Neighbor Rupture Degree of Gear Graphs

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Abstract

Various structures such as communication networks, internet networks, transportation networks, etc. can be demonstrated with the aid of graphs. The common feature of these networks is the presence of one or more centers and the connection between them partially or completely. These networks can be modelled with the help of a graph showing the centers by vertices and the connections by edges. In case of a problem encountered with the centers or the connections of a network, the resistance of the network is called vulnerability. There are some graph parameters to measure the vulnerability such as connectivity, integrity, toughness, tenacity, rupture degree and neighbor rupture degree. Some of these parameters only consider vertices, some of them take into account the neighbors of the vertices. Neighbor rupture degree is a vulnerability parameter that considers the neighborhoods. In this study, the general solutions for the neighbor rupture degree of gear graphs $G_n, \overline{G_n}, G_{n,k}, \overline{G_{n,k}}$ are obtained.

Keywords — Gear graph, generalized gear graph, graph theory, neighbour rupture degree, vulnerability

1 Introduction

Various structures like communication networks, internet networks and transportation networks can be demonstrated with the help of graphs. How long these structures can sustain their functionality in case of damage or attack in these structures is important. The measurement of this time or resistance of the structure is related to the term vulnerability. Varios parameters are proposed in order to find out the vulnerability value in the graphs. Parameters like connectivity [1], integrity [2], toughness [3], scattering number [4], tenacity [5] and rupture degree [6] have just taken into account the vertices dismissed from the graph. However most of these parameters do not consider the neighborhoods of the affected vertices. On the other hand, in spy networks, if a spy or a station is captured, then adjacent stations are unreliable. Therefore neighborhoods should be taken into consideration in spy networks [7]. Parameters like neighbor connectivity [8], neighbor integrity [9] and neighbor scattering number [10] have also taken

into account those vertices' neighbors along with dismissed vertices. Neighbor rupture degree is one of the parameters which takes into account the neighborhoods.

Definition 1.1: The neighbor rupture degree of a noncomplete connected graph G is defined to be

$$Nr(G) = \max_{S \subset V(G)} \{w(G/S) - |S| - c(G/S); w(G/S) \geq 1\}$$

where S is any vertex subversion strategy of G, $w(G/S)$ is the number of connected components in G/S, and $c(G/S)$ is the maximum order of the components of G/S [7].

In this study general information was given about gear graphs and neighbor rupture degree of gear graphs were calculated.

2 Gear Graphs and Neighbor Rupture Degree

In this section, general results for neighbor rupture degree of gear graphs $G_n, \overline{G_n}, G_{n,k}$ and $\overline{G_{n,k}}$ are obtained and the proofs are given in detailed.
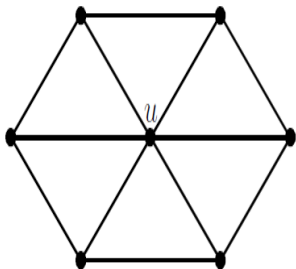

Figure 1. Wheel Graph (W$_6$)

Definition 2.1: The wheel graph with n spokes, W$_n$, is the graph that consists of an n-cycle and one additional vertex, say u, that is adjacent to all the vertices of the cycle [11].

Definition 2.2: The gear graph is a wheel graph with a vertex added between each pair of adjacent graph vertices of the outer cycle. The gear graph G$_n$ has 2n + 1 vertices and 3n edges [12].
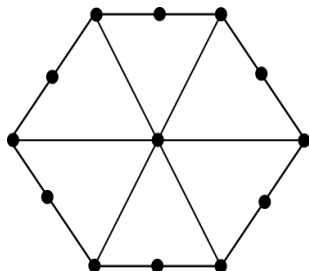

Figure 2. Gear Graf (G$_6$)

Theorem 2.1: Let G$_n$ be a gear graph with $2n + 1$ vertices. Then the neighbor rupture degree is

$$Nr(G_n) = n - 2.$$

Proof: Let S be a subversion strategy of G_n and let $|S| = r$ where $1 \leq r \leq 2n - 3$. There are two cases according to the elements of S.

Case 1: Let $u \in S$ where $u \in V(G_n)$ and $\deg(u) = n$. Since u is adjacent to n vertices, the survival subgraph has at most $n - r + 1$ compenents and they all are isolated vertices. That is $w(G_n/S) \leq n - r + 1$ and

$c(G_n/S) = 1$. Thus, we have

$$w(G_n/S) - |S| - c(G_n/S) \leq n - r + 1 - r - 1$$
$$= n - 2r.$$

Let $f(r) = n - 2r$ and since $f(r)$ is a decreasing function it takes its maximum value at $r = 1$. Hence taking the maximum of both sides of the inequality we get

$$Nr(G_n) \leq n - 2 \quad (2.1)$$

Case 2: Let $u \notin S$ where $u \in V(G_n)$ and $\deg(u) = n$. When r vertices are removed, the survival subgraph has at most r components. Thus, we have

$$w(G_n/S) \leq r, \quad c(G_n/S) \geq \tfrac{2n-3r}{r} \text{ and}$$

$$w(G_n/S) - |S| - c(G_n/S) \leq r - r - \tfrac{2n-3r}{r} = 3 - \tfrac{2n}{r}.$$

Assume $f(r) = 3 - \tfrac{2n}{r}$ and since $f(r)$ is an increasing function, it takes its maximum value at $r = 2n - 3$. Hence $f(2n - 3) = 3 - \tfrac{2n}{2n-3}$ and

$$Nr(G_n) \leq 3 - \tfrac{2n}{2n-3} \leq 1 \quad (2.2)$$

By (2.1) and (2.2) we get

$$Nr(G_n) \leq n - 2 \quad (2.3)$$

On the other hand, if we choose $S^* = \{u\}$ then we get $w(G_n/S^*) = n$ and $c(G_n/S^*) = 1$. Thus by the definition, we obtain

$$Nr(G_n) \geq w(G_n/S^*) - |S^*| - c(G_n/S^*)$$

$$= n - 1 - 1 = n - 2 \quad (2.4)$$

Therefore by (2.3) and (2.4) we get the result.

Theorem 2.2: Let $\overline{G_n}$ be the complement of a gear graph G$_n$. Then the neighbor rupture degree is

$$Nr\big(\overline{G_n}\big) = -2.$$

Proof: The coplement of a gear graph $\overline{G_n}$ is composed of two complete graphs K_n and K_{n+1} where vertices

of K_n are connected to $n-2$ vertices of K_{n+1}. Let S be a subversion strategy of $\overline{G_n}$ and let $|S| = r$.

Case 1: Let $S \subset V(K_n)$.

Subcase 1: If $r = 1$, then $w(\overline{G_n}/S) = 1$ and $c(\overline{G_n}/S) = 3$. Thus we have

$$w(\overline{G_n}/S) - |S| - c(\overline{G_n}/S) = 1 - 1 - 3 = -3$$

Hence
$$Nr(\overline{G_n}) = -3 \text{ (2.5)}$$

Subcase 2: If $r \geq 2$, then $w(\overline{G_n}/S) = 1$ and $c(\overline{G_n}/S) \geq 1$. Then we get

$$w(\overline{G_n}/S) - |S| - c(\overline{G_n}/S) \leq 1 - r - 1 = -r \leq -2.$$

$$Nr(\overline{G_n}) \leq -2 \text{ (2.6)}$$

Case 2: Let $S \subset V(K_{n+1})$.

Subcase 1: If $r = 1$, then $w(\overline{G_n}/S) = 1$ and

$c(\overline{G_n}/S) \geq 2$. Hence

$$w(\overline{G_n}/S) - |S| - c(\overline{G_n}/S) \leq 1 - 1 - 2 = -2.$$

$$Nr(\overline{G_n}) \leq -2. \text{ (2.7)}$$

Subcase 2: If $r \geq 2$, then $w(\overline{G_n}/S) = 1$ and $c(\overline{G_n}/S) \geq 1$. Thus we obtain

$$w(\overline{G_n}/S) - |S| - c(\overline{G_n}/S) \leq 1 - r - 1 = -r \leq -2$$

$$Nr(\overline{G_n}) \leq -2 \text{ (2.8)}$$

By (2.5), (2.6), (2.7) and (2.8) we get
$$Nr(\overline{G_n}) \leq -2. \text{ (2.9)}$$

Conversely, let $S^* = \{u\}$ where $deg_{G_n}(u) = 2$. Then the survival subgraph is $\overline{G_n}/\{u\} \cong K_2$. Hence $w(\overline{G_n}/S^*) = 1$, $c(\overline{G_n}/S^*) = 2$ and

$$w(\overline{G_n}/S^*) - |S^*| - c(\overline{G_n}/S^*) = 1 - 1 - 2 = -2.$$

By the definition of neighbor rupture degree, we have

$$Nr(\overline{G_n}) \geq -2 \text{ (2.10)}$$

By (2.9) and (2.10) we obtain the result.

Definition 2.3: The generalized gear graph $G_{n,k}$ is obtained from a wheel graph by introducing k vertices between every pair of adjacent vertices on the cycle. [13]
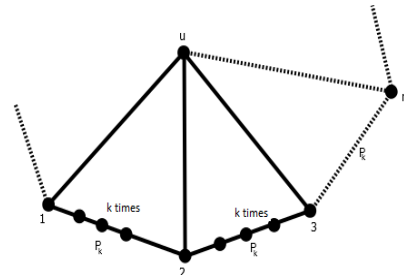

Figure 3. Generalized Gear Graph $G_{n,k}$

In a generalized gear graph $G_{n,k}$ the vertex of degree n is labeled by u, called the center of a gear graph, the neighbors of u is labeled by $1,2,3,\dots,n$, and k vertices are added between two adjacent vertices labeled by two concecutive numbers.

Theorem 2.3: Let $G_{n,k}$ be a generalized gear graph where $k \geq 12$. Then the neighbor rupture degree is

$$Nr(G_{n,k}) = \begin{cases} n-2, & k \equiv 1 \ (mod \ 4) \\ n-3, & k \ T \ 1 \ (mod \ 4). \end{cases}$$

Proof: Let S be a subversion strategy of $G_{n,k}$ and let $|S| = r$. There are two cases according to the vertex $u \in V(G_{n,k})$ with $deg_{G_{n,k}}(u) = n$ whether or not it is an element of S.

Case 1: Let $u \notin S$ where $u \in V(G_{n,k})$ with $deg_{G_{n,k}}(u) = n$ and $r \leq \left\lfloor \frac{n.k+n-4}{4} \right\rfloor$. Then we have $w(G_{n,k}/S) \leq r$ and $c(G_{n,k}/S) \geq \frac{n.k+n-4r}{r}$

$$w(G_{n,k}/S) - |S| - c(G_{n,k}/S)$$
$$\leq r - r - \frac{n.k + n - 4r}{r}$$
$$= 4 - \frac{n.k + n}{r}$$

By taking the maximum of both sides we get

$$Nr(G_{n,k}) \leq -1. \ (2.11)$$

Case 2: Let $u \in S$ where $u \in V(G_{n,k})$ with $deg_{G_{n,k}}(u) = n$.

Subcase 1: If $r \leq \left\lfloor \frac{k-1}{4} \right\rfloor . n + 1$, then

$w(G_{n,k}/S) \leq n + r - 1$ and $c(G_{n,k}/S) \geq \frac{k-3.\frac{r-1}{n}}{\frac{r-1}{n}+1}$.

Then we get

$$w(G_{n,k}/S) - |S| - c(G_{n,k}/S)$$

$$\leq n + r - 1 - r - \frac{k - 3.\frac{r-1}{n}}{\frac{r-1}{n} + 1}$$

$$= n - 1 - \frac{kn - 3r + 3}{r - 1 + n} \ (2.12)$$

Let $f(r) = n - 1 - \frac{kn-3r+3}{r-1+n}$. Since $f(r)$ is an increasing function, it takes its maximum value at $r = \left\lfloor \frac{k-1}{4} \right\rfloor . n + 1$. Thus maximum value of $f(r) \leq n - 2$ where $k = 1$, $f(r) \leq n - 3$ otherwise. Therefore

taking the maximum of both sides of the inequality in (2.12) we obtain

$$Nr(G_{n,k}) \leq \begin{cases} n - 2, \ k \equiv 1 (mod \ 4) \\ n - 3, \ k \ \text{T} \ 1 (mod \ 4). \end{cases} \ (2.13)$$

Subcase 2: If $r \geq \left\lceil \frac{k}{4} \right\rceil . n + 1$, then we have

$$w(G_{n,k}/S) \leq r - 1 \text{ and } c(G_{n,k}/S) \geq 1.$$

Thus $w(G_{n,k}/S) - |S| - c(G_{n,k}/S) \leq r - 1 - r - 1 = -2$

$$Nr(G_{n,k}) \leq -2 \ (2.14)$$

On the other hand, there is a set S^* such that $|S^*| = \left\lfloor \frac{k-1}{4} \right\rfloor . n + 1$, $w(G_{n,k}/S^*) = n. \left(\left\lfloor \frac{k-1}{4} \right\rfloor + 1 \right)$ and $c(G_{n,k}/S^*) = 1$ where $k = 1$, $c(G_{n,k}/S^*) = 2$ otherwise.

Therefore by the definition of neighbor rupture degree

$$Nr(G_{n,k}) \geq w(G_{n,k}/S^*) - |S^*| - c(G_{n,k}/S^*)$$

$$Nr(G_{n,k}) \geq \begin{cases} n - 2, \ k \equiv 1 \ (mod 4) \\ n - 3, \ k \ \text{T} \ 1 \ (mod 4) \end{cases} \ (2.15)$$

By (2.11), (2.12), (2.13), (2.14) and (2.15) we get the result.

Theorem 2.4: Let $\overline{G_{n,k}}$ be the complement of a generalized gear graph $G_{n,k}$. Then the neighbor rupture degree is

$$Nr(\overline{G_{n,k}}) = -2.$$

Proof: Let S be a subversion strategy of $\overline{G_{n,k}}$ and let $|S| = r$.

Case 1: Let $r = 1$.

Subcase 1: Let $s = \{u\}$ where $u \in V(\overline{G_{n,k}})$ and $deg_{G_{n,k}}(u) = n$. Then $\overline{G_{n,k}}/S \cong K_n$.

$$w(\overline{G_{n,k}}/S) - |S| - c(\overline{G_{n,k}}/S) = 1 - 1 - n = -n \ (2.16)$$

Subcase 2: Let $s = \{u\}$ where $u \in V(\overline{G_{n,k}})$ and $deg_{G_{n,k}}(u) = 3$. Then $\overline{G_{n,k}}/S \cong K_3$.

$$w(\overline{G_{n,k}}/S) - |S| - c(\overline{G_{n,k}}/S) = 1 - 1 - 3 = -3 \ (2.17)$$

Subcase 3: Let $s = \{u\}$ where $u \in V(\overline{G_{n,k}})$ and

$deg_{G_{n,k}}(u) = 2$. Then $\overline{G_{n,k}}/S \cong K_2$.

$$w(\overline{G_{n,k}}/S) - |S| - c(\overline{G_{n,k}}/S) = 1 - 1 - 2 = -2 \ (2.18)$$

Case 2: Let $r \geq 2$. $w(\overline{G_{n,k}}/S) \leq 1$ and $c(\overline{G_{n,k}}/S) \geq 1$

$$w(\overline{G_{n,k}}/S) - |S| - c(\overline{G_{n,k}}/S) \leq 1 - r - 1 = -r \leq -2 \ (2.19)$$

By (2.16), (2.17), (2.18), and (2.19) we get the result.

3 Conclusion

Geared systems can be used for dynamic analysis. Since gear graphs are contained in geared networks, to examine the vulnerability of a geared system, gear graphs can be used for exploring the vulnerability of the network. In this study, the neighbor rupture degree of gear graphs, the generalized gear graphs and their

complements are examined, and general results are obtained.

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