 \le i \le m-1. \end{cases}$$

Next we evaluate the weights of vertices of degree s. We get

$$wt_{f_\varepsilon}(u_0) = \sum_{j=1}^{s} f_\varepsilon(b_1^j u_0) = 2ms^2 + s^2 + s,$$

$$wt_{f_\varepsilon}(u_m) = \sum_{j=1}^{s} f_\varepsilon(b_m^j u_m) = \begin{cases} 4ms^2 - s^2, & \text{when } m \text{ is odd,} \\ 4ms^2 - s^2 + s, & \text{when } m \text{ is even.} \end{cases}$$

Finally, according to (3.2) and (3.3). We get

$$wt_{f_\varepsilon}(v_0) = \sum_{j=1}^{s} f_\varepsilon(a_0^j v_0) + \sum_{j=1}^{s} f_\varepsilon(a_1^j v_0) = \frac{s(s+1)}{2} + 3s^2 + s + \varepsilon,$$

$$wt_{f_\varepsilon}(v_{m-1}) = \sum_{j=1}^{s} f_\varepsilon(a_m^j v_{m-1}) + \sum_{j=1}^{s} f_\varepsilon(a_{m-1}^j v_{m-1}) = \begin{cases} 2ms^2 + \frac{s(s+1)}{2} - \varepsilon, & \text{when } m \text{ is even,} \\ 2ms^2 + \frac{s(s+1)}{2} + s - \varepsilon, & \text{when } m \text{ is odd.} \end{cases}$$

Note that only the weights of vertices v_0, v_{m-1} and a_0^j, a_m^j, $j = 1, 2, \dots, s$, depend on the value of ε.

Evidently, the weights of the vertices of degree 1 are distinct and they are different (smaller) from all the other vertex weights.

Moreover, as the weights of the vertices of degree 2 are odd, they are different from the weights of the vertices v_i, $i = 1, 2, \dots, m-2$ and u_i, $i = 1, 2, \dots, m-1$, as they are all even numbers. Moreover, it is easy to see that

$$wt_{f_\varepsilon}(v_0) < wt_{f_\varepsilon}(v_1) < wt_{f_\varepsilon}(v_2) < \dots < wt_{f_\varepsilon}(v_{m-2}) < wt_{f_\varepsilon}(u_1) < wt_{f_\varepsilon}(u_2) < \dots < wt_{f_\varepsilon}(u_{m-1}),$$
$$wt_{f_\varepsilon}(u_0) < wt_{f_\varepsilon}(u_m) < wt_{f_\varepsilon}(u_1).$$

Moreover, for $m \ge 3$ and $s \ge 2$ we get that for every $\varepsilon \in \{0, 1 \dots, s^2\}$ also holds

$$wt_{f_\varepsilon}(v_0) < wt_{f_\varepsilon}(v_{m-1}) < wt_{f_\varepsilon}(u_0).$$

Now, we prove that also the other vertex weights are distinct. To prove it we show the following.

(1) $wt_{f_\varepsilon}(u_m) \ne wt_{f_\varepsilon}(v_i)$ for every $i = 1, 2, \dots, m-2$. This follows from the fact that,

$$wt_{f_\varepsilon}(v_{m-2}) \le 4ms^2 - 4s^2 + 2s < 4ms^2 - s^2 \le wt_{f_\varepsilon}(u_m).$$

(2) $wt_{f_\varepsilon}(u_m)$ is distinct from the weight of the vertices of degree 2. When m is even then $wt_{f_\varepsilon}(u_m) = 4ms^2 - s^2 + s$ is even. Also when m is odd and s is even we get that $wt_{f_\varepsilon}(u_m) = 4ms^2 - s^2$ is even. Thus in these cases $wt_{f_\varepsilon}(u_m)$ is different from the weights of vertices of degree 2 as these weights are odd.

When both m and s are odd, $s \geq 3$ then

$$wt_{f_\varepsilon}(u_m) = s^2(4m-1) \geq 3s(4m-1) > 8ms > 8ms - 1 = wt_{f_\varepsilon}(b_m^s).$$

Thus also in this case $wt_{f_\varepsilon}(u_m)$ is different from the weights of the vertices of degree 2.

(3) $wt_{f_\varepsilon}(u_0) \neq wt_{f_\varepsilon}(v_i)$ for every $i = 1, 2, \dots, m-2$.

By contradiction. Consider that there exists $t \in \{1, 2, \dots, m-2\}$ such that $wt_{f_\varepsilon}(u_0) = wt_{f_\varepsilon}(v_t)$. Then

$$2ms^2 + s^2 + s = wt_{f_\varepsilon}(u_0) = wt_{f_\varepsilon}(v_t) = \begin{cases} 4ts^2 + 4s^2 + 2s, & \text{when } t \text{ is even}, \\ 4ts^2 + 4s^2, & \text{when } t \text{ is odd}, \end{cases}$$

$$s(2m - 4t - 3) = \begin{cases} 1, & \text{when } t \text{ is even}, \\ -1, & \text{when } t \text{ is odd}. \end{cases}$$

However, this is not possible when $s \geq 2$.

(4) $wt_{f_\varepsilon}(u_0)$ is distinct from the weight of the vertices of degree 2.

This follows from the fact that $wt_{f_\varepsilon}(u_0) = 2ms^2 + s^2 + s$ is even and the weights of the vertices of degree 2 are odd.

(5) Now, we prove that for at least one integer ε^* from the set $\{0, 1, 2, 3\}$ under the labeling f_{ε^*} the weight of the vertex v_0 is different from the weights of the vertices of degree 2, the weight of the vertex v_{m-1} is different from the weight of the vertices of degree 2 and also $wt_{f_\varepsilon}(v_{m-1}) \neq wt_{f_\varepsilon}(v_i)$ for every $i = 1, 2, \dots, m-2$.

But this follows from the fact that the difference between two weights of vertices of degree 2 is four and the difference between the weight of vertices v_i, $i = 1, 2, \dots, m-2$, is at least $4s^2 + 2s$.

Figure 4 illustrates an antimagic labeling of $K_{1,3} \times P_7$.

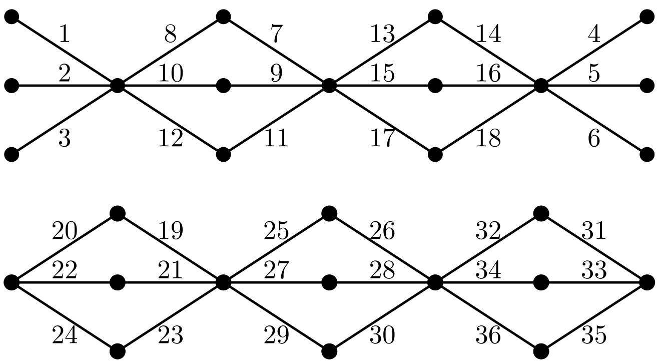

Figure 4. An antimagic labeling of of $K_{1,3} \times P_7$.

Combining the previous the proof is completed. □

According to Lemmas 3.1, 3.2 and Theorem 3.1 we obtain the following result for the direct product of a star and an odd path.

Theorem 3.2. *The graph $K_{1,s} \times P_{2m+1}$ is antimagic for $s \geq 2$, $m \geq 1$.*

4. Conclusion

Our main motivation in this paper is to study antimagicness of the disconnected graphs which are constructed using some known graph operations. We used the direct product as an operation to construct new classes of disconnected graphs. We gave a characterization of antimagicness of the direct product of a star and a path. As the main result we obtained that the graph $K_{1,s} \times P_n$ is antimagic for all positive integers $s \geq 1$, $n \geq 2$ except three cases when $(s, n) \in \{(1, 2), (1, 3), (2, 2)\}$.

Examining the antimagicness of the direct product of a star and a cycle presents a logical direction for further study. Note that the resulting graph is disconnected if and only if the cycle is even. In light of the Antimagic Conjecture we present the following problem.

Problem 4.1. Prove that the graph $K_{1,s} \times C_n$ is antimagic.

Finally, we conclude our paper with the following question.

Problem 4.2. What are the other classes of disconnected graphs constructed from the direct product of graphs or from any graph operation which admit an antimagic labeling?

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