

RESEARCH ARTICLE

# On topological complexity of Gorenstein spaces

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# Abstract

In this paper, using Sullivan's approach to rational homotopy theory of simply connected finite type CW complexes, we endow the Q-vector space  $\mathcal{E}xt_{C^*(X;\mathbb{Q})}(\mathbb{Q}, C^*(X;\mathbb{Q}))$  with a graded commutative algebra structure. Thus, we introduce new algebraic invariants referred to as the Ext-versions of the ordinary higher, module, and homology Topological Complexities of  $X_0$ , the rationalization of X. For Gorenstein spaces, we establish, under additional hypotheses, that the new homology topological complexity, denoted  $HTC_n^{\mathcal{E}xt}(X,\mathbb{Q})$ , lowers the ordinary  $HTC_n(X)$  and, in case of equality, we extend Carasquel's characterization for  $HTC_n(X)$  to some class of Gorenstein spaces (Theorem 1.2). We also highlight, in this context, the benefit of Adams-Hilton models over a field of odd characteristic especially through two cases, the first one when the space is a 2-cell CW-complex and the second one when it is a suspension.

Mathematics Subject Classification (2020). Primary 55M30, Secondary 55P62

**Keywords.** higher topological complexity, Eilenberg-Moore functor, Sullivan algebra, Gorenstein spaces

# 1. Introduction

The concept of topological complexity TC(-) was introduced by Michael Farber in [6]. It is a numerical invariant that measures the difficulty of planning continuous motions of a mechanical system along its configuration space X. Formally,  $TC(X) = secat(\Delta_X)$ , the sectional category of the diagonal map  $\Delta_X : X \to X \times X$ , defined as the smallest integer m such that there exist m + 1 local homotopy sections  $s_i : U_i \to X$  such that  $\{U_i\}$  form an open cover of  $X \times X$ . Intuitively, this means that we can divide  $X \times X$  into m + 1 parts, and on each part, the system has a well-defined continuous motion. Since its introduction, the concept TC has been studied in various contexts such as robotics, algorithmic topology, and algebraic topology. Later, Y. Rudyak ([19]) generalizes Farber's concept by introducing its higher analogous denoted  $TC_n(X)$   $(n \ge 2)$  and defined as the sectional category,  $secat(\Delta_X^n)$ , of the n-diagonal  $\Delta_X^n : X \to X^n$  so that  $TC(X) = TC_2(X)$ .

In [4], J. Carrasquel used the characterization à la Félix-Halperin to give an explicit definition of the *(higher)* rational topological complexity  $TC_n(X_0)$ , of the rationalization

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 $X_0$  of X, which turns out to lower  $TC_n(X)$  (cf. Definition 4.1 below). He also introduced higher (rational) homology (resp. module) topological complexity  $HTC_n(X)$ , (resp.  $mTC_n(X)$ ) and showed that they interpolate  $zcl_n(X) := nilkerH^*(\Delta_X^n; \mathbb{Q})$  and  $TC_n(X_0)$ .

In this paper, we approach the study (rational versions) of the aforementioned invariants for *Gorenstein spaces*. These were introduced in [11] with the aim of deepening the study, in the context of local algebra, of the *Lusternik-Scnirelmann category*  $cat_{LS}(X) = secat(* \hookrightarrow X)$  [17]. Recall from [20] that a finite *n*-dimensional subcomplex X of  $\mathbb{R}^{n+k}$  is a Poincaré complex if and only if its *Spivak fiber*  $F_X$  (see §4) is a homotopy sphere. In particular, when X has finite  $cat_{LS}(X)$  and  $\mathbb{K} = \mathbb{Z}_p$  for p prime or zero, a *local analogue of the Spivak characterization* in terms of the Eilenberg-Moore Ext functor  $\mathcal{E}xt$ reads as follows [11, Theorem 3.1]:

$$\begin{aligned} H^*(X, \mathbb{Z}_p) \text{ is a Poincaré duality algebra } &\Leftrightarrow (F_X)_{(p)} \simeq (\mathbb{S}^k)_{(p)} \\ &\Leftrightarrow \dim \mathbb{E} xt_{C^*(X; \mathbb{Z}_p)}(\mathbb{Z}_p, C^*(X; \mathbb{Z}_p)) = 1 \end{aligned}$$
(1.1)

(cf. §2 for the definition of Poincaré duality algebras and the functor  $\mathcal{E}xt$ ). The equivalences in (1.1) make sense to the following definition [11]:

**Definition 1.1.** X is a Gorenstein space over  $\mathbb{K}$  if dim  $\mathcal{E}xt_{C^*(X;\mathbb{K})}(\mathbb{K}, C^*(X;\mathbb{K})) = 1$ .

For instance, if X is finite dimensional then [11, Proposition 4.5, Theorem 3.1]:

E.g. every closed manifold is Gorenstein over any field  $\mathbb{K}$ . In the rational case, if  $\dim \pi_*(X) \otimes \mathbb{Q}$  is finite dimensional then X is Gorenstein over  $\mathbb{Q}$  [11, Proposition 3.4]. In particular, every rationally elliptic space X (i.e. if its rational homology and rational homotopy are both finite dimensional) is Gorenstein over  $\mathbb{Q}$ . Another advantageous homotopy invariant that allows to better (algebraically) characterize Poincaré duality spaces is the morphism of  $\mathbb{K}$ -graded vector spaces:

$$ev_{C^*(X;\mathbb{K})} : \mathcal{E}xt_{C^*(X;\mathbb{K})}(\mathbb{K}, C^*(X;\mathbb{K})) \to H^*(X,\mathbb{K})$$

called the *evaluation map* of X over  $\mathbb{K}$  [11] (cf. §2 for more details).

In fact, we have [13, Corollary 2]:

$$X \text{ is a Poincaré duality space} \Leftrightarrow \begin{cases} (i) X \text{ is a Gorenstein space}, \\ (ii) ev_{C^*(X,\mathbb{K})} \neq 0, \\ (iii) H^*(X;\mathbb{Q}) \text{ is a Gorenstein algebra.} \end{cases}$$

In order to highlight our goal, notice that while the (ordinary) rational higher Topological Complexities  $TC_n(X_0)$  has a simple geometric interpretation (as it is the case for  $TC_n(X)$ ), its calculation is among NP-hard problems [16]. The complexities  $mTC_n(X)$ and  $HTC_n(X)$  are algebraic approximations of  $TC_n(X_0)$  that can be determined using the same techniques as those used to calculate or at least approximate  $cat_{LS}(X_0)$  ([11],[10],[8]). In fact, fewer are spaces for which the invariants  $HTC_n(X)$ ,  $mTC_n(X)$  and  $TC_n(X_0)$  are determined. However, in [15], a new invariant  $L(X_0)$  is introduced and it is shown that  $cat_{LS}(X_0) + L(X_0) \leq TC_n(X_0)$  for any pure elliptic coformal space. It is also established that  $TC_n(X_0) = \dim \pi_*(X) \otimes \mathbb{Q}$  for certain particular families of spaces.

In this work, inspired by Carrasquel's characterizations, we will introduce new algebraic approximations of  $TC_n(X)$  which we call *Ext-versions topological complexities* and establish some inequalities between these invariants. To this end, we first endow, in section 3, the graded Q-vector space

$$\mathbf{A} =: Hom_{C^*(X;\mathbb{Q})}((P,d), C^*(X;\mathbb{Q}))$$

with a homotopy multiplicative structure (see §3 below) denoted in all that follows by

$$\mu_{\mathbf{A}}: \mathbf{A} \otimes \mathbf{A} \to \mathbf{A},$$

where (P, d) is a semi-free resolution of  $(\mathbb{Q}, 0)$  (cf. §2 for more details).

The cohomology of A is

$$\mathcal{A} = \mathcal{E}xt_{C^*(X;\mathbb{Q})}(\mathbb{Q}, C^*(X;\mathbb{Q})),$$

and our first main result reads:

**Theorem 1.2.** The graded vector space A endowed with the product  $\mu_A := H^*(\mu_A)$  is a graded commutative algebra with unit. Moreover, the evaluation map

$$ev_{(X,\mathbb{Q})}: \mathcal{E}xt_{C^*(X;\mathbb{Q})}(\mathbb{Q}, C^*(X;\mathbb{Q})) \to H^*(X;\mathbb{Q})$$

is a morphism of graded algebras.

Next, recall that ([5]) a morphism of graded commutative differential algebra (cdga for short)  $\varpi : (C,d) \to (D,d)$  admits a homotopy retraction if there exists a map  $r : (\Lambda V \otimes C, d)) \to (C,d)$  such that  $r \circ \iota = Id_C$ , where  $\iota : (C,d) \to (\Lambda V \otimes C, d)$  is the relative model of  $\varpi$ . As a particular case, consider any Sullivan algebra ( $\Lambda V, d$ ) and denote by  $\mu_{A,n} : A^{\otimes n} \to A$  the *n*-fold product of  $\mu_A : A \otimes A \to A$ . Using [4, Definition 9], we state in terms of the cdga projection

$$\Gamma_m : (\mathbf{A}^{\otimes n}, d) \to \left(\frac{\mathbf{A}^{\otimes n}}{\left(\ker\left(\mu_{\mathbf{A}, n}\right)\right)^{m+1}}, \overline{d}\right)$$

the following

- **Definition 1.3.** (a):  $sc(\mu_{A,n})$  is the least m such that  $\Gamma_m$  admits a homotopy retraction.
  - (b):  $msc(\mu_{A,n})$  is the least m such that  $\Gamma_m$  admits a homotopy retraction as  $(A^{\otimes n}, d)$ -modules.
  - (c):  $Hsc(\mu_{A,n})$  is the least m such that  $H(\Gamma_m)$  is injective.
  - (d): nil ker  $(\mu_{\mathcal{A},n}, \mathbb{Q})$  is the longest non trivial product of elements of ker  $(\mu_{\mathcal{A},n})$ .

As a particular case, given  $A = (\Lambda V, d)$  a Sullivan model of X, the aforementioned (rational) *Ext*-version invariants are denoted respectively  $\operatorname{TC}_{n}^{\mathcal{E}xt}(X,\mathbb{Q})$ ,  $\operatorname{mTC}_{n}^{\mathcal{E}xt}(X,\mathbb{Q})$ ,  $\operatorname{mTC}_{n}^{\mathcal{E}xt}(X,\mathbb{Q})$ ,  $\operatorname{mTC}_{n}^{\mathcal{E}xt}(X,\mathbb{Q})$ . We call theme respectively, the rational, module, homology topological complexity and *Ext*-version zero cup length.

Then, we prove the following

**Theorem 1.4.** Let X be a 1-connected finite type CW-complex. If X is a Gorenstein space over  $\mathbb{Q}$  and  $ev_{C^*(X,\mathbb{Q})} \neq 0$ , then

$$HTC_n^{\mathcal{E}xt}(X,\mathbb{Q}) \le HTC_n(X)$$

for any integer  $n \geq 2$ . Furthermore, if  $(\Lambda V, d)$  is a Sullivan minimal model of X, [f] is the generating class of  $\mathcal{A}$  and  $m = HTC_n^{\mathcal{E}xt}(X, \mathbb{Q})$ , then:

$$HTC_n(X) = HTC_n^{\mathcal{E}xt}(X, \mathbb{Q}) =: m \Leftrightarrow f(1)^{\otimes n} \in (\ker \mu_n)^m \setminus (\ker \mu_n)^{m+1}.$$

Notice that the hypotheses on X imply that it is either (i) a Poincaré duality space over  $\mathbb{Q}$  or else (ii)  $H^*(X, \mathbb{Q})$  is not noetherian and not a Gorenstein graded algebra [13, Theorem 3]. Moreover, it is well known that if X satisfies the Poincaré duality property then  $\omega = f(1)$  is a cocycle representing its fundamental class. Thus, (cf. Remark 4.6), our theorem extends, Corollary 5.5 in [5], to spaces of the class (ii).

The following corollary presents some essential classes of spaces satisfying the previous theorem

**Corollary 1.5.** The hypothesis of Theorem 1.4 and hence its conclusions are satisfied in the following cases:

- (a) X is rationally elliptic,
- (b)  $H_{>N}(X,\mathbb{Z}) = 0$ , for some N, and  $H^*(X,\mathbb{Q})$  is a Poincaré duality algebra,
- (c) X is a finite 1-connected CW-complex and its Spivak fiber  $F_X$  has finite dimensional cohomology.

The rest of the paper is organized as follows. In section 2 we summarize the tools essential for the rest of the document, while section 3 is dedicated to the proof of Theorem 1.1. In section 4, we formally introduce the Ext-version of higher, module and homology topological complexities. In Section 5 we will extend a part of Theorem 1.2 and its corollary to the sub-category  $CW_r(R)$  when  $R = \mathbb{K}$  is a field of odd characteristic and implement the advantageous computational properties of Adams-Hilton models [1] to obtain explicit calculations of the homotopy invariant  $\mathcal{A} = \mathcal{E}xt_{C^*(X,\mathbb{K})}(\mathbb{K}, C^*(X,\mathbb{K}))$  when X is a suspension or a two-cell CW-complex.

# 2. Preliminaries

Let  $\mathbb{K}$  denote an arbitrary ground field unless otherwise stated.

# 2.1. Eilenberg-Moore Ext

A graded module is a family  $A = (A^i)_{i \in \mathbb{Z}}$  of  $\mathbb{K}$ -modules denoted also  $A = \bigoplus_{i \in \mathbb{Z}} A^i$ . Every  $a \in A^i$  is of degree *i* denoted thereafter |a|.

A linear map of graded modules  $f : A \to B$  of degree |f| is a K-linear map sending each  $A^i$  to  $B^{i+|f|}$ . If |f| = 0 we call it a morphism of graded modules.

In all that follows, unless otherwise stated, modules are over  $\mathbb{K}$  and we will assume that  $A^i = 0$  if i < 0.

A graded algebra A is a graded module together with an associative multiplication  $\mu_A : A \otimes A \to A$  that has an identity element  $1_A =: 1 \in A^0$ . We will put  $\mu_A(x \otimes y) =: xy$ . Notice that  $|\mu_A| = 0$ . Moreover, if we have  $ab = (-1)^{|a||b|} ba$  for all  $a, b \in A$ , then A is said to be commutative.

A differential graded algebra (A, d) (dga for short) is a graded algebra A together with a linear map  $d: A \to A$  of degree |d| = +1 that is a derivation  $d(ab) = d(a)b + (-1)^{|a|}ad(b)$ , and satisfying  $d \circ d = 0$ .

A morphism of dga  $f : (A, d) \to (B, d)$  is a linear map of degree zero satisfying f(aa') = f(a)f(a'), and compatible with the differential d: f(da) = d(f(a)).

A dga algebra A is said to be augmented if it is endowed with a morphism  $\varepsilon : A \to \mathbb{K}$  of graded algebras.

A (left) graded (A, d) module is a graded module M equipped with a linear map  $A \otimes M \to M$ ,  $a \otimes m \mapsto am$  of degree zero such that a(bm) = (ab)m and 1m = m, and a differential d satisfying  $d(am) = (da)m + (-1)^{|a|}a(dm)$ ,  $m \in M$ ,  $a \in A$ .

A morphism of (left) graded modules over a dga (A, d) is a morphism  $f : (M, d) \to (N, d)$  compatible with the differential:  $d \circ f = f \circ d$ .

A left (A, d)-module (M, d) is said to be semi-free if it is the union of an increasing sequence  $M(0) \subset M(1) \subset M(2) \subset \cdots \subset M(n) \subset \cdots$  of sub (A, d)-modules such that M(0) and each M(i)/M(i-1) is A-free on a basis of cycles. Such an increasing sequence is called a semi-free filtration of (M, d).

A semi-free resolution of an (A, d)-module (M, d) is an (A, d)-semi-free module (P, d) together with a quasi-isomorphism (i.e. a morphism inducing an isomorphism in homology)  $m: (P, d) \xrightarrow{\simeq} (M, d)$  of (A, d)-modules. Each of P(0) and P(i)/P(i-1) has the form  $(A, d) \otimes (V(i), 0)$  where V(i) is a free K-module. Thus the surjections  $P(n) \to A \otimes V(n)$  split and the differential d satisfies:

$$P(n) = P(n-1) \oplus (A \otimes V(n))$$
 and  $d: V(n) \to P(n-1)$ .

Every (A, d)-module (M, d) has a semi-free resolution  $m : (P, d) \xrightarrow{\simeq} (M, d)$  ([12, Prop. 6.6]) and if  $m' : (P', d) \xrightarrow{\simeq} (M, d)$  is a second semi-free resolution, then, there exists an equivalence

$$\alpha: (P', d) \longrightarrow (P, d)$$

of (A, d)-modules such that  $m \circ \alpha$  and m' are homotopic morphisms, denoted  $m \circ \alpha \simeq_A m'$ .

Particularly, let (A, d) be a differential graded algebra and  $(P, d) \xrightarrow{\simeq} (\mathbb{Q}, 0)$  an (A, d)-semi-free resolution of  $(\mathbb{Q}, 0)$ . This defines the graded (A, d)-module

$$Hom_{A}((P,d),(A,d)) = \bigoplus_{p \ge 0} Hom_{A}^{p,*}((P,d),(A,d)) = \bigoplus_{p \ge 0} \bigoplus_{i \ge 0} Hom_{A}(P^{i}, A^{i+p}),$$

which, endowed with the differential

$$D(f) = d \circ f - (-1)^{p} f \circ d; \quad f \in Hom_{A}^{p,*}((P,d), (A,d)),$$

yields the Eilenberg-Moore Ext functor:

$$\mathcal{E}xt_{(A,d)}(\mathbb{K}, (A,d)) = H^*(Hom_A((P,d), (A,d)), D).$$

This is an invariant up to homotopy of differential graded algebras (see [11, Appendix] or [7, Appendix]). Moreover, [11, Remark 1.3], if  $(A, d) \xrightarrow{\simeq} (B, d)$  is a quasi-isomorphism of differential graded algebras, then  $\mathcal{E}xt_{(A,d)}(\mathbb{K}, (A, d))$  is identified with  $\mathcal{E}xt_{(B,d)}(\mathbb{K}, (B, d))$  via natural (induced) isomorphisms

$$\mathcal{E}xt_{(A,d)}(\mathbb{K}, (A,d)) \xrightarrow{\cong} \mathcal{E}xt_{(A,d)}(\mathbb{K}, (B,d)) \xleftarrow{\cong} \mathcal{E}xt_{(B,d)}(\mathbb{K}, (B,d)).$$
(2.1)

Particularly,  $\mathcal{E}xt_{C^*(X;\mathbb{K})}(\mathbb{K}, C^*(X;\mathbb{K}))$  and  $\mathcal{E}xt_{C^*(\Omega X;\mathbb{K})}(\mathbb{K}, C_*(\Omega X;\mathbb{K}))$  depend only on the homotopy class of X.

The highest N such that  $[\mathcal{E}xt_{C^*(X;\mathbb{K})}(\mathbb{K}, C^*(X;\mathbb{K}))]^N \neq 0$  is called the formal dimension of X. It is denoted  $fd(X,\mathbb{K})$ .

# 2.2. Evaluation map and Gorenstein spaces

Let  $\rho: (P,d) \xrightarrow{\simeq} (\mathbb{K},0)$  be a minimal (A,d)-semi-free resolution of  $(\mathbb{K},0)$ . Consider the chain map

$$cev_{(A,d)} : \operatorname{Hom}_{(A,d)}((P,d), (A,d)) \longrightarrow (A,d)$$

given by  $f \mapsto f(z)$ , where  $z \in P$  is a cocycle representing 1 in K. We call it the *chain* evaluation map of (A, d). Passing to homology, we obtain the natural map

$$ev_{(A,d)}: \mathcal{E}xt_{(A,d)}(\mathbb{K}, (A,d)) \longrightarrow H^*(A,d),$$

called the evaluation map of (A, d). The definition of  $ev_{(A,d)}$  is independent of the choice of (P, d) and z. The evaluation map  $ev_{(X,\mathbb{K})}$  of X over  $\mathbb{K}$  is by definition the evaluation map of  $C^*(X,\mathbb{K})$ .

A Poincaré duality algebra over  $\mathbb{K}$  is a graded algebra  $H = \{H^k\}_{0 \le k \le N}$  such that  $H^N = \mathbb{K}\alpha$  and the pairing  $\langle \beta, \gamma \rangle \alpha = \beta\gamma, \ \beta \in H^k, \ \gamma \in H^{N-k}$  defines an isomorphism  $H^k \xrightarrow{\cong} Hom_{\mathbb{K}}(H^{N-k}, \mathbb{K}), \ 0 \le k \le N$ . In particular,  $H = Hom_{\mathbb{K}}(Hom_{\mathbb{K}}(H, \mathbb{K}), \mathbb{K})$  is necessarily finite dimensional.

A Poincaré duality space at  $\mathbb{K}$  is a space whose cohomology with coefficients in  $\mathbb{K}$  is a Poincaré duality algebra. In this case, the cohomology class  $\alpha$  such that  $H^N(X;\mathbb{K}) = \mathbb{K}\alpha$  has degree  $N = fd(X,\mathbb{K})[11, \text{Proposition 5.1}]$ . It is called the fundamental class of X.

A Gorenstein algebra over  $\mathbb{K}$  is a differential graded algebra (A, d) whose associated graded vector space  $\mathcal{E}xt_{(A,d)}(\mathbb{K}, (A, d))$  is one dimensional.

A space X is Gorenstein over K if the cochain algebra  $C^*(X; \mathbb{K})$  is a Gorenstein algebra. For instance, let X be a simply connected CW complex. If in addition X is finite dimensional, then,  $C^*(X; \mathbb{K})$  is Gorenstein if and only if  $H^*(X; \mathbb{K})$  is a Poincaé duality algebra [11, Theorem 3.1]. However, if dim  $\pi_*(X) \otimes \mathbb{Q} < \infty$  then, on one hand, X is a Gorenstein space even though X has infinite dimension [11, Proposition 3.4] (see also Corollary 1 above) and, on the other hand, dim  $H^*(X, \mathbb{Q}) < \infty$  if and only if  $ev_{C^*(X; \mathbb{Q})} \neq 0$ ([18]).

# **3.** The Q-algebra $\mathcal{E}xt_{C^*(X;\mathbb{Q})}(\mathbb{Q}, C^*(X;\mathbb{Q}))$

Along this section, the ground field is  $\mathbb{Q}$ . Recall that to every finite-type simplyconnected space X, it is associated a quasi-isomorphism  $(\Lambda V, d) \xrightarrow{\simeq} A_{PL}(X)$  from a free commutative differential graded algebra (cdga for short)  $(\Lambda V, d)$  to the commutative graded algebra  $A_{PL}(X)$  of polynomial forms with rational coefficients [12]. This latter is connected to  $C^*(X, \mathbb{Q})$  via a sequence of quasi-isomorphisms. More explicitly,  $\Lambda V = TV/I$ where I is the graded ideal spanned by  $\{v \otimes w - (-1)^{deg(u)deg(v)}w \otimes v, v, w \in V\},$  $V = \bigoplus_{n\geq 2}V^n$  is a finite-type graded vector space and the differential d is a derivation defined on V satisfying  $d \circ d = 0$ .  $(\Lambda V, d)$  is called a *Sullivan model* of X. This model is said to be *minimal* if d is decomposable, i.e.  $d: V \to \Lambda^{\geq 2}V$  where  $\Lambda^{\geq 2}V$  denotes the graded vector space spanned by all the monomials  $v_1 \dots v_r$   $(r \geq 2)$ , such a model is unique up to isomorphisms [12].

Let  $(X, x_0)$  be a based simply-connected finite type CW-complex and denote by

$$m: (\Lambda V, d) \xrightarrow{\simeq} A_{PL}(X)$$

its minimal Sullivan model. In fact, the multiplicative structure of  $(\Lambda V, d)$ ,  $\mu_{\Lambda V} : \Lambda V \otimes_{\mathbb{Q}} \Lambda V \to \Lambda V$  is compatible with the one induced on  $C^*(X, \mathbb{Q})$  by the diagonal map  $\Delta_X : X \to X \times X$ , and the same holds for the augmentation  $\varepsilon_{\Lambda V} : (\Lambda V, d) \to (\mathbb{Q}, 0)$  and the inclusion  $\iota : \{x_0\} \hookrightarrow X$ .

A  $(\Lambda V, d)$ -semi-free resolution of  $(\mathbb{Q}, 0)$  has the form  $(P, d) = (\Lambda V \otimes \Lambda sV, d) \xrightarrow{\simeq} (\mathbb{Q}, 0)$ where sV is the suspension of V defined by  $(sV)^k = V^{k+1}$  and d(sv) = -s(dv) for all  $v \in V$ , and is called the acyclic closure of  $(\mathbb{Q}, 0)$ , [9].

We are now ready to define a homotopy multiplication on  $A =: Hom_{(\Lambda V,d)}(\mathbb{Q}, (\Lambda V, d)):$ 

$$\mu_{A} : A \otimes A \to A$$

in the sens that it only depends on the homotopy class of the model  $m : (\Lambda V, d) \to A_{PL}(X)$ (cf. the uniqueness property bellow). Passing to cohomology  $\mu_A$  induce an associative multiplicative structure with unit on  $\mathcal{A} =: \mathcal{E}xt_{(\Lambda V,d)}(\mathbb{Q}, (\Lambda V, d))$  denoted

$$\mu_{\mathcal{A}}: \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$$

The proof of Theorem 1.1 follows from the following

**Theorem 3.1.** The  $\mathbb{Q}$ -vector space  $\mathcal{A}$ , endowed with  $\mu_{\mathcal{A}}$ , is a graded commutative algebra with unit. Moreover, the evaluation map is a morphism of graded algebras.

**Proof.** Let  $f, g: P \to \Lambda V$  be elements in A representing two classes in  $\mathcal{A}$ . As  $\Lambda V$  is commutative, the left  $\Lambda V$ -module P is also a right  $\Lambda V$ -module by setting  $x \cdot a = (-1)^{|x||a|} a \cdot x$ ,  $x \in P$  and  $a \in \Lambda V$ .

## Multiplicative structure:

First, we consider

$$egin{aligned} f\otimes g:P\otimes_{\mathbb{Q}}P&\longrightarrow\Lambda V\otimes_{\mathbb{Q}}\Lambda V\ &x\otimes y&\longmapsto(-1)^{|g||x|}f(x)\otimes g(y), \end{aligned}$$

and *I* the ideal generated by  $x \cdot a \otimes y - x \otimes a \cdot y$ ;  $x, y \in P$  and  $a \in \Lambda V$ . It is straightforward that the map  $\mu_{\Lambda V} \circ (f \otimes g)$  sends *I* to zero, which then induces, on the quotient  $P \otimes_{\Lambda V} P = P \otimes_{\mathbb{Q}} P/I$ , the dashed map

defined by:

$$\mu_{\mathcal{A}}(f \otimes g) \coloneqq f.g : P \otimes_{\Lambda V} P \longrightarrow \Lambda V$$
$$x \otimes y \longmapsto (-1)^{|g||x|} f(x)g(y)$$

For any  $a \in \Lambda V$  and  $x, y \in P$ , we have  $(f \cdot g)((a \cdot x) \otimes y) = (-1)^{|f \cdot g||a|} a(f \cdot g)(x \otimes y)$ . Therefore  $f \cdot g$  is a  $\Lambda V$ -morphism.

Next, we show that  $Q = P \otimes_{\Lambda V} P$  is a  $\Lambda V$ -semi-free resolution. Recall that a semi-free resolution (P, d) of  $\mathbb{Q}$  has the form  $W \otimes_{\mathbb{Q}} \Lambda V$  with  $W = \bigoplus_{i=0}^{+\infty} W(i)$  and each W(i) is a free graded  $\mathbb{Q}$ -module and  $d: W(k) \to P(k-1)$ , with the semi-free filtration given by  $P(k) = \bigoplus_{i=0}^{k} W(i) \otimes_{\mathbb{Q}} \Lambda V$  [12]. Therefore,

$$Q = (W \otimes_{\mathbb{Q}} \Lambda V) \otimes_{\Lambda V} (W \otimes_{\mathbb{Q}} \Lambda V) = (W \otimes_{\mathbb{Q}} W) \otimes_{\mathbb{Q}} \Lambda V.$$

Let  $Z = W \otimes_{\mathbb{Q}} W$  and put  $Q(k) = \bigotimes_{i=0}^{k} Z(i) \otimes_{\mathbb{Q}} \Lambda V$  where  $Z(l) = \bigoplus_{i+j=l} W(i) \otimes_{\mathbb{Q}} W(j)$  is obviously a free graded  $\mathbb{Q}$ -module since each W(i) is. For any  $x \otimes y \in W(i) \otimes W(j)$ , we easily verify that

$$dx \otimes y \in P(i-1) \otimes W(j) \subseteq Q(k-1)$$
 and  $x \otimes dy \in W(i) \otimes P(j-1) \subseteq Q(k-1)$ ,

whence  $D: Z(k) \to Q(k-1)$ . It results that  $(Q, D) \xrightarrow{\simeq} (\mathbb{Q}, 0)$  is a  $\Lambda V$ -semi-free resolution of  $(\mathbb{Q}, 0)$ . This defines a multiplication

$$\mu_{A} : A \otimes A \to A$$

on  $A = Hom_{(\Lambda V, d)}(\Lambda V \otimes \Lambda sV, \Lambda V).$ 

Now, since  $(\Lambda V, d)$  has an associative structure, we deduce that  $\mu_A$  is associative. Passing to cohomology we acquire a well-defined map of vector spaces:

$$\mu_{\mathcal{A}} : \mathcal{E}xt_{(\Lambda V,d)}(\mathbb{Q}, (\Lambda V, d)) \otimes_{\mathbb{Q}} \mathcal{E}xt_{(\Lambda V,d)}(\mathbb{Q}, (\Lambda V, d)) \longrightarrow \mathcal{E}xt_{(\Lambda V,d)}(\mathbb{Q}, (\Lambda V, d))$$
$$[f] \otimes [g] \longmapsto [f \cdot g]$$

that is an associative multiplication on  $\mathcal{E}xt_{(\Lambda V,d)}(\mathbb{Q}, (\Lambda V, d))$ .

#### Uniqueness:

Let  $m' : (\Lambda V', d') \xrightarrow{\simeq} A_{PL}(X)$  be another minimal Sullivan model of X i.e. in the homotopy class of  $m : (\Lambda V, d) \xrightarrow{\simeq} A_{PL}(X)$  [12, proposition 12.7]. We know that  $(\Lambda V, d)$ and  $(\Lambda V', d')$  are isomorphic [12, Proposition 12. 10]. Clearly, the same applies for associated semi-free resolutions  $Q = P \otimes_{\Lambda V} P$  and  $Q' = P' \otimes_{\Lambda V} P'$ . It results, from the above commutative triangle, that  $\mu_A$  is independent of the choice of a minimal model of X.

#### Unit element:

Let  $\varepsilon : \Lambda V \to \mathbb{Q}$  be the augmentation. Recall that  $P = \Lambda V \otimes \Lambda s V$  is a  $(\Lambda V, d)$ -semi-free resolution of  $(\mathbb{Q}, 0)$ .

We extend  $\varepsilon$  to  $\varepsilon' = \varepsilon \otimes \varepsilon_{\Lambda sV} : \Lambda V \otimes \Lambda sV \to \mathbb{Q}$ , then we compose it with the injection  $i : \mathbb{Q} \to \Lambda V$  and obtain  $\tilde{\varepsilon} : \Lambda V \otimes \Lambda sV \to \Lambda V$  a representative of a class in

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 $\mathcal{E}xt_{(\Lambda V,d)}(\mathbb{Q}, (\Lambda V, d))$ . Now, for  $f : \Lambda V \otimes \Lambda s V \to \Lambda V$ , a representative of an arbitrary class in  $\mathcal{E}xt_{(\Lambda V,d)}(\mathbb{Q}, (\Lambda V, d))$ , we have

$$f \cdot \tilde{\varepsilon} : (\Lambda V \otimes \Lambda s V) \otimes_{\Lambda V} (\Lambda V \otimes \Lambda s V) = \Lambda V \otimes \Lambda s V \otimes \Lambda s V \to \Lambda V$$

and the map

$$\begin{aligned} \theta &= Id_{\Lambda V \otimes \Lambda s V} \otimes \varepsilon_{\Lambda s V} : \quad \Lambda V \otimes \Lambda s V \otimes \Lambda s V &\longrightarrow \quad \Lambda V \otimes \Lambda s V \\ & 1 \otimes s v \otimes 1 &\longmapsto \quad 1 \otimes s v; \\ & 1 \otimes s v \otimes s w &\longmapsto \quad 0; \\ & 1 \otimes 1 \otimes s v &\longmapsto \quad 0 \end{aligned}$$

makes the following diagram commutative:



Thus, it defines a homotopy unit element for  $A = Hom_{(\Lambda V,d)}((P,d), (\Lambda V,d))$ . Passing to cohomology, we get  $[f] \cdot [\tilde{\varepsilon}] = [f]$  and similarly  $[\tilde{\varepsilon}] \cdot [f] = [f]$ . Henceforth, the class  $[\tilde{\varepsilon}]$  defines a unit element for  $\mu_{\mathcal{A}}$ .

#### Commutativity:

Let  $\tau$  be the flip map  $\tau: P \otimes_{\Lambda V} P \to P \otimes_{\Lambda V} P; x \otimes y \mapsto (-1)^{|x||y|} y \otimes x$ . The diagram



is commutative.

clearly  $\tau$  being a quasi-isomorphism,  $f \cdot g \sim g \cdot f$  and  $[f \cdot g] = (-1)^{|f||g|} [g \cdot f]$  so that, the multiplication on A is homotopy commutative and consequently, it is commutative on  $\mathcal{A}$ .

We respectively conclude that A is a homotopy commutative differential graded algebra with unit and  $\mathcal{E}xt_{(\Lambda V,d)}(\mathbb{Q}, (\Lambda V, d))$  is a graded commutative  $\mathbb{Q}$ -algebra with unit.

Finally, it is clear that the following diagram, where cev is the chain evaluation map of  $(\Lambda V, d)$ , is commutative:

$$\begin{array}{ccc} \mathbf{A} \otimes \mathbf{A} & & \stackrel{\mu_{\mathbf{A}}}{\longrightarrow} \mathbf{A} \\ & & \downarrow_{cev \otimes cev} & & \downarrow_{cev} \\ (\Lambda V, d) \otimes (\Lambda V, d) & \stackrel{\mu_{\Lambda V}}{\longrightarrow} (\Lambda V, d). \end{array}$$

Thus, passing to cohomology



we deduce that the evaluation map is a morphism of graded algebras.

# 4. Ext-versions approximations and the main theorem

Recall that we still assume that  $\mathbb{K} = \mathbb{Q}$ , and let (A, d) be any commutative differential graded model for a space X, i.e. (A, d) is quasi-isomorphic to the cdga  $A_{PL}(X)$  (cf. the beginning of Section 3), and  $(\Lambda V, d)$  its minimal Sullivan model given by the quasiisomorphism  $\theta : (\Lambda V, d) \xrightarrow{\simeq} (A, d)$  [12]. Referring to [5], the cdga morphism

$$\mu_n^{\theta} := (\mathrm{Id}_A, \theta, \dots, \theta) : (A, d) \otimes (\Lambda V, d)^{\otimes n-1} \to (A, d)$$

is a special model, called an *s*-model, for the path fibration  $\pi_n : X^I \to X^n$ , the substitute of the *n*-fold diagonal map  $\Delta_X^n : X \to X^n$ . This allows the following

**Definition 4.1.** (a):  $TC_n(X_0)$  is the least m such that the projection

$$\rho_m: \left(A \otimes (\Lambda V)^{\otimes n-1}, d\right) \to \left(\frac{A \otimes (\Lambda V)^{\otimes n-1}}{\left(\ker \mu_n^{\theta}\right)^{m+1}}, \overline{d}\right)$$

admits an algebra retraction.

- (b): mTC<sub>n</sub>(X) is the least m such that  $\rho_m$  admits a retraction as  $(A \otimes (\Lambda V)^{\otimes n-1}, d)$ -module.
- (c):  $\operatorname{HTC}_n(X)$  is the least m such that  $H(\rho_m)$  is injective.
- (d): nil ker  $H^*(\Delta_X^n, \mathbb{Q})$  is the longest non trivial product of elements of ker  $H^*(\Delta_X^n, \mathbb{Q})$ .

These invariants are ordered as follows ([5])

$$nil \ker H^*(\Delta_X^n, \mathbb{Q}) \le \operatorname{HTC}_n(X) \le \operatorname{mTC}_n(X) \le \operatorname{TC}_n(X_0) \le \operatorname{TC}_n(X).$$
 (4.1)

Inspired by the previous definition, we introduce the *Ext-version* of the original invariants. Now if we take  $\theta$  to be the identity of  $\Lambda V$ ,  $\mu_n^{\theta} = \mu_n^{Id_{\Lambda V}}$  becomes the *n*-fold multiplication in  $\Lambda V$  denoted by

$$u_n: (\Lambda V)^{\otimes n} \to \Lambda V_n$$

Therefore,  $nil \ker H^*(\Delta_X^n, \mathbb{Q}) = nil \ker H^*(\mu_n).$ 

In a similar way, we put

$$\mu_{\mathbf{A},n}: \mathbf{A}^{\otimes n} \to \mathbf{A} \qquad \text{and} \qquad \mu_{\mathcal{A},n}: \mathcal{A}^{\otimes n} \to \mathcal{A}$$

where  $A := Hom_{\Lambda V}((P, d), (\Lambda V, d))$  and  $\mathcal{A} := H(A) = \mathcal{E}xt_{(\Lambda V, d)}((P, d), (\Lambda V, d))$ . Then (cf. Definition 1.3) we have

**Definition 4.2.** (a):  $\operatorname{TC}_{n}^{\mathcal{E}xt}(X,\mathbb{Q})$  is the least *m* such that the projection

$$\Gamma_m : (\mathbf{A}^{\otimes n}, d) \to \left(\frac{\mathbf{A}^{\otimes n}}{(\ker(\mu_{\mathbf{A}, n}))^{m+1}}, \overline{d}\right)$$

admits a homotopy retraction.

- (b):  $mTC_n^{\mathcal{E}xt}(\mathbf{X}, \mathbb{Q})$  is the least m such that  $\Gamma_m$  admits a homotopy retraction as  $(\mathbf{A}^{\otimes n}, d)$ -module.
- (c):  $\operatorname{HTC}_{n}^{\mathcal{E}xt}(X,\mathbb{Q})$  is the least m such that  $H(\Gamma_{m})$  is injective.
- (d): nil ker  $(\mu_{\mathcal{A},n}, \mathbb{Q})$  is the longest non trivial product of elements of ker  $(\mu_{\mathcal{A},n})$ .

The same arguments used to establish the inequalities in (4.1) allow the following:

$$nil \ker (\mu_{\mathcal{A},n}, \mathbb{Q}) \le \operatorname{HTC}_{n}^{\mathcal{E}xt}(X, \mathbb{Q}) \le \operatorname{mTC}_{n}^{\mathcal{E}xt}(X, \mathbb{Q}) \le \operatorname{TC}_{n}^{\mathcal{E}xt}(X, \mathbb{Q})$$

**Remark 4.3.** Note that when X is a Poincaré duality space, the same process followed to prove that  $mTC_n(X) = HTC_n(X)$  ([5]) allows the following equality  $mTC_n^{\mathcal{E}xt}(X, \mathbb{Q}) = HTC_n^{\mathcal{E}xt}(X, \mathbb{Q})$ .

We are now in place to state the main theorem.

**Theorem 4.4.** Let X be a 1-connected finite type CW-complex. If X is a Gorenstein space over  $\mathbb{Q}$  and  $ev_{C^*(X,\mathbb{Q})} \neq 0$ , then

$$HTC_n^{\mathcal{E}xt}(X,\mathbb{Q}) \le HTC_n(X)$$

for any integer  $n \ge 2$ . Furthermore, if  $(\Lambda V, d)$  is a minimal Sullivan model of X, [f] is the generating class of  $\mathcal{A}$  and  $m = HTC_n^{\mathcal{E}xt}(X, \mathbb{Q})$ , then:

$$HTC_n(X) = HTC_n^{\mathcal{E}xt}(X, \mathbb{Q}) =: m \Leftrightarrow f(1)^{\otimes n} \in (\ker \mu_n)^m \setminus (\ker \mu_n)^{m+1}.$$

Before giving the proof of the theorem, let us first recall that if X is a finite ndimensional sub-complex of  $\mathbb{R}^{n+k}$ , k > n+1 and M its regular neighborhood, the homotopy fiber  $F_X$  of the inclusion  $\partial M \hookrightarrow M$  is called the *Spivak fiber* for X and it is a homotopy invariant of X. It is introduced in [20] where it is shown that X is a Poincaré complex if and only if  $F_X$  is a homotopy sphere.

The following corollary presents some essential classes of spaces satisfying the previous theorem:

**Corollary 4.5.** The hypothesis of 1.4 and hence its conclusions are satisfied in the following cases:

- (a) X is rationally elliptic,
- (b)  $H_{>N}(X,\mathbb{Z}) = 0$ , for some N, and  $H^*(X,\mathbb{Q})$  is a Poincaré duality algebra,
- (c) X is a finite 1-connected CW-complex and its Spivak fiber  $F_X$  has finite dimensional cohomology.

For the sake of completeness, we present below a sketch of the proof of the corollary:

- (a) If X is rationally elliptic, then it is Gorenstein and  $ev_{C^*(X,\mathbb{Q})} \neq 0$  thanks to [11, Proposition 3.4] and [18, Theorem A] respectively.
- (b) Referring to [7] (cf. also [11, Theorem 3.6]), under the hypothesis,  $H^*(X, \mathbb{Q})$  is a Poincaré duality algebra if and only if X is a Gorenstein space over  $\mathbb{Q}$ . Thus, by [11, Theorem 2.2] we have dim  $\pi_*(X) \otimes \mathbb{Q} < \infty$  and by [18, Theorem A] we obtain  $ev_{C^*(X,\mathbb{Q})} \neq 0$ .
- (c) Here using [11, Corollary 4.5], we have  $H^*(X, \mathbb{Q})$  is a Poincaré duality algebra, hence it is a Gorenstein algebra. It results that X is Gorenstein over  $\mathbb{Q}$  [11, Proposition 3.2] and  $ev_{C^*(X,\mathbb{Q})} \neq 0$  as in the previous case.

**Proof.** (of Theorem 4.1): Let  $(\Lambda V, d)$  be a Sullivan minimal model of X. The projections

$$\Gamma_m : (\mathbf{A}^{\otimes n}, d) \to \left(\frac{\mathbf{A}^{\otimes n}}{\left(\ker\left(\mu_{\mathbf{A}, n}\right)\right)^{m+1}}, \overline{d}\right) \quad \text{and} \quad \rho_m : \left((\Lambda V)^{\otimes n}, d\right) \to \left(\frac{(\Lambda V)^{\otimes n}}{\left(\ker\mu_n\right)^{m+1}}, \overline{d}\right)$$

induce two short exact sequences linked by chain evaluation map

Since X is Gorenstein,  $\mathcal{A} \cong \mathbb{Q}\Omega$  where  $\Omega$  is the generating class represented by a cocycle  $f \in \mathcal{A}^N$  of degree N = fd(X), the formal dimension of X (cf. §5, [11]). Therefore, the

diagram (4.2) induces, in cohomology, the following one

$$\begin{array}{cccc} 0 & \longrightarrow & H^{nN}(\ker{(\mu_{\mathrm{A},n})^{m+1}}) & \longrightarrow & (\mathcal{A}^{N})^{\otimes n} \stackrel{H^{nN}(\Gamma_{\mathrm{m}})}{\longrightarrow} H^{nN}(\frac{\mathrm{A}^{\otimes n}}{\ker{(\mu_{\mathrm{A},n})^{m+1}}}) & \longrightarrow & 0 \\ & & & \downarrow & & \downarrow \\ & & & \downarrow & & \downarrow e^{v_{(\Lambda V,d)}^{\otimes n}} & \downarrow H^{nN}(\theta) \\ H^{nN-1}(\frac{(\Lambda V)^{\otimes n}}{(\ker{\mu_n})^{m+1}}) & \longrightarrow & H^{nN}((\ker{\mu_n})^{m+1}) & \longrightarrow & (H^N(\Lambda V))^{\otimes n} \stackrel{H^{nN}(\rho_m)}{\longrightarrow} H^{nN}(\frac{(\Lambda V)^{\otimes n}}{(\ker{\mu_n})^{m+1}}) & \longrightarrow & 0. \end{array}$$

Now, since  $ev_{(\Lambda V,d)} = ev_{C^*(X,\mathbb{Q})} \neq 0$ , this is also the case for the horizontal arrow  $ev_{(\Lambda V,d)}^{\otimes n}$ . Thus, if  $H^{nN}(\rho_m)$  is injective then  $H^{nN}(\Gamma_m)$  is also injective. It results that:  $HTC_n^{\mathcal{E}xt}(X,\mathbb{Q}) \leq HTC_n(X)$ .

$$\begin{split} HTC_{n}^{(\Lambda V,a)} \\ HTC_{n}^{\ell xt}(X,\mathbb{Q}) &\leq HTC_{n}(X). \\ \text{Next, let } m \text{ denote the smallest integer such that } H^{nN}(\Gamma_{m}) \text{ is injective or, equivalently,} \\ f^{\otimes n} \text{ is a cocycle in } \mathbf{A}^{\otimes n} \text{ and } f^{\otimes n} \in \ker(\mu_{\mathbf{A},n})^{m} \backslash \ker(\mu_{\mathbf{A},n})^{m+1} \text{ (see Remark below). More$$
 $over, since } (\mathcal{A}^{N})^{\otimes n} \text{ is one dimensional, } H^{nN}(\Gamma_{m}) \text{ is indeed a bijection. Hence, } H^{nN}(\rho_{m}) \text{ is injective if and only if } H^{nN}(\theta) \text{ is injective. But, } ev_{(\Lambda V,d)}^{\otimes n} \text{ being non-zero, the commutativ$  $ity of the right diagram implies that this is equivalent to <math>f(1)^{\otimes n} \in (\ker \mu_{n})^{m} \backslash (\ker \mu_{n})^{m+1}. \\ \text{Notice that } \mu_{n}(f(1)^{\otimes n}) \text{ is a cocycle in } (\Lambda V)^{\otimes n} \text{ and } ev_{(\Lambda V,d)}^{\otimes n} [f^{\otimes n}] = [f(1)]^{\otimes n} \neq 0. \text{ It results} \\ \text{that } HTC_{n}(X) = HTC_{n}^{\ell xt}(X, \mathbb{Q}) =: m \Leftrightarrow f(1)^{\otimes n} \in (\ker \mu_{n})^{m} \backslash (\ker \mu_{n})^{m+1}. \end{split}$ 

**Remark 4.6.** An equivalent definition of  $HTC_n(X)$ , when X is a Poincaré duality space, reads as follows: It is the smallest integer  $m \ge 0$  such that some cocycle  $\omega$  representing the fundamental class of  $(\Lambda V, d)^{\otimes n}$ , can be written as a product of m elements of  $ker(\mu_n)$ (not necessarily cocycles). Similarly, for any Gorenstein space X,  $HTC_n^{\mathcal{E}xt}(X, \mathbb{Q})$  is the smallest integer m such that some cocycle representing the fundamental class of  $\Lambda^{\otimes n}$ , namely  $\Omega = [f]^{\otimes n}$  where [f] designates the generating element of  $\mathcal{A}^N$ , can be written as a product of length m of elements in  $ker(\mu_{A,n})$ . Therefore, in order to determine  $HTC_n(X)$ we may, using the precedent theorem, calculate  $m = HTC_n^{\mathcal{E}xt}(X, \mathbb{Q})$ , which is quite simpler since  $\mathcal{A}^*$  is one dimensional, and afterwards deal with the obstruction to have the equality.

Now, if dim  $V < \infty$  and dim  $H(\Lambda V, d) = \infty$  then, by [18, Theorem A], we have  $(\Lambda V, d)$  is a Gorenstein but not a Poincaré duality algebra. Moreover  $ev_{(\Lambda V,d)} = 0$ . Hence, in this case, to compare the invariants  $HTC_n^{\mathcal{E}xt}(X, \mathbb{Q})$  and  $HTC_n(X)$ , we determine them separately.

## 5. Use of the Adams-Hilton model

Let R be a principal ideal domain containing  $\frac{1}{2}$ , and let  $\rho(R)$  denote the least non invertible prime ( or  $\infty$ ) in R.  $CW_r(R)$  is the sub-category of finite r-connected CW-complexes X ( $r \geq 1$ ) satisfying dim $(X) \leq r\rho(R)$ .

In his attempt to extend Sullivan's theory to arbitrary rings ([14], see also [2]), S. Halperin associated to every X in  $CW_r(R)$  an appropriate differential graded Lie algebra  $(L,\partial)$  and showed that its Cartan-Eilenberg-Chevally complex  $C^*(L,\partial)$  is, on one hand, linked with the cochains algebra  $C^*(X; R)$  by a series of quasi-isomorphisms ([14, p. 274]) and, on the other hand, is quasi-isomorphic to a free commutative differential graded algebra  $(\Lambda W, d)$  [14, §7].  $(\Lambda W, d)$  is then called a free commutative model of X or a minimal Sullivan model of X.

Now, if  $\mathbb{K}$  is a field with odd characteristic (containing  $\frac{1}{2}$ ) and X is a q-connected  $(q \ge 1)$  finite CW-complex such that dim  $X \le q \cdot char(\mathbb{K})$ , i.e.  $X \in CW_q(\mathbb{K})$ , it has a minimal Sullivan model  $(\Lambda W, d)$  [14, Theorem 7.1].

The Adams-Hilton model ([1]) of X over  $\mathbb{K}$  is a chain algebra quasi-isomorphism  $\theta_X : (TV, d) \xrightarrow{\simeq} C_*(\Omega X; \mathbb{K})$ , i.e.  $H_*(\theta_X)$  is an isomorphism of graded algebras. Here V satisfies

 $H_{i-1}(V, d_1) \cong H_i(X; \mathbb{K})$  and  $d_1: V \to V$  is the linear part of d. (TV, d) is called a *free* model of X.

Therefore, using the isomorphism (2.1) we have successively:

$$\mathcal{E}xt_{C^*(X;\mathbb{K})}(\mathbb{K}, C^*(X;\mathbb{K})) \xrightarrow{\cong} \mathcal{E}xt_{(\Lambda W,d)}(\mathbb{K}, (\Lambda W, d)).$$

and

$$\mathcal{E}xt_{(TV,d)}(\mathbb{K}, (TV,d)) \stackrel{\cong}{\to} \mathcal{E}xt_{C_*(\Omega X;\mathbb{K})}(\mathbb{K}, C_*(\Omega X;\mathbb{K})).$$

Now, combining these two models via the isomorphism of graded  $\mathbb{K}$ -vector spaces [11, Theorem 2.1] yields

$$\mathcal{E}xt_{C^*(X;\mathbb{K})}(\mathbb{K}, C^*(X;\mathbb{K})) \xrightarrow{\cong} \mathcal{E}xt_{C_*(\Omega X;\mathbb{K})}(\mathbb{K}, C_*(\Omega X;\mathbb{K})).$$
(5.1)

gives the isomorphism of graded  $\mathbb{K}$ -vector spaces:

$$\mathcal{E}xt_{(\Lambda W,d)}(\mathbb{K}, (\Lambda W, d)) \cong \mathcal{E}xt_{(TV,d)}(\mathbb{K}, (TV, d)).$$

Argument used in the rational case allows us to conclude that  $\mathcal{E}xt_{(\Lambda W,d)}(\mathbb{K}, (\Lambda W, d))$  has the structure of a graded commutative algebra with unit. The latter isomorphism serves to endow  $\mathcal{E}xt_{(TV,d)}(\mathbb{K}, (TV, d))$  with the same structure. It results the following

**Proposition 5.1.** Let  $\mathbb{K}$  be a field with odd characteristic and  $X \in CW_q(\mathbb{K})$ . Then, the graded vector spaces  $\mathcal{E}xt_{C^*(X;\mathbb{K})}(\mathbb{K}, C^*(X;\mathbb{K}))$  and  $\mathcal{E}xt_{C^*(\Omega X;\mathbb{K})}(\mathbb{K}, C_*(\Omega X;\mathbb{K}))$  have isomorphic graded commutative algebra structures with unit. In particular, the Adams-Hilton model can be used to make this structure explicit.

Recall that  $\mathcal{E}xt_{(TV,d)}(\mathbb{K}, (TV, d))$  is, as in the rational case, obtained in terms of the acyclic closure of  $\mathbb{K}$  of the form  $(TV \otimes (\mathbb{K} \oplus sV), \delta)$ , where the differential  $\delta$  satisfies  $\delta s + sd = id$ , d being the differential of TV. That is, for any element  $z \otimes sv$  of  $TV \otimes (\mathbb{K} \oplus sV)$ , we have

$$\delta(z \otimes sv) = dz \otimes sv + (-1)^{|z|} zv \otimes 1 - (-1)^{|z|} z \otimes sdv.$$

Notice that any element f in  $Hom^p_{(TV,d)}((TV \otimes (\mathbb{K} \oplus sV), \delta), (TV, d))$  is entirely determined by its image of  $1 \otimes (\mathbb{K} \oplus sV)$  since  $TV \otimes (\mathbb{K} \oplus sV)$  is a left (TV, d)-module acting on the first factor. Thus we have

$$(D(g))(1 \otimes sv) = d \circ f(1 \otimes sv) - (-1)^p f \circ \delta(1 \otimes sv)$$
$$= df(1 \otimes sv) - (-1)^{p(|v|+1)} v f(1) + (-1)^p f(1 \otimes sdv)$$

Therefore,

(a) An element g in  $Hom_{(TV,d)}^{p-1}((TV \otimes (\mathbb{K} \oplus sV), \delta), (TV, d))$  is in Im(D) if and only if g = D(f) for some f in  $Hom_{(TV,d)}^{p}((TV \otimes (\mathbb{K} \oplus sV), \delta), (TV, d))$ , i.e.

$$g(1 \otimes sv) = df(1 \otimes sv) - (-1)^{p(|v|+1)} vf(1) + (-1)^p f(1 \otimes sdv).$$

Consequently:

$$g \in Im(D) \Leftrightarrow g(1 \otimes sv) = df(1 \otimes sv) - (-1)^{p(|v|+1)}vf(1) + (-1)^{p}f(1 \otimes sdv) \text{ for some } f.$$

$$(5.2)$$
(b) An element  $f \in Hom^{p}_{(TV,d)}((TV \otimes (\mathbb{K} \oplus sV), \delta), (TV, d)) \text{ is in } Ker(D) \text{ if and only if}$ 

$$D(f) = 0, \text{ that is, } df(1 \otimes sv) = (-1)^{p(|v|+1)}vf(1) - (-1)^{p}f(1 \otimes sdv). \text{ Consequently:}$$

$$f \in Ker(D) \Leftrightarrow df(1 \otimes sv) = (-1)^{p(|v|+1)}vf(1) - (-1)^{p}f(1 \otimes sdv). \quad (5.3)$$

Now, since deg(d) = -1,  $\mathcal{A}_* = (Hom_{(TV,d)}((TV \otimes (\mathbb{K} \oplus sV), \delta), (TV, d)), D)$  is a  $dga_*$ in the sense of [11]. Using the standard convention  $\mathcal{A}^{-q} = \mathcal{A}_q$ , for all  $q \in \mathbb{Z}$ , we obtain a  $dga^*$  whose cohomology at -p is the  $\mathbb{K}$ -module:

$$\mathcal{E}xt_{(TV,d)}^{-p}(\mathbb{Q},(TV,d)) = H_p\left(Hom_{(TV,d)}((TV\otimes(\mathbb{K}\oplus sV),\delta),(TV,d)),D\right).$$

More explicitly, if  $f \in Hom_{(TV,d)}(TV \otimes (\mathbb{K} \oplus sV), \Lambda V)$  is a cycle of (homological) degree p, it defines a cohomological class  $[f] \in \mathcal{E}xt^{-p}_{(TV,d)}(\mathbb{Q}, (TV, d))$  of degree -p.

# 5.1. Case of a suspension

Assume that  $\mathbb{K} = \mathbb{Q}$  and let X be a simply connected space, and  $Y = \Sigma X$  its suspension. The morphism of graded modules  $\sigma_* : H_*(X; \mathbb{Q}) \longrightarrow H_*(\Omega \Sigma X; \mathbb{Q})$  induced by the adjoint  $\sigma : X \to \Omega \Sigma X$  of  $id_{\Sigma X}$  extends to a morphism of graded algebras  $T(\sigma_*) : TH_*(X; \mathbb{Q}) \longrightarrow H_*(\Omega \Sigma X; \mathbb{Q})$ . In fact, this is an isomorphism of graded algebras since  $H_*(X; \mathbb{Q})$  is a free graded  $\mathbb{Q}$ -module. Therefore ([3])

$$(TH_*(X;\mathbb{Q}),0) \xrightarrow{\simeq} C_*(\Omega\Sigma X;\mathbb{Q})$$

is an Adams-Hilton model of  $Y = \Sigma X$ . It results that

$$\mathcal{E}xt_{C_*(\Omega\Sigma X;\mathbb{Q})}(\mathbb{Q}, C_*(\Omega\Sigma X;\mathbb{Q})) \cong \operatorname{Ext}_{TV}(\mathbb{Q}, TV),$$

where  $V = H_*(X; \mathbb{Q})$ . Now if  $(\Lambda W, d)$  is a Sullivan model of  $Y = \Sigma X$ , using (5.1) we obtain a commutative diagram

This permits the use of Adams-Hilton models to explicitly describe the algebra structure on  $\mathcal{A}$ , since it restricts to ordinary Ext which loosen the calculations. Notice that  $\Omega(\Sigma X)$ is weakly equivalent to the James space J(X) and, referring to [12, Example 7], that there exists a minimal Sullivan model for  $\Sigma X$  of the form  $(\Lambda Z, d)$  with quadratic differential, i.e. such that  $d(Z) \subset \Lambda^2 Z$ .

# 5.2. When X is a 2-cell CW complex

Let  $\mathbb{K}$  any field containing  $\frac{1}{2}$ .

In this subsection, we showcase another use of the Adams-Hiton models to help picture the algebra structure on  $\mathcal{A}$ . Let then  $X = S^q \cup_{\varphi} e^{q+1}$ ,  $q \ge 2$ , be the space where the cell  $e^{q+1}$  is attached by a map  $\varphi$  of degree r. The Adams-Hilton model of X has the form (TV, d), where V is a  $\mathbb{K}$ -vector space generated by a and a' with deg(a) = q - 1, deg(a') = q, da = 0 and da' = -ra.

Let us go back to where we left off at the beginning of this section and apply, in this case, the obtained formulas (5.2) and (5.3).

$$g \in Im(D) \Leftrightarrow \begin{cases} g(1) = df(1), \\ g(1 \otimes sa) = df(1 \otimes sa) - (-1)^{pq} af(1), & (\text{ for some } f) \\ g(1 \otimes sa') = df(1 \otimes sa') - (-1)^{p} rf(1 \otimes sa) - (-1)^{p(q+1)} a'f(1), \end{cases}$$
(5.4)

and

$$f \in Ker(D) \Leftrightarrow \begin{cases} df(1) = 0, \\ df(1 \otimes sa) = (-1)^{pq} a f(1), \\ df(1 \otimes sa') = (-1)^{p} r f(1 \otimes sa) + (-1)^{p(q+1)} a' f(1). \end{cases}$$
(5.5)

Recall that to any pointed topological space X, it is associated in [11] an invariant called the *formal dimension* of X (with respect to a field  $\mathbb{K}$ ) defined as follows:

$$fd(X,\mathbb{K}) = \sup\{r \in \mathbb{Z} \mid [\mathcal{E}xt^p_{C^*(X;\mathbb{K})}(\mathbb{K}, C^*(X;\mathbb{K}))]^r \neq 0\},\$$

or  $fd(X, \mathbb{K}) = -\infty$  if such integer does not exist. In particular [11, Proposition 5.1], if  $H^*(X; \mathbb{K})$  is finite dimensional,

$$fd(X,\mathbb{K}) = \sup\{r \in \mathbb{Z} \mid H^r(X;\mathbb{K}) \neq 0\}.$$

Notice that, using cellular homology, we see that  $H_*(X,\mathbb{Z}) = H_0(X,\mathbb{Z}) \oplus H_q(X,\mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}/r\mathbb{Z}$ . We should then discuss two cases:

- (i) If  $char(\mathbb{K}) = 0$  or co-prime with r, we have  $H^*(X, \mathbb{K}) = H^0(X, \mathbb{K}) \cong \mathbb{K}$ . In this case,  $H^*(X, \mathbb{K})$  has formal dimension fd(X) = 0, thus, it is a Poincaré duality space. Moreover, since it has finite dimensional cohomology, it is also a Gorenstein space [11, Theorem 3.1].
- (ii) If  $char(\mathbb{K})$  divides r then,  $H^*(X, \mathbb{K}) = H^0(X, \mathbb{K}) \oplus H^q(X, \mathbb{K}) \oplus H^{q+1}(X, \mathbb{K}) \cong \mathbb{K} \oplus \mathbb{K} \oplus \mathbb{K}$ . Thus, since  $q \ge 2$ , X is neither a Poincaré duality space nor a Gorenstein space [7, Theorem 1]. In this case, fd(X) = q + 1, so that  $\mathcal{E}xt^k_{(TV,d)}(\mathbb{K}, (TV, d)) = 0, \forall k > q + 1$ .

**Example:** In this example, we specify the case where q = 2, i.e.  $X = S^2 \cup_{\varphi} e^3$ . Thus  $V = \mathbb{K}a \oplus \mathbb{K}a'$  with |a| = 1 and |a'| = 2. We give below an explicit computation of it to illustrate the use of Adams-Hilton models.

i. Assume that  $char(\mathbb{K}) = 0$  or co-prime with r (we specialize in the case where  $\mathbb{K} = \mathbb{Q}$ ). Let f be a cycle of degree 0, we have df(1) = 0, then f(1) is necessarily a scalar  $f(1) = \gamma$ . The second equation in (5.5) implies that  $df(sa) = \gamma a$ , therefore  $f(sa) = -\frac{\gamma}{r}a' + \gamma'a^2$ . The last equation in (5.5) gives, after a simple simplification,  $df(sa') = r\gamma'a^2$ , then  $f(sa') = -\gamma'_1a' \cdot a + \gamma'_2a \cdot a' + \gamma''a^3$ ,

$$Ker(D) = \frac{\mathbb{Q} \oplus \mathbb{Q}a' \oplus \mathbb{Q}a^2 \oplus \mathbb{Q}a' \cdot a \oplus \mathbb{Q}a \cdot a' \oplus \mathbb{Q}a^3}{\langle x_1 = -rx_2; x_3 = -x_4 + x_5 \rangle}$$

Now let g be an arbitrary element of degree 1:

$$\begin{cases} g(1) = \alpha_1 a \\ g(sa) = \alpha_2 a' \cdot a + \alpha_3 a \cdot a' + \alpha_4 a^3 \\ g(sa') = \alpha_5 a'^2 + \alpha_6 a' \cdot a^2 + \alpha_7 a \cdot a' \cdot a + \alpha_8 a^2 \cdot a' + \alpha_9 a^4 \end{cases}$$

by (5.4) we have

$$\begin{cases} D(g)(1) = 0\\ D(g)(sa) = (-\alpha_1 - r\alpha_2 + r\alpha_3)a^2\\ D(g)(sa') = (\alpha_1 + r\alpha_2 - r\alpha_5)a' \cdot a + (r\alpha_3 - r\alpha_5)a \cdot a' + (r\alpha_4 - r\alpha_6 + r\alpha_7 - r\alpha_8)a^3, \end{cases}$$

therefore

$$Im(D) = \frac{\mathbb{Q}a^2 \oplus \mathbb{Q}a' \cdot a \oplus \mathbb{Q}a \cdot a' \oplus \mathbb{Q}a^3}{\langle x_1 = -x_2 + x_3 \rangle}$$

We consequently obtain  $\mathcal{E}xt^0_{(TV,d)}(\mathbb{Q}, (TV, d)) = \mathbb{Q}.$ 

Applying the same process for  $i \neq 0$ , we recover the previously stated fact  $\mathcal{E}xt^i_{(TV,d)}(\mathbb{Q}, (TV,d)) = 0.$ 

ii. Assume that  $char(\mathbb{K})$  divides r, so that  $r = 0 \pmod{char(\mathbb{K})}$  and da = da' = 0. Recall that in general, we have:

$$(D(f))(1 \otimes sa) = df(1 \otimes sa) - (-1)^{pq}af(1) = 0,$$

and

$$D(f))(1 \otimes sa') = df(1 \otimes sa') - (-1)^p rf(1 \otimes sa) - (-1)^{p(q+1)}a'f(1) = 0.$$

These become respectively in this case:

$$(D(f))(1 \otimes sa) = -(-1)^{pq}af(1) = 0$$
, and  $(D(f))(1 \otimes sa') = -(-1)^{p(q+1)}a'f(1) = 0$ .

Notice that in this case, for an element f to be in ker(D), it is necessarily that f(1) = 0.

Let f be a cycle of degree 0, then we have f(1) = 0, consequently df(sa) = df(sa') = 0, which implies that  $f(sa) = \gamma_1 a' + \gamma_2 a^2$  and  $f(sa') = \gamma_3 a' \cdot a + \gamma_4 a \cdot a' + \gamma_5 a^3$ . Therefore

$$Ker(D) = \mathbb{K}a^2 \oplus \mathbb{K}a' \oplus \mathbb{K}a' \cdot a \oplus \mathbb{K}a \cdot a' \oplus \mathbb{K}a^3.$$

Now let g be an arbitrary element of degree 1:

$$\begin{cases} g(1) = \alpha_1 a \\ g(sa) = \alpha_2 a' \cdot a + \alpha_3 a \cdot a' + \alpha_4 a^3 \\ g(sa') = \alpha_5 a'^2 + \alpha_6 a' \cdot a^2 + \alpha_7 a \cdot a' \cdot a + \alpha_8 a^2 \cdot a' + \alpha_9 a^4 \end{cases}$$

hence

$$\left\{ \begin{array}{l} D(g)(1) = 0 \\ D(g)(sa) = -ag(1) = -\alpha_1 a^2 \\ D(g)(sa') = a'g(1) = \alpha_1 a' \cdot a, \end{array} \right.$$

therefore

$$Im(D) = \frac{\mathbb{K}a^2 \oplus \mathbb{K}a' \cdot a}{\langle x_1 = -x_2 \rangle} \cong \mathbb{K}(a' \cdot a - a^2).$$

we obtain  $\mathcal{E}xt^0_{(TV,d)}(\mathbb{K}, (TV,d)) \cong \mathbb{K}a^2 \oplus \mathbb{K}a' \oplus \mathbb{K}a \cdot a' \oplus \mathbb{K}a^3 \cong \mathbb{K}^4$ . An application of the same argument yields:

$$\mathcal{E}xt^{-1}_{(TV,d)}(\mathbb{K}, (TV,d)) \cong \mathbb{K}a^3 \oplus \mathbb{K}a \cdot a' \oplus \mathbb{K}a' \cdot a \oplus \mathbb{K}a \cdot a' \oplus \mathbb{K}a^2 \cdot a' \oplus \mathbb{K}a^4 \cong \mathbb{K}^6,$$

$$\begin{aligned} & \mathcal{E}xt_{(TV,d)}^{-2}(\mathbb{K},(TV,d)) &\cong & \mathbb{K}a^4 \oplus \mathbb{K}a^2 \cdot a' \oplus \mathbb{K}a \cdot a' \cdot a \oplus \mathbb{K}a' \cdot a^2 \oplus \mathbb{K}a'^2 \oplus \\ & & \mathbb{K}a \cdot a'^2 \oplus \mathbb{K}a^3 \cdot a' \oplus \mathbb{K}a^2 \cdot a' \cdot a \oplus \mathbb{K}a \cdot a' \cdot a^2 \oplus \mathbb{K}a^5 \\ &\cong & \mathbb{K}^{10}. \end{aligned}$$

From the previous cases, it is obvious that  $\mathcal{E}xt^{-i}_{(TV,d)}(\mathbb{K}, (TV,d)) \neq 0, \forall i \geq 0$ . Since  $fd(X,\mathbb{K}) = 3$ , we have  $\mathcal{E}xt^{i}_{(TV,d)}(\mathbb{K}, (TV,d)) = 0, \forall i \geq 4$ . It remains then to calculate the cases i = -1, -2, -3 which are given successively by applying the same process as follows:  $\mathcal{E}xt^{1}_{(TV,d)}(\mathbb{K}, (TV,d)) \cong \mathbb{K}a' \oplus \mathbb{K}a^{2}, \mathcal{E}xt^{2}_{(TV,d)}(\mathbb{K}, (TV,d)) \cong \mathbb{K} \oplus \mathbb{K}a$  and  $\mathcal{E}xt^{3}_{(TV,d)}(\mathbb{K}, (TV,d)) \cong \mathbb{K}$ .

Now notice that for q > 2, Adams-Hilton model for  $X = S^q \cup_{\varphi} e^{q+1}$  has two generators a and a' of degrees respectively q-1 and q, thus the degree of a' does not double that of a, so the previous example is somewhat a special case. However the computational process still holds, and we have, for the case  $\mathbf{i}$ ,  $\mathcal{E}xt^0_{(TV,d)}(\mathbb{K}, (TV, d)) = \mathbb{K}$  and  $\mathcal{E}xt^i_{(TV,d)}(\mathbb{K}, (TV, d)) = 0$  for  $i \neq 0$ . Whereas for the case  $\mathbf{i}$ , computation process holds but the results differ, since we might have  $\mathcal{E}xt^{-i}_{(TV,d)}(\mathbb{K}, (TV, d)) = 0$  for finitely many  $i \geq -(q+1)$  (e.g. for q = 7 we have  $\mathcal{E}xt^7_{(TV,d)}(\mathbb{K}, (TV, d)) = 0$ ), on the other hand, we always have  $\mathcal{E}xt^{-i}_{(TV,d)}(\mathbb{K}, (TV, d)) = 0$ ,  $\forall i < -(q+1)$  since fd(X) = q + 1.

Acknowledgment. We would like to express our sincere gratitude to the referee and the editor for their efforts in proofreading and valuable comments, which improved the quality of this paper.

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