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LYAPUNOV-TYPE INEQUALITIES FOR LINEAR HYPERBOLIC AND ELLIPTIC EQUATIONS ON A RECTANGULAR DOMAIN

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Abstract. In the case of oscillatory potential, we present some new Lyapunov -type inequalities for linear hyperbolic and elliptic equations on a rectangular domain in $\mathbb{R}^2$. No sign restriction is imposed on the potential function. As applications of the Lyapunov-type inequalities obtained, we give some estimations for disconjugacy of hyperbolic and elliptic Dirichlet boundary value problems.

1. Introduction

In the paper, we first obtain a Lyapunov-type inequality for the linear hyperbolic equation of the form

$$u_{tt}(x,t) - u_{xx}(x,t) + q(t)u(x,t) = 0, \qquad (x,t) \in \mathcal{R} \tag{1}$$

satisfying the Dirichlet boundary condition

$$u(x,t) = 0, \qquad (x,t) \in \partial\mathcal{R}, \tag{2}$$

where

$$\mathcal{R} = \{(x,t) : x \in [x_1, x_2], t \in [t_1, t_2]\}, \tag{3}$$

and that no sign restriction is imposed on the potential function $q(t) \in \mathrm{L}^1[t_1, t_2]$.

Secondly, we give an analogous result for the linear elliptic equation of the form

$$u_{tt}(x,t) + u_{xx}(x,t) + q(t)u(x,t) = 0, \qquad (x,t) \in \mathcal{R} \tag{4}$$

satisfying the Dirichlet boundary condition (2).

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The well-known Lyapunov inequality [13] for Hill's equation

$$x''(t) + \nu(t)x(t) = 0 \tag{5}$$

states that if t_1 and t_2 ($t_1 < t_2$) are consecutive zeros of a nontrivial solution $x(t)$ of this equation, then the inequality

$$\int_{t_1}^{t_2} |\nu(t)|\mathrm{d}t > \frac{4}{t_2 - t_1} \tag{6}$$

holds. Inequality (6) was later strengthened by replacement of $|\nu|$ by ν^+, i.e.,

$$\int_{t_1}^{t_2} \nu^+(t)\mathrm{d}t > \frac{4}{t_2 - t_1}, \tag{7}$$

cf. Wintner [17], and thereafter by some other authors, where $\nu^+ = \max\{\nu, 0\}$. Inequality (7) is the best possible in the sense that the constant "4" can not be replaced by any larger constant in (7) due to Hartman [8, Theorem 5.1]. Inequalities (6) and (7) and their several generalizations to Hamiltonian systems, higher order differential equations, nonlinear and half-linear differential equations, difference and dynamic equations, functional and impulsive differential equations, have found many applications in areas like oscillation and Sturmian theory, disconjugacy, asymptotic theory, eigenvalue problems, boundary value problems, and various properties of the solutions of related differential equations, see [5, 12, 16] and their references. We also refer reader to recently published monograph by Agarwal et al. [1] for the historical development of Lyapunov inequalities and its applications.

The classical result of Lyapunov is usually connected with the disconjugacy of Eq. (5), i.e. the inequality

$$\int_{t_1}^{t_2} \nu^+(t)\mathrm{d}t \leq \frac{4}{t_2 - t_1} \tag{8}$$

implies that (5) is disconjugate in $[t_1, t_2]$.

There has been an increasing interest for the Lyapunov-type inequalities for partial differential equations in the last few decades; see for example [2–4, 6, 7, 9, 10, 14, 15] and their references. In 2006, Canada et al. [2] considered the linear partial differential equations

$$\begin{cases} -\Delta u(x) = a(x)u(x), & x \in \Omega \\ \dfrac{\partial u}{\partial n}(x) = 0, & x \in \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ ($N \geq 2$) is a bounded and regular domain and the function $a : \Omega \to \mathbb{R}$. They proved how the relation between the quantity p and $N/2$ play a crucial role by considering the sub-critical ($1 < p < N/2$), super-critical ($p > N/2$) and the critical ($p = N/2$) cases.

In 2016, de Nápoli and Pinasco [7] proved Lyapunov-type inequalities for the p-Laplacian equations

$$\begin{cases} \Delta_p u(x) + w(x)|u(x)|^{p-2}u(x) = 0, & x \in \Omega \\ u(x) = 0, & x \in \partial\Omega, \end{cases}$$

where $p > 1$ and the weight function $w \in \mathrm{L}^s$ for some s depending on p and N. They obtained Lyapunov-type inequalities for two separate cases $p < N$ and $p > N$, and the case $p = N$ was given as an open problem for the reader. Recently, Kumar and Tyagi [11] solved this open problem and established Lyapunov-type inequality for a class of the following N-Laplace equations:

$$\begin{cases} \Delta_N v(x) + f(x)|v(x)|^{N-2}v(x) = 0, & x \in \Omega, \\ v(x) = 0, & x \in \partial\Omega \end{cases}$$

under some conditions on μ, g, R and b, where

$$f(x) = \mu g(x) \left(|x| \log \frac{R}{|x|} \right)^{-N} + b(x).$$

In 2020, Jleli at al. [10] established Lyapunov-type inequalities for the partial differential equations of the form

$$\begin{cases} -G_\gamma u(x,y) = w(x)u(x,y), & (x,y) \in \Gamma \\ u(x,y) = 0, & (x,y) \in \partial\Gamma, \end{cases}$$

where $\Gamma = (a,b) \times \mathcal{O}$; $(a,b) \in \mathbb{R}^2$ and $\mathcal{O}$ is an open bounded subset in $\mathbb{R}^N$ for $N \geq 1$. Here G_γ is the Grushin operator

$$G_\gamma u(x,y) = \frac{\partial^2 u}{\partial x^2}(x,y) + x^{2\gamma} \Delta_y u(x,y), \qquad (x,y) \in \Gamma.$$

When $\gamma = 0$, Grushin operator reduces to the standart Laplace operator but the presence of $x^{2\gamma}$, this operator can not be elliptic on Γ. In 2020, Jleli et al. [10] proved the Lyapunov-type inequality for the Grushin operator via sign change criteria.

In this paper, we obtain a Lyapunov-type inequality for the hyperbolic equation (1) satisfying the Dirichlet boundary condition (2). Moreover, we extend this result to the elliptic equation (4) satisfying the Dirichlet boundary condition (2). To obtain such type of inequalities, we use the separation of variables technique in problems both (1)–(2) and (4)–(2). In Section 3, we present several examples which illustrate how easily the results obtained can be applied to the related equations. At the end of the paper, we impose some open problems.

2. Main Results

Throughout this section, we denote $h^+ = \max\{h, 0\}$ and we shall assume that the potential q is in the set $\mathrm{L}^1[t_1, t_2]$.

Now, let us restate Prb. (1)–(2) as

$$\begin{cases} u_{tt}(x,t) - u_{xx}(x,t) + q(t)u(x,t) = 0, & (x,t) \in \mathcal{R}; \\ u(x,t_1) = u(x,t_2) = 0, & x_1 \le x \le x_2; \\ u(x_1,t) = u(x_2,t) = 0, & t_1 \le t \le t_2, \end{cases} \tag{9}$$

where $\mathcal{R}$ is defined in (3).

The first main result of the paper is the following.

Theorem 1 (Lyapunov-type inequality)**.** *If u is a nontrivial solution of Prb.* (9), *then the inequality*

$$\int_{t_1}^{t_2} \left[(x_2 - x_1)^2 q(t) + \pi^2\right]^+ \mathrm{d}t > \frac{4}{t_2 - t_1}(x_2 - x_1)^2 \tag{10}$$

holds.

Proof. Let u be a nontrivial solution of Prb. (9). The method of separation of variables starts by looking the solutions of Eq. (1) of the form

$$u(x,t) = y(x)z(t), \tag{11}$$

where the variables separate with $y(x) \not\equiv 0$ on (x_1,x_2) and $z(t) \not\equiv 0$ on (t_1,t_2). Substituting (11) in (1), we obtain

$$y(x)z''(t) - y''(x)z(t) + q(t)y(x)z(t) = 0 \tag{12}$$

for $x \in (x_1,x_2)$ and $t \in (t_1,t_2)$. Since $y(x)z(t) \not\equiv 0$, dividing both sides of (12) by it, we separate the variables x and t as

$$\frac{z''(t)}{z(t)} + q(t) = \frac{y''(x)}{y(x)}. \tag{13}$$

The left-hand side of (13) is a function of t only, whereas the right-hand side has just x. But x and t are independent variables so (13) is possible only when both sides of it are constant; that is

$$\frac{z''(t)}{z(t)} + q(t) = \frac{y''(x)}{y(x)} = \lambda \tag{14}$$

for some real number λ. On the other hand, we have boundary conditions to be satisfied. The first boundary conditions in (9) imply that

$$y(x)z(t_1) = 0 \quad \text{and} \quad y(x)z(t_2) = 0 \tag{15}$$

for all $x \in (x_1,x_2)$. Since $y(x) \not\equiv 0$ on (x_1,x_2), (15) is possible only when $z(t_1) = z(t_2) = 0$. Applying the similar argument to the second boundary conditions in (9), we must have $y(x_1) = y(x_2) = 0$. Using these conditions together with (14), we can conclude that $z(t)$ is a nontrivial solution of the boundary value problem

$$\begin{cases} z''(t) + [q(t) - \lambda]z(t) = 0, \\ z(t_1) = z(t_2) = 0 \end{cases} \tag{16}$$

and $y(x)$ is a nontrivial solution of the boundary value problem

$$\begin{cases} y''(x) - \lambda y(x) = 0, \\ y(x_1) = y(x_2) = 0. \end{cases} \tag{17}$$

We note that t_1, t_2 ($t_1 < t_2$) and x_1, x_2 ($x_1 < x_2$) are consecutive zeros of $z(t)$ and $y(x)$, respectively. Now consider the boundary value problem

$$\begin{cases} w''(x) + kw(x) = 0, \\ w(x_1) = w(x_2) = 0, \end{cases} \tag{18}$$

where k is a constant. It is known that the eigenvalues k_n of Prb. (18) are

$$k_n = \frac{n^2\pi^2}{(x_2 - x_1)^2}, \qquad n = 1, 2, \ldots,$$

and hence the smallest eigenvalue of it is $k_1 = \pi^2/(x_2 - x_1)^2$. Since Prb. (17) has a nontrivial solution, we take $\lambda = -k_1$. Now replacing λ by $-k_1$ in Prb. (16), it turns out that

$$\begin{cases} z''(t) + Q(t)z(t) = 0, \\ z(t_1) = z(t_2) = 0, \end{cases} \tag{19}$$

where

$$Q(t) = q(t) + \frac{\pi^2}{(x_2 - x_1)^2}.$$

Applying Lyapunov's result to Prb. (19), we see that inequality (10) holds. The proof of Theorem 1 is complete. □

In case

$$q(t) > -\frac{\pi^2}{(x_2 - x_1)^2} \quad \text{for} \quad t \in (t_1, t_1),$$

inequality (10) turns out to be

$$\int_{t_1}^{t_2} \left[(x_2 - x_1)^2 q(t) + \pi^2\right] \mathrm{d}t > \frac{4}{t_2 - t_1}(x_2 - x_1)^2$$

which requires that

$$\int_{t_1}^{t_2} q(t)\mathrm{d}t > \frac{4}{t_2 - t_1} - \frac{t_2 - t_1}{(x_2 - x_1)^2}\pi^2.$$

Corollary 1. *If the inequality*

$$\int_{t_1}^{t_2} \left[(x_2 - x_1)^2 q(t) + \pi^2\right]^+ \mathrm{d}t \le \frac{4}{t_2 - t_1}(x_2 - x_1)^2 \tag{20}$$

holds, then Prb. (9) *has no nontrivial solution.*

Now consider the elliptic equation (4) satisfying the Dirichlet boundary condition (2) by restating it as

$$\begin{cases} u_{tt}(x,t) + u_{xx}(x,t) + q(t)u(x,t) = 0, & (x,t) \in \mathcal{R}; \\ u(x,t_1) = u(x,t_2) = 0, & x_1 \le x \le x_2, \\ u(x_1,t) = u(x_2,t) = 0, & t_1 \le t \le t_2, \end{cases} \tag{21}$$

where $\mathcal{R}$ is defined in (3).

The following is the second main result of the paper.

Theorem 2 (Lyapunov-type inequality)**.** *If u is a nontrivial solution of Prb.* (21)*, then the inequality*

$$\int_{t_1}^{t_2} \left[(x_2-x_1)^2 q(t) - \pi^2\right]^+ \mathrm{d}t > \frac{4}{t_2-t_1}(x_2-x_1)^2 \tag{22}$$

holds.

Proof. The proof Theorem 2 is analogous to that of Theorem 1, and hence it is left to the reader. □

When

$$q(t) > \frac{\pi^2}{(x_2-x_1)^2} \quad \text{for} \quad t \in (t_1,t_1),$$

inequality (10) turns out to be

$$\int_{t_1}^{t_2} \left[(x_2-x_1)^2 q(t) - \pi^2\right] \mathrm{d}t > \frac{4}{t_2-t_1}(x_2-x_1)^2$$

which requires that

$$\int_{t_1}^{t_2} q(t)\mathrm{d}t > \frac{4}{t_2-t_1} + \frac{t_2-t_1}{(x_2-x_1)^2}\pi^2.$$

Corollary 2. *If the inequality*

$$\int_{t_1}^{t_2} \left[(x_2-x_1)^2 q(t) - \pi^2\right]^+ \mathrm{d}t \le \frac{4}{t_2-t_1}(x_2-x_1)^2 \tag{23}$$

holds, then Prb. (21) *has no nontrivial solution.*

3. Applications

In this section, we give some disconjugacy estimations for hyperbolic and elliptic Dirichlet boundary value problems, by applying the Lyapunov-type inequalities obtained in Section 2.

Example 1. Consider the hyperbolic boundary value problem

$$\begin{cases} u_{tt}(x,t) - u_{xx}(x,t) + (1-\pi^2)u(x,t) = 0, & (x,t) \in \mathcal{R}_0, \\ u(x,0) = u(x,\pi) = 0, & 0 \le x \le 1, \\ u(0,t) = u(1,t) = 0, & 0 \le t \le \pi, \end{cases} \tag{24}$$

where $\mathcal{R}_0$ is the rectangular region

$$\mathcal{R}_0 = \{(x,t) : x \in [0,1], t \in [0,\pi]\}.$$

Substituting $q(t) = 1 - \pi^2$, $x_2 - x_1 = 1$ and $t_2 - t_1 = \pi$ in Lyapunov-type inequality (10), we see that it is satisfied by $\pi^2 > 4$. Note that the solution of Prb. (24) is the function $u(x,t) = \sin(\pi x)\sin t$.

Example 2. Consider the hyperbolic boundary value problem

$$\begin{cases} u_{tt}(x,t) - u_{xx}(x,t) + \mu(2 + t - t^2)u(x,t) = 0, & (x,t) \in \mathcal{R}_1, \\ u(x,0) = u(x,1) = 0, & 0 \le x \le \pi, \\ u(0,t) = u(\pi,t) = 0, & 0 \le t \le 1, \end{cases} \tag{25}$$

where μ is a positive constant and $\mathcal{R}_1$ is the rectangular region

$$\mathcal{R}_1 = \{(x,t) : x \in [0,\pi], t \in [0,1]\}. \tag{26}$$

In the view of Lyapunov-type inequality (10), the following inequality must be satisfied:

$$\int_0^1 h^+(t;\mu)\mathrm{d}t > 4, \qquad (\mu > 0) \tag{27}$$

where $h(t;\mu) = 2\mu + 1 + \mu t - \mu t^2$. It can be shown that $h(t;\mu) > 0$ for all $t \in [-1,2]$, and hence (27) turns to

$$\int_0^1 h(t;\mu)\mathrm{d}t = \int_0^1 [2\mu + 1 + \mu t - \mu t^2]\mathrm{d}t = \frac{13}{6}\mu + \frac{1}{6} > 4. \tag{28}$$

So Prb. (25) has no nontrivial solution, if $\mu \le 23/13 \approx 1,76923$ by Corollary 1. In particular if $\mu = 4$, then Prb. (25) has a nontrivial solution

$$u(x,t) = t(1-t)e^{t(1-t)}\sin x, \qquad (x,t) \in \mathcal{R}_1.$$

Example 3. Consider the elliptic boundary value problem

$$\begin{cases} u_{tt}(x,t) + u_{xx}(x,t) + \sigma(5/2 + t - t^2)u(x,t) = 0, & (x,t) \in \mathcal{R}_1, \\ u(x,0) = u(x,1) = 0, & 0 \le x \le \pi, \\ u(0,t) = u(\pi,t) = 0, & 0 \le t \le 1, \end{cases} \tag{29}$$

where σ is a positive constant and $\mathcal{R}_1$ is the rectangular region defined in (26). In the view of Lyapunov-type inequality (22), the inequality

$$\int_0^1 \nu^+(t;\sigma)\mathrm{d}t > 4 \qquad (\sigma > 0) \tag{30}$$

must be hold, where $\nu(t;\sigma) = 5\sigma/2 - 1 + \sigma t - \sigma t^2$, $\sigma > 0$. It is clear that $\nu(t;\sigma) < 0$ for all $\sigma \in (0, 4/11)$. Moreover, if $\sigma \ge 4/11$, then $\nu(t;\sigma) \ge 0$ for all $t \in [(1 - \sqrt{11})/2, (1 + \sqrt{11})/2]$, and hence (30) turns to

$$\int_0^1 \nu(t;\sigma)\mathrm{d}t = \int_0^1 [5\sigma/2 - 1 + \sigma t - \sigma t^2]\mathrm{d}t = \frac{8}{3}\sigma - 1 > 4. \tag{31}$$

So Prb. (29) has no nontrivial solution, if $\sigma \leq 15/8 \approx 1,875$ by Corollary 2. In particular if $\sigma = 4$, then Prb. (29) has a nontrivial solution

$$u(x,t) = t(1-t)e^{t(1-t)} \sin x, \qquad (x,t) \in \mathcal{R}_1.$$

Finally, we present some open problems concerning possible extensions of Theorem 1 and Theorem 2. It will be of interest to find a Lyapunov-type inequalities for the linear parabolic equation of the form

$$u_t(x,t) - u_{xx}(x,t) + p(t)u(x,t) = 0, \qquad (x,t) \in \mathcal{R} \tag{32}$$

satisfying the Dirichlet boundary condition (2), where $\mathcal{R}$ is defined in (3), and that no sign restriction is imposed on the potential function $p(t) \in \mathrm{L}^1[t_1, t_2]$. In fact, the nonlinear cases

$$u_{tt}(x,t) \pm u_{xx}(x,t) + F(t, u(x,t)) = 0, \qquad (x,t) \in \mathcal{R}$$

and

$$u_t(x,t) - u_{xx}(x,t) + G(t, u(x,t)) = 0, \qquad (x,t) \in \mathcal{R}$$

are of immense interest. Moreover, Lyapunov-type inequalities for elliptic, hyperbolic and parabolic equations of the form

$$u_{tt}(x,t) \pm \Delta u(x,t) + F(t, u(x,t)) = 0, \qquad (x,t) \in \Omega$$

and

$$u_t(x,t) - \Delta u(x,t) + G(t, u(x,t)) = 0, \qquad (x,t) \in \Omega$$

may give remarkable results under some appropriate boundary conditions, where Ω is any closed subset of $\mathbb{R}^n$.

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