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Research Article

On Topological Complexity of Gorenstein spaces

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Abstract

In this paper, using Sullivan's approach to rational homotopy theory of simply connected finite type CW complexes, we endow the $\mathbb{Q}$-vector space $\mathcal{E}xt_{C^*(X;\mathbb{Q})}(\mathbb{Q}, C^*(X;\mathbb{Q}))$ with a graded commutative algebra structure. Thus, we introduce new algebraic invariants referred to as the Ext-versions of the ordinary higher, module, and homology Topological Complexities of X_0, the rationalization of X. For Gorenstein spaces, we establish, under additional hypotheses, that the new homology topological complexity, denoted $HTC_n^{\mathcal{E}xt}(X,\mathbb{Q})$, lowers the ordinary $HTC_n(X)$ and, in case of equality, we extend Carasquel's characterization for $HTC_n(X)$ to some class of Gorenstein spaces (Theorem 1.2). We also highlight, in this context, the benefit of Adams-Hilton models over a field of odd characteristic especially through two cases, the first one when the space is a 2-cell CW-complex and the second one when it is a suspension.

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1. Introduction

The concept of *topological complexity* $TC(-)$ was introduced by Michael Farber in [6]. It is a numerical invariant that measures the difficulty of planning continuous motions of a mechanical system along its configuration space X. Formally, $TC(X) = secat(\Delta_X)$, the *sectional category* of the diagonal map $\Delta_X : X \to X \times X$, defined as the smallest integer m such that there exist $m+1$ local homotopy sections $s_i : U_i \to X$ such that $\{U_i\}$ form an open cover of $X \times X$. Intuitively, this means that we can divide $X \times X$ into $m+1$ parts, and on each part, the system has a well-defined continuous motion. Since its introduction, the concept TC has been studied in various contexts such as robotics, algorithmic topology, and algebraic topology. Later, Y. Rudyak ([19]) generalizes Farber's concept by introducing its higher analogous denoted $TC_n(X)$ $(n \geq 2)$ and defined as the sectional category, $secat(\Delta_X^n)$, of the n-diagonal $\Delta_X^n : X \to X^n$ so that $TC(X) = TC_2(X)$.

In [4], J. Carrasquel used the characterization à la Félix-Halperin to give an explicit definition of the *(higher) rational topological complexity* $TC_n(X_0)$, of the rationalization

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X_0 of X, which turns out to lower $TC_n(X)$ (cf. Definition 4.1 below). He also introduced higher (rational) *homology (resp. module) topological complexity* $HTC_n(X)$, (resp. $mTC_n(X)$) and showed that they interpolate $zcl_n(X) := nilker H^*(\Delta_X^n;\mathbb{Q})$ and $TC_n(X_0)$.

In this paper, we approach the study (rational versions) of the aforementioned invariants for *Gorenstein spaces*. These were introduced in [11] with the aim of deepening the study, in the context of local algebra, of the *Lusternik-Scnirelmann category* $cat_{LS}(X) = secat(* \hookrightarrow X)$ [17]. Recall from [20] that a finite n-dimensional subcomplex X of $\mathbb{R}^{n+k}$ is a Poincaré complex if and only if its *Spivak fiber* F_X (see §4) is a homotopy sphere. In particular, when X has finite $cat_{LS}(X)$ and $\mathbb{K} = \mathbb{Z}_p$ for p prime or zero, a *local analogue of the Spivak characterization* in terms of the Eilenberg-Moore Ext functor $\mathcal{E}xt$ reads as follows [11, Theorem 3.1]:

$$\begin{aligned} H^*(X,\mathbb{Z}_p) \text{ is a Poincaré duality algebra } &\Leftrightarrow (F_X)_{(p)} \simeq (\mathbb{S}^k)_{(p)} \\ &\Leftrightarrow \dim \mathcal{E}xt_{C^*(X;\mathbb{Z}_p)}(\mathbb{Z}_p, C^*(X;\mathbb{Z}_p)) = 1 \end{aligned} \tag{1.1}$$

(cf. §2 for the definition of Poincaré duality algebras and the functor $\mathcal{E}xt$). The equivalences in (1.1) make sense to the following definition [11]:

Definition 1.1. X is a Gorenstein space over $\mathbb{K}$ if $\dim \mathcal{E}xt_{C^*(X;\mathbb{K})}(\mathbb{K}, C^*(X;\mathbb{K})) = 1$.

For instance, if X is finite dimensional then [11, Proposition 4.5, Theorem 3.1]:

$$\begin{aligned} X \text{ is Gorenstein over } \mathbb{K} &\Leftrightarrow H^*(X,\mathbb{K}) \text{ is a Gorenstein algebra} \\ &\Leftrightarrow H^*(X,\mathbb{K}) \text{ is a Poincaré duality algebra.} \end{aligned}$$

E.g. every closed manifold is Gorenstein over any field $\mathbb{K}$. In the rational case, if $\dim \pi_*(X) \otimes \mathbb{Q}$ is finite dimensional then X is Gorenstein over $\mathbb{Q}$ [11, Proposition 3.4]. In particular, every rationally elliptic space X (i.e. if its rational homology and rational homotopy are both finite dimensional) is Gorenstein over $\mathbb{Q}$. Another advantageous homotopy invariant that allows to better (algebraically) characterize Poincaré duality spaces is the morphism of $\mathbb{K}$-graded vector spaces:

$$ev_{C^*(X;\mathbb{K})} : \mathcal{E}xt_{C^*(X;\mathbb{K})}(\mathbb{K}, C^*(X;\mathbb{K})) \to H^*(X,\mathbb{K})$$

called the *evaluation map* of X over $\mathbb{K}$ [11] (cf. §2 for more details).

In fact, we have [13, Corollary 2]:

$$X \text{ is a Poincaré duality space} \Leftrightarrow \begin{cases} (i)\ X \text{ is a Gorenstein space,} \\ (ii)\ ev_{C^*(X,\mathbb{K})} \neq 0, \\ (iii)\ H^*(X;\mathbb{Q}) \text{ is a Gorenstein algebra.} \end{cases}$$

In order to highlight our goal, notice that while the (ordinary) rational higher Topological Complexities $TC_n(X_0)$ has a simple geometric interpretation (as it is the case for $TC_n(X)$), its calculation is among NP-hard problems [16]. The complexities $mTC_n(X)$ and $HTC_n(X)$ are algebraic approximations of $TC_n(X_0)$ that can be determined using the same techniques as those used to calculate or at least approximate $cat_{LS}(X_0)$ ([11],[10],[8]). In fact, fewer are spaces for which the invariants $HTC_n(X)$, $mTC_n(X)$ and $TC_n(X_0)$ are determined. However, in [15], a new invariant $L(X_0)$ is introduced and it is shown that $cat_{LS}(X_0) + L(X_0) \leq TC_n(X_0)$ for any pure elliptic coformal space. It is also established that $TC_n(X_0) = \dim \pi_*(X) \otimes \mathbb{Q}$ for certain particular families of spaces.

In this work, inspired by Carrasquel's characterizations, we will introduce new algebraic approximations of $TC_n(X)$ which we call *Ext-versions topological complexities* and establish some inequalities between these invariants. To this end, we first endow, in section 3, the graded $\mathbb{Q}$-vector space

$$\mathrm{A} =: Hom_{C^*(X;\mathbb{Q})}((P,d), C^*(X;\mathbb{Q}))$$

with a homotopy multiplicative structure (see §3 below) denoted in all that follows by

$$\mu_{\mathrm{A}} : \mathrm{A} \otimes \mathrm{A} \to \mathrm{A},$$

where (P, d) is a semi-free resolution of $(\mathbb{Q}, 0)$ (cf. §2 for more details).

The cohomology of A is

$$\mathcal{A} = \mathcal{E}xt_{C^*(X;\mathbb{Q})}(\mathbb{Q}, C^*(X;\mathbb{Q})),$$

and our first main result reads:

Theorem 1.2. *The graded vector space $\mathcal{A}$ endowed with the product $\mu_{\mathcal{A}} := H^*(\mu_{\mathrm{A}})$ is a graded commutative algebra with unit. Moreover, the evaluation map*

$$ev_{(X,\mathbb{Q})} : \mathcal{E}xt_{C^*(X;\mathbb{Q})}(\mathbb{Q}, C^*(X;\mathbb{Q})) \to H^*(X;\mathbb{Q})$$

is a morphism of graded algebras.

Next, recall that ([5]) a morphism of graded commutative differential algebra (cdga for short) $\varpi : (C, d) \to (D, d)$ admits a *homotopy retraction* if there exists a map $r : (\Lambda V \otimes C, d)) \to (C, d)$ such that $r \circ \iota = Id_C$, where $\iota : (C, d) \to (\Lambda V \otimes C, d)$ is the relative model of ϖ. As a particular case, consider any Sullivan algebra $(\Lambda V, d)$ and denote by $\mu_{\mathrm{A},n} : \mathrm{A}^{\otimes n} \to \mathrm{A}$ the n-fold product of $\mu_{\mathrm{A}} : \mathrm{A} \otimes \mathrm{A} \to \mathrm{A}$. Using [4, Definition 9], we state in terms of the cdga projection

$$\Gamma_m : (\mathrm{A}^{\otimes n}, d) \to \left(\frac{\mathrm{A}^{\otimes n}}{(\ker(\mu_{\mathrm{A},n}))^{m+1}}, \overline{d} \right)$$

the following

Definition 1.3. (a): *$sc(\mu_{\mathrm{A},n})$ is the least m such that Γ_m admits a homotopy retraction.*

(b): *$msc(\mu_{\mathrm{A},n})$ is the least m such that Γ_m admits a homotopy retraction as $(\mathrm{A}^{\otimes n}, d)$-modules.*

(c): *$Hsc(\mu_{\mathrm{A},n})$ is the least m such that $H(\Gamma_m)$ is injective.*

(d): *$nil \ker(\mu_{\mathcal{A},n}, \mathbb{Q})$ is the longest non trivial product of elements of $\ker(\mu_{\mathcal{A},n})$.*

As a particular case, given $\mathrm{A} = (\Lambda V, d)$ a Sullivan model of X, the aforementioned (rational) *Ext*-version invariants are denoted respectively $\mathrm{TC}_n^{\mathcal{E}xt}(X, \mathbb{Q})$, $\mathrm{mTC}_n^{\mathcal{E}xt}(X, \mathbb{Q})$, $\mathrm{HTC}_n^{\mathcal{E}xt}(X, \mathbb{Q})$ and $zcl_n^{\mathcal{E}xt}(X, \mathbb{Q})$. We call theme respectively, *the rational, module, homology topological complexity* and *Ext-version zero cup length.*

Then, we prove the following

Theorem 1.4. *Let X be a 1-connected finite type CW-complex. If X is a Gorenstein space over $\mathbb{Q}$ and $ev_{C^*(X,\mathbb{Q})} \neq 0$, then*

$$HTC_n^{\mathcal{E}xt}(X, \mathbb{Q}) \leq HTC_n(X)$$

for any integer $n \geq 2$. Furthermore, if $(\Lambda V, d)$ is a Sullivan minimal model of X, $[f]$ is the generating class of $\mathcal{A}$ and $m = HTC_n^{\mathcal{E}xt}(X, \mathbb{Q})$, then:

$$HTC_n(X) = HTC_n^{\mathcal{E}xt}(X, \mathbb{Q}) =: m \Leftrightarrow f(1)^{\otimes n} \in (\ker \mu_n)^m \backslash (\ker \mu_n)^{m+1}.$$

Notice that the hypotheses on X imply that it is either (i) a Poincaré duality space over $\mathbb{Q}$ or else (ii) $H^*(X, \mathbb{Q})$ is not noetherian and not a Gorenstein graded algebra [13, Theorem 3]. Moreover, it is well known that if X satisfies the Poincaré duality property then $\omega = f(1)$ is a cocycle representing its fundamental class. Thus, (cf. Remark 4.6), our theorem extends, Corollary 5.5 in [5], to spaces of the class (ii).

The following corollary presents some essential classes of spaces satisfying the previous theorem

Corollary 1.5. *The hypothesis of Theorem 1.4 and hence its conclusions are satisfied in the following cases:*

(a) X *is rationally elliptic,*
(b) $H_{>N}(X,\mathbb{Z})=0$*, for some* N*, and* $H^*(X,\mathbb{Q})$ *is a Poincaré duality algebra,*
(c) X *is a finite* 1*-connected CW-complex and its Spivak fiber* F_X *has finite dimensional cohomology.*

The rest of the paper is organized as follows. In section 2 we summarize the tools essential for the rest of the document, while section 3 is dedicated to the proof of Theorem 1.1. In section 4, we formally introduce the *Ext*-version of higher, module and homology topological complexities. In Section 5 we will extend a part of Theorem 1.2 and its corollary to the sub-category $CW_r(R)$ when $R=\mathbb{K}$ is a field of odd characteristic and implement the advantageous computational properties of Adams-Hilton models [1] to obtain explicit calculations of the homotopy invariant $\mathcal{A}=\mathcal{E}xt_{C^*(X,\mathbb{K})}(\mathbb{K},C^*(X,\mathbb{K}))$ when X is a suspension or a two-cell CW-complex.

2. Preliminaries

Let $\mathbb{K}$ denote an arbitrary ground field unless otherwise stated.

2.1. Eilenberg-Moore Ext

A graded module is a family $A=(A^i)_{i\in\mathbb{Z}}$ of $\mathbb{K}$-modules denoted also $A=\oplus_{i\in\mathbb{Z}}A^i$. Every $a\in A^i$ is of degree i denoted thereafter $|a|$.

A linear map of graded modules $f:A\to B$ of degree $|f|$ is a $\mathbb{K}$-linear map sending each A^i to $B^{i+|f|}$. If $|f|=0$ we call it a morphism of graded modules.

In all that follows, unless otherwise stated, modules are over $\mathbb{K}$ *and we will assume that* $A^i=0$ *if* $i<0$.

A graded algebra A is a graded module together with an associative multiplication $\mu_A:A\otimes A\to A$ that has an identity element $1_A=:1\in A^0$. We will put $\mu_A(x\otimes y)=:xy$. Notice that $|\mu_A|=0$. Moreover, if we have $ab=(-1)^{|a||b|}ba$ for all $a,b\in A$, then A is said to be commutative.

A differential graded algebra (A,d) (dga for short) is a graded algebra A together with a linear map $d:A\to A$ of degree $|d|=+1$ that is a derivation $d(ab)=d(a)b+(-1)^{|a|}ad(b)$, and satisfying $d\circ d=0$.

A morphism of dga $f:(A,d)\to(B,d)$ is a linear map of degree zero satisfying $f(aa')=f(a)f(a')$, and compatible with the differential d: $f(da)=d(f(a))$.

A dga algebra A is said to be augmented if it is endowed with a morphism $\varepsilon:A\to\mathbb{K}$ of graded algebras.

A (left) graded (A,d) module is a graded module M equipped with a linear map $A\otimes M\to M$, $a\otimes m\mapsto am$ of degree zero such that $a(bm)=(ab)m$ and $1m=m$, and a differential d satisfying $d(am)=(da)m+(-1)^{|a|}a(dm)$, $m\in M$, $a\in A$.

A morphism of (left) graded modules over a dga (A,d) is a morphism $f:(M,d)\to(N,d)$ compatible with the differential: $d\circ f=f\circ d$.

A left (A,d)-module (M,d) is said to be semi-free if it is the union of an increasing sequence $M(0)\subset M(1)\subset M(2)\subset\cdots\subset M(n)\subset\cdots$ of sub (A,d)-modules such that $M(0)$ and each $M(i)/M(i-1)$ is A-free on a basis of cycles. Such an increasing sequence is called a semi-free filtration of (M,d).

A semi-free resolution of an (A,d)-module (M,d) is an (A,d)-semi-free module (P,d) together with a quasi-isomorphism (i.e. a morphism inducing an isomorphism in homology) $m:(P,d)\xrightarrow{\simeq}(M,d)$ of (A,d)-modules. Each of $P(0)$ and $P(i)/P(i-1)$ has the form $(A,d)\otimes(V(i),0)$ where $V(i)$ is a free $\mathbb{K}$-module. Thus the surjections $P(n)\to A\otimes V(n)$

split and the differential d satisfies:

$$P(n) = P(n-1) \oplus (A \otimes V(n)) \quad and \quad d : V(n) \to P(n-1).$$

Every (A, d)-module (M, d) has a semi-free resolution $m : (P, d) \xrightarrow{\simeq} (M, d)$ ([12, Prop. 6.6]) and if $m' : (P', d) \xrightarrow{\simeq} (M, d)$ is a second semi-free resolution, then, there exists an equivalence

$$\alpha : (P', d) \longrightarrow (P, d)$$

of (A, d)-modules such that $m \circ \alpha$ and m' are homotopic morphisms, denoted $m \circ \alpha \simeq_A m'$.

Particularly, let (A, d) be a differential graded algebra and $(P, d) \xrightarrow{\simeq} (\mathbb{Q}, 0)$ an (A, d)-semi-free resolution of $(\mathbb{Q}, 0)$. This defines the graded (A, d)-module

$$Hom_A((P, d), (A, d)) = \bigoplus_{p \geq 0} Hom_A^{p,*}((P, d), (A, d)) = \bigoplus_{p \geq 0} \bigoplus_{i \geq 0} Hom_A(P^i, A^{i+p}),$$

which, endowed with the differential

$$D(f) = d \circ f - (-1)^p f \circ d; \quad f \in Hom_A^{p,*}((P, d), (A, d)),$$

yields the Eilenberg-Moore Ext functor:

$$\mathcal{E}xt_{(A,d)}(\mathbb{K}, (A, d)) = H^*(Hom_A((P, d), (A, d)), D).$$

This is an invariant up to homotopy of differential graded algebras (see [11, Appendix] or [7, Appendix]). Moreover, [11, Remark 1.3], if $(A, d) \xrightarrow{\simeq} (B, d)$ is a quasi-isomorphism of differential graded algebras, then $\mathcal{E}xt_{(A,d)}(\mathbb{K}, (A, d))$ is identified with $\mathcal{E}xt_{(B,d)}(\mathbb{K}, (B, d))$ via natural (induced) isomorphisms

$$\mathcal{E}xt_{(A,d)}(\mathbb{K}, (A, d)) \xrightarrow{\cong} \mathcal{E}xt_{(A,d)}(\mathbb{K}, (B, d)) \xleftarrow{\cong} \mathcal{E}xt_{(B,d)}(\mathbb{K}, (B, d)). \tag{2.1}$$

Particularly, $\mathcal{E}xt_{C^*(X;\mathbb{K})}(\mathbb{K}, C^*(X; \mathbb{K}))$ and $\mathcal{E}xt_{C_*(\Omega X;\mathbb{K})}(\mathbb{K}, C_*(\Omega X; \mathbb{K}))$ depend only on the homotopy class of X.

The highest N such that $[\mathcal{E}xt_{C^*(X;\mathbb{K})}(\mathbb{K}, C^*(X; \mathbb{K}))]^N \neq 0$ is called the formal dimension of X. It is denoted $fd(X, \mathbb{K})$.

2.2. Evaluation map and Gorenstein spaces

Let $\rho : (P, d) \xrightarrow{\simeq} (\mathbb{K}, 0)$ be a minimal (A, d)-semi-free resolution of $(\mathbb{K}, 0)$. Consider the chain map

$$cev_{(A,d)} : \mathrm{Hom}_{(A,d)}((P, d), (A, d)) \longrightarrow (A, d)$$

given by $f \mapsto f(z)$, where $z \in P$ is a cocycle representing 1 in $\mathbb{K}$. We call it the *chain evaluation map of* (A, d). Passing to homology, we obtain the natural map

$$ev_{(A,d)} : \mathcal{E}xt_{(A,d)}(\mathbb{K}, (A, d)) \longrightarrow H^*(A, d),$$

called the *evaluation map of* (A, d). The definition of $ev_{(A,d)}$ is independent of the choice of (P, d) and z. The evaluation map $ev_{(X,\mathbb{K})}$ of X over $\mathbb{K}$ is by definition the *evaluation map* of $C^*(X, \mathbb{K})$.

A *Poincaré duality algebra* over $\mathbb{K}$ is a graded algebra $H = \{H^k\}_{0 \leq k \leq N}$ such that $H^N = \mathbb{K}\alpha$ and the pairing $< \beta, \gamma > \alpha = \beta\gamma$, $\beta \in H^k$, $\gamma \in H^{N-k}$ defines an isomorphism $H^k \xrightarrow{\cong} Hom_{\mathbb{K}}(H^{N-k}, \mathbb{K})$, $0 \leq k \leq N$. In particular, $H = Hom_{\mathbb{K}}(Hom_{\mathbb{K}}(H, \mathbb{K}), \mathbb{K})$ is necessarily finite dimensional.

A *Poincaré duality space* at $\mathbb{K}$ is a space whose cohomology with coefficients in $\mathbb{K}$ is a Poincaré duality algebra. In this case, the cohomology class α such that $H^N(X; \mathbb{K}) = \mathbb{K}\alpha$ has degree $N = fd(X, \mathbb{K})$[11, Proposition 5.1]. It is called the *fundamental class* of X.

A *Gorenstein algebra* over $\mathbb{K}$ is a differential graded algebra (A, d) whose associated graded vector space $\mathcal{E}xt_{(A,d)}(\mathbb{K}, (A, d))$ is one dimensional.

A space X is *Gorenstein* over $\mathbb{K}$ if the cochain algebra $C^*(X;\mathbb{K})$ is a Gorenstein algebra.

For instance, let X be a simply connected CW complex. If in addition X is finite dimensional, then, $C^*(X;\mathbb{K})$ is Gorenstein if and only if $H^*(X;\mathbb{K})$ is a Poincaé duality algebra [11, Theorem 3.1]. However, if $\dim \pi_*(X)\otimes\mathbb{Q}<\infty$ then, on one hand, X is a Gorenstein space even though X has infinite dimension [11, Proposition 3.4] (see also Corollary 1 above) and, on the other hand, $\dim H^*(X,\mathbb{Q})<\infty$ if and only if $ev_{C^*(X;\mathbb{Q})}\neq 0$ ([18]).

3. The $\mathbb{Q}$-algebra $\mathcal{E}xt_{C^*(X;\mathbb{Q})}(\mathbb{Q}, C^*(X;\mathbb{Q}))$

Along this section, the ground field is $\mathbb{Q}$. Recall that to every finite-type simply-connected space X, it is associated a quasi-isomorphism $(\Lambda V,d)\xrightarrow{\simeq} A_{PL}(X)$ from a free commutative differential graded algebra (cdga for short) $(\Lambda V,d)$ to the commutative graded algebra $A_{PL}(X)$ of polynomial forms with rational coefficients [12]. This latter is connected to $C^*(X,\mathbb{Q})$ via a sequence of quasi-isomorphisms. More explicitly, $\Lambda V = TV/I$ where I is the graded ideal spanned by $\{v\otimes w-(-1)^{deg(u)deg(v)}w\otimes v,\ v,\ w\in V\}$, $V=\oplus_{n\geq 2}V^n$ is a finite-type graded vector space and the differential d is a derivation defined on V satisfying $d\circ d=0$. $(\Lambda V,d)$ is called a *Sullivan model* of X. This model is said to be *minimal* if d is decomposable, i.e. $d:V\to\Lambda^{\geq 2}V$ where $\Lambda^{\geq 2}V$ denotes the graded vector space spanned by all the monomials $v_1\dots v_r$ $(r\geq 2)$, such a model is unique up to isomorphisms [12].

Let (X,x_0) be a based simply-connected finite type CW-complex and denote by

$$m:(\Lambda V,d)\xrightarrow{\simeq} A_{PL}(X)$$

its minimal Sullivan model. In fact, the multiplicative structure of $(\Lambda V,d)$, $\mu_{\Lambda V}:\Lambda V\otimes_{\mathbb{Q}}\Lambda V\to\Lambda V$ is compatible with the one induced on $C^*(X,\mathbb{Q})$ by the diagonal map $\Delta_X: X\to X\times X$, and the same holds for the augmentation $\varepsilon_{\Lambda V}:(\Lambda V,d)\to(\mathbb{Q},0)$ and the inclusion $\iota:\{x_0\}\hookrightarrow X$.

A $(\Lambda V,d)$-semi-free resolution of $(\mathbb{Q},0)$ has the form $(P,d)=(\Lambda V\otimes\Lambda sV,d)\xrightarrow{\simeq}(\mathbb{Q},0)$ where sV is the suspension of V defined by $(sV)^k=V^{k+1}$ and $d(sv)=-s(dv)$ for all $v\in V$, and is called the acyclic closure of$(\mathbb{Q},0)$, [9].

We are now ready to define a homotopy multiplication on $\mathrm{A} =: Hom_{(\Lambda V,d)}(\mathbb{Q},(\Lambda V,d))$:

$$\mu_{\mathrm{A}}:\mathrm{A}\otimes\mathrm{A}\to\mathrm{A}$$

in the sens that it only depends on the homotopy class of the model $m:(\Lambda V,d)\to A_{PL}(X)$ (cf. the uniqueness property bellow). Passing to cohomology μ_A induce an associative multiplicative structure with unit on $\mathcal{A} =: \mathcal{E}xt_{(\Lambda V,d)}(\mathbb{Q},(\Lambda V,d))$ denoted

$$\mu_{\mathcal{A}}:\mathcal{A}\otimes\mathcal{A}\to\mathcal{A}.$$

The proof of Theorem 1.1 follows from the following

Theorem 3.1. *The $\mathbb{Q}$-vector space $\mathcal{A}$, endowed with $\mu_{\mathcal{A}}$, is a graded commutative algebra with unit. Moreover, the evaluation map is a morphism of graded algebras.*

Proof. Let $f,g:P\to\Lambda V$ be elements in A representing two classes in $\mathcal{A}$. As ΛV is commutative, the left ΛV-module P is also a right ΛV-module by setting $x\cdot a=(-1)^{|x||a|}a\cdot x$, $x\in P$ and $a\in\Lambda V$.

Multiplicative structure:

First, we consider

$$\begin{aligned} f\otimes g: P\otimes_{\mathbb{Q}}P &\longrightarrow \Lambda V\otimes_{\mathbb{Q}}\Lambda V\\ x\otimes y &\longmapsto (-1)^{|g||x|}f(x)\otimes g(y),\end{aligned}$$

and I the ideal generated by $x \cdot a \otimes y - x \otimes a \cdot y$; $x, y \in P$ and $a \in \Lambda V$. It is straightforward that the map $\mu_{\Lambda V} \circ (f \otimes g)$ sends I to zero, which then induces, on the quotient $P \otimes_{\Lambda V} P = P \otimes_{\mathbb{Q}} P/I$, the dashed map

$$\begin{array}{ccccc} P \otimes_{\mathbb{Q}} P & \xrightarrow{f \otimes g} & \Lambda V \otimes_{\mathbb{Q}} \Lambda V & \xrightarrow{\mu_{\Lambda V}} & \Lambda V \\ \downarrow & & & \nearrow_{\mu_{\mathrm{A}}(f \otimes g)} & \\ P \otimes_{\Lambda V} P & & & & \end{array}$$

defined by:

$$\begin{aligned} \mu_{\mathrm{A}}(f \otimes g) =: f.g : P \otimes_{\Lambda V} P &\longrightarrow \Lambda V \\ x \otimes y &\longmapsto (-1)^{|g||x|} f(x) g(y), \end{aligned}$$

For any $a \in \Lambda V$ and $x, y \in P$, we have $(f \cdot g)((a \cdot x) \otimes y) = (-1)^{|f \cdot g||a|} a (f \cdot g)(x \otimes y)$. Therefore $f \cdot g$ is a ΛV-morphism.

Next, we show that $Q = P \otimes_{\Lambda V} P$ is a ΛV-semi-free resolution. Recall that a semi-free resolution (P, d) of $\mathbb{Q}$ has the form $W \otimes_{\mathbb{Q}} \Lambda V$ with $W = \bigoplus_{i=0}^{+\infty} W(i)$ and each $W(i)$ is a free graded $\mathbb{Q}$-module and $d : W(k) \to P(k-1)$, with the semi-free filtration given by $P(k) = \bigoplus_{i=0}^{k} W(i) \otimes_{\mathbb{Q}} \Lambda V$ [12]. Therefore,

$$Q = (W \otimes_{\mathbb{Q}} \Lambda V) \otimes_{\Lambda V} (W \otimes_{\mathbb{Q}} \Lambda V) = (W \otimes_{\mathbb{Q}} W) \otimes_{\mathbb{Q}} \Lambda V.$$

Let $Z = W \otimes_{\mathbb{Q}} W$ and put $Q(k) = \bigotimes_{i=0}^{k} Z(i) \otimes_{\mathbb{Q}} \Lambda V$ where $Z(l) = \bigoplus_{i+j=l} W(i) \otimes_{\mathbb{Q}} W(j)$ is obviously a free graded $\mathbb{Q}$-module since each $W(i)$ is. For any $x \otimes y \in W(i) \otimes W(j)$, we easily verify that

$$dx \otimes y \in P(i-1) \otimes W(j) \subseteq Q(k-1) \text{ and } x \otimes dy \in W(i) \otimes P(j-1) \subseteq Q(k-1),$$

whence $D : Z(k) \to Q(k-1)$. It results that $(Q, D) \xrightarrow{\simeq} (\mathbb{Q}, 0)$ is a ΛV-semi-free resolution of $(\mathbb{Q}, 0)$. This defines a multiplication

$$\mu_{\mathrm{A}} : \mathrm{A} \otimes \mathrm{A} \to \mathrm{A}$$

on $\mathrm{A} = Hom_{(\Lambda V, d)}(\Lambda V \otimes \Lambda sV, \Lambda V)$.

Now, since $(\Lambda V, d)$ has an associative structure, we deduce that μ_A is associative. Passing to cohomology we acquire a well-defined map of vector spaces:

$$\begin{aligned} \mu_{A} : \mathcal{E}xt_{(\Lambda V, d)}(\mathbb{Q}, (\Lambda V, d)) \otimes_{\mathbb{Q}} \mathcal{E}xt_{(\Lambda V, d)}(\mathbb{Q}, (\Lambda V, d)) &\longrightarrow \mathcal{E}xt_{(\Lambda V, d)}(\mathbb{Q}, (\Lambda V, d)) \\ [f] \otimes [g] &\longmapsto [f \cdot g] \end{aligned}$$

that is an associative multiplication on $\mathcal{E}xt_{(\Lambda V, d)}(\mathbb{Q}, (\Lambda V, d))$.

Uniqueness:

Let $m' : (\Lambda V', d') \xrightarrow{\simeq} A_{PL}(X)$ be another minimal Sullivan model of X i.e. in the homotopy class of $m : (\Lambda V, d) \xrightarrow{\simeq} A_{PL}(X)$ [12, proposition 12.7]. We know that $(\Lambda V, d)$ and $(\Lambda V', d')$ are isomorphic [12, Proposition 12. 10]. Clearly, the same applies for associated semi-free resolutions $Q = P \otimes_{\Lambda V} P$ and $Q' = P' \otimes_{\Lambda V} P'$. It results, from the above commutative triangle, that μ_A is independent of the choice of a minimal model of X.

Unit element:

Let $\varepsilon : \Lambda V \to \mathbb{Q}$ be the augmentation. Recall that $P = \Lambda V \otimes \Lambda sV$ is a $(\Lambda V, d)$-semi-free resolution of $(\mathbb{Q}, 0)$.

We extend ε to $\varepsilon' = \varepsilon \otimes \varepsilon_{\Lambda sV} : \Lambda V \otimes \Lambda sV \to \mathbb{Q}$, then we compose it with the injection $i : \mathbb{Q} \hookrightarrow \Lambda V$ and obtain $\tilde{\varepsilon} : \Lambda V \otimes \Lambda sV \to \Lambda V$ a representative of a class in

$\mathcal{E}xt_{(\Lambda V,d)}(\mathbb{Q},(\Lambda V,d))$. Now, for $f:\Lambda V\otimes\Lambda sV\to\Lambda V$, a representative of an arbitrary class in $\mathcal{E}xt_{(\Lambda V,d)}(\mathbb{Q},(\Lambda V,d))$, we have

$$f\cdot\tilde{\varepsilon}:(\Lambda V\otimes\Lambda sV)\otimes_{\Lambda V}(\Lambda V\otimes\Lambda sV)=\Lambda V\otimes\Lambda sV\otimes\Lambda sV\to\Lambda V$$

and the map

$$\begin{array}{rccc}\theta=Id_{\Lambda V\otimes\Lambda sV}\otimes\varepsilon_{\Lambda sV}: & \Lambda V\otimes\Lambda sV\otimes\Lambda sV & \longrightarrow & \Lambda V\otimes\Lambda sV\\ & 1\otimes sv\otimes 1 & \longmapsto & 1\otimes sv;\\ & 1\otimes sv\otimes sw & \longmapsto & 0;\\ & 1\otimes 1\otimes sv & \longmapsto & 0\end{array}$$

makes the following diagram commutative:

$$\begin{array}{ccc}\mathbb{Q} & \xleftarrow{\simeq} & \Lambda V\otimes\Lambda sV\\ \uparrow{\scriptstyle\simeq} & \nearrow{\scriptstyle\theta} & \downarrow{\scriptstyle f}\\ \Lambda V\otimes\Lambda sV\otimes\Lambda sV & \xrightarrow[f\cdot\tilde{\varepsilon}]{} & \Lambda V.\end{array}$$

Thus, it defines a homotopy unit element for $\mathrm{A}=Hom_{(\Lambda V,d)}((P,d),(\Lambda V,d))$. Passing to cohomology, we get $[f]\cdot[\tilde{\varepsilon}]=[f]$ and similarly $[\tilde{\varepsilon}]\cdot[f]=[f]$. Henceforth, the class $[\tilde{\varepsilon}]$ defines a unit element for $\mu_{\mathcal{A}}$.

Commutativity:

Let τ be the flip map $\tau:P\otimes_{\Lambda V}P\to P\otimes_{\Lambda V}P$; $x\otimes y\mapsto(-1)^{|x||y|}y\otimes x$. The diagram

$$\begin{array}{ccc} & & P\otimes_{\Lambda V}P\\ & \nearrow{\scriptstyle\tau} & \downarrow{\scriptstyle(-1)^{|f||g|}g\cdot f}\\ P\otimes_{\Lambda V}P & \xrightarrow[f\cdot g]{} & \Lambda V\end{array}$$

is commutative.

clearly τ being a quasi-isomorphism, $f\cdot g\sim g\cdot f$ and $[f\cdot g]=(-1)^{|f||g|}[g\cdot f]$ so that, the multiplication on A is homotopy commutative and consequently, it is commutative on $\mathcal{A}$.

We respectively conclude that A is a homotopy commutative differential graded algebra with unit and $\mathcal{E}xt_{(\Lambda V,d)}(\mathbb{Q},(\Lambda V,d))$ is a graded commutative $\mathbb{Q}$-algebra with unit.

Finally, it is clear that the following diagram, where cev is the chain evaluation map of $(\Lambda V,d)$, is commutative:

$$\begin{array}{ccc}\mathrm{A}\otimes\mathrm{A} & \xrightarrow{\mu_{\mathrm{A}}} & \mathrm{A}\\ \downarrow{\scriptstyle cev\otimes cev} & & \downarrow{\scriptstyle cev}\\ (\Lambda V,d)\otimes(\Lambda V,d) & \xrightarrow{\mu_{\Lambda V}} & (\Lambda V,d).\end{array}$$

Thus, passing to cohomology

$$\begin{array}{ccc}\mathcal{A}\otimes\mathcal{A} & \xrightarrow{\mu_{\mathcal{A}}} & \mathcal{A}\\ {\scriptstyle ev\otimes ev}\downarrow & & \downarrow{\scriptstyle ev}\\ H(\Lambda V,d)\otimes H(\Lambda V,d) & \longrightarrow & H(\Lambda V,d),\end{array}$$

we deduce that the evaluation map is a morphism of graded algebras. □

4. *Ext*-versions approximations and the main theorem

Recall that we still assume that $\mathbb{K} = \mathbb{Q}$, and let (A, d) be any commutative differential graded model for a space X, i.e. (A, d) is quasi-isomorphic to the cdga $A_{PL}(X)$ (cf. the beginning of Section 3), and $(\Lambda V, d)$ its minimal Sullivan model given by the quasi-isomorphism $\theta : (\Lambda V, d) \xrightarrow{\simeq} (A, d)$ [12]. Referring to [5], the cdga morphism

$$\mu_n^\theta := (\mathrm{Id}_A, \theta, \ldots, \theta) : (A, d) \otimes (\Lambda V, d)^{\otimes n-1} \to (A, d)$$

is a special model, called an *s-model*, for the path fibration $\pi_n : X^I \to X^n$, the substitute of the n-fold diagonal map $\Delta_X^n : X \to X^n$. This allows the following

Definition 4.1. (a): $\mathrm{TC}_n(X_0)$ *is the least m such that the projection*

$$\rho_m : \left(A \otimes (\Lambda V)^{\otimes n-1}, d\right) \to \left(\frac{A \otimes (\Lambda V)^{\otimes n-1}}{(\ker \mu_n^\theta)^{m+1}}, \bar{d}\right)$$

admits an algebra retraction.

(b): $\mathrm{mTC}_n(\mathrm{X})$ *is the least m such that ρ_m admits a retraction as $(A \otimes (\Lambda V)^{\otimes n-1}, d)$-module.*

(c): $\mathrm{HTC}_n(X)$ *is the least m such that $H(\rho_m)$ is injective.*

(d): *nil* $\ker H^*(\Delta_X^n, \mathbb{Q})$ *is the longest non trivial product of elements of* $\ker H^*(\Delta_X^n, \mathbb{Q})$.

These invariants are ordered as follows ([5])

$$nil \ker H^*(\Delta_X^n, \mathbb{Q}) \leq \mathrm{HTC}_n(X) \leq \mathrm{mTC}_n(\mathrm{X}) \leq \mathrm{TC}_n(X_0) \leq \mathrm{TC}_n(X). \qquad (4.1)$$

Inspired by the previous definition, we introduce the *Ext-version* of the original invariants. Now if we take θ to be the identity of ΛV, $\mu_n^\theta = \mu_n^{Id_{\Lambda V}}$ becomes the n-fold multiplication in ΛV denoted by

$$\mu_n : (\Lambda V)^{\otimes n} \to \Lambda V.$$

Therefore, $nil \ker H^*(\Delta_X^n, \mathbb{Q}) = nil \ker H^*(\mu_n)$.

In a similar way, we put

$$\mu_{\mathrm{A},n} : \mathrm{A}^{\otimes n} \to \mathrm{A} \qquad \text{and} \qquad \mu_{\mathcal{A},n} : \mathcal{A}^{\otimes n} \to \mathcal{A}$$

where $\mathrm{A} := Hom_{\Lambda V}((P, d), (\Lambda V, d))$ and $\mathcal{A} := H(\mathrm{A}) = \mathcal{E}xt_{(\Lambda V, d)}((P, d), (\Lambda V, d))$. Then (cf. Definition 1.3) we have

Definition 4.2. (a): $\mathrm{TC}_n^{\mathcal{E}xt}(X, \mathbb{Q})$ *is the least m such that the projection*

$$\Gamma_m : (\mathrm{A}^{\otimes n}, d) \to \left(\frac{\mathrm{A}^{\otimes n}}{(\ker(\mu_{\mathrm{A},n}))^{m+1}}, \bar{d}\right)$$

admits a homotopy retraction.

(b): $\mathrm{mTC}_n^{\mathcal{E}xt}(\mathrm{X}, \mathbb{Q})$ *is the least m such that Γ_m admits a homotopy retraction as $(\mathrm{A}^{\otimes n}, d)$-module.*

(c): $\mathrm{HTC}_n^{\mathcal{E}xt}(X, \mathbb{Q})$ *is the least m such that $H(\Gamma_m)$ is injective.*

(d): *nil* $\ker(\mu_{\mathcal{A},n}, \mathbb{Q})$ *is the longest non trivial product of elements of* $\ker(\mu_{\mathcal{A},n})$.

The same arguments used to establish the inequalities in (4.1) allow the following:

$$nil \ker(\mu_{\mathcal{A},n}, \mathbb{Q}) \leq \mathrm{HTC}_n^{\mathcal{E}xt}(X, \mathbb{Q}) \leq \mathrm{mTC}_n^{\mathcal{E}xt}(X, \mathbb{Q}) \leq \mathrm{TC}_n^{\mathcal{E}xt}(X, \mathbb{Q}).$$

Remark 4.3. Note that when X is a Poincaré duality space, the same process followed to prove that $\mathrm{mTC}_n(X) = \mathrm{HTC}_n(X)$ ([5]) allows the following equality $\mathrm{mTC}_n^{\mathcal{E}xt}(X, \mathbb{Q}) = \mathrm{HTC}_n^{\mathcal{E}xt}(X, \mathbb{Q})$.

We are now in place to state the main theorem.

Theorem 4.4. *Let X be a 1-connected finite type CW-complex. If X is a Gorenstein space over $\mathbb{Q}$ and $ev_{C^*(X,\mathbb{Q})} \neq 0$, then*

$$HTC_n^{\mathcal{E}xt}(X,\mathbb{Q}) \leq HTC_n(X)$$

for any integer $n \geq 2$. Furthermore, if $(\Lambda V, d)$ is a minimal Sullivan model of X, $[f]$ is the generating class of $\mathcal{A}$ and $m = HTC_n^{\mathcal{E}xt}(X,\mathbb{Q})$, then:

$$HTC_n(X) = HTC_n^{\mathcal{E}xt}(X,\mathbb{Q}) =: m \Leftrightarrow f(1)^{\otimes n} \in (\ker \mu_n)^m \backslash (\ker \mu_n)^{m+1}.$$

Before giving the proof of the theorem, let us first recall that if X is a finite n-dimensional sub-complex of $\mathbb{R}^{n+k}$, $k > n+1$ and M its regular neighborhood, the homotopy fiber F_X of the inclusion $\partial M \hookrightarrow M$ is called the *Spivak fiber* for X and it is a homotopy invariant of X. It is introduced in [20] where it is shown that X is a Poincaré complex if and only if F_X is a homotopy sphere.

The following corollary presents some essential classes of spaces satisfying the previous theorem:

Corollary 4.5. *The hypothesis of 1.4 and hence its conclusions are satisfied in the following cases:*

(a) *X is rationally elliptic,*
(b) *$H_{>N}(X,\mathbb{Z}) = 0$, for some N, and $H^*(X,\mathbb{Q})$ is a Poincaré duality algebra,*
(c) *X is a finite 1-connected CW-complex and its Spivak fiber F_X has finite dimensional cohomology.*

For the sake of completeness, we present below a sketch of the proof of the corollary:

(a) If X is rationally elliptic, then it is Gorenstein and $ev_{C^*(X,\mathbb{Q})} \neq 0$ thanks to [11, Proposition 3.4] and [18, Theorem A] respectively.
(b) Referring to [7] (cf. also [11, Theorem 3.6]), under the hypothesis, $H^*(X,\mathbb{Q})$ is a Poincaré duality algebra if and only if X is a Gorenstein space over $\mathbb{Q}$. Thus, by [11, Theorem 2.2] we have $\dim \pi_*(X) \otimes \mathbb{Q} < \infty$ and by [18, Theorem A] we obtain $ev_{C^*(X,\mathbb{Q})} \neq 0$.
(c) Here using [11, Corollary 4.5], we have $H^*(X,\mathbb{Q})$ is a Poincaré duality algebra, hence it is a Gorenstein algebra. It results that X is Gorenstein over $\mathbb{Q}$ [11, Proposition 3.2] and $ev_{C^*(X,\mathbb{Q})} \neq 0$ as in the previous case.

Proof. **(of Theorem 4.1)**: Let $(\Lambda V, d)$ be a Sullivan minimal model of X. The projections

$$\Gamma_m : (\mathrm{A}^{\otimes n}, d) \to \left(\frac{\mathrm{A}^{\otimes n}}{(\ker(\mu_{\mathrm{A},n}))^{m+1}}, \bar{d}\right) \quad \text{and} \quad \rho_m : ((\Lambda V)^{\otimes n}, d) \to \left(\frac{(\Lambda V)^{\otimes n}}{(\ker \mu_n)^{m+1}}, \bar{d}\right)$$

induce two short exact sequences linked by chain evaluation map

$$\begin{array}{ccccccccc}
0 & \longrightarrow & (\ker(\mu_{\mathrm{A},n}))^{m+1} & \longrightarrow & \mathrm{A}^{\otimes n} & \xrightarrow{\Gamma_m} & \dfrac{\mathrm{A}^{\otimes n}}{(\ker(\mu_{\mathrm{A},n}))^{m+1}} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \theta & & \\
0 & \longrightarrow & (ker\mu_n)^{m+1} & \longrightarrow & (\Lambda V)^{\otimes n} & \xrightarrow{\rho_m} & \dfrac{(\Lambda V)^{\otimes n}}{(\ker \mu_n)^{m+1}} & \longrightarrow & 0.
\end{array} \tag{4.2}$$

Since X is Gorenstein, $\mathcal{A} \cong \mathbb{Q}\Omega$ where Ω is the generating class represented by a cocycle $f \in \mathrm{A}^N$ of degree $N = fd(X)$, the formal dimension of X (cf. §5, [11]). Therefore, the diagram (4.2) induces, in cohomology, the following one

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^{nN}(\ker(\mu_{A,n})^{m+1}) & \longrightarrow & (\mathcal{A}^N)^{\otimes n} & \xrightarrow{H^{nN}(\Gamma_m)} & H^{nN}\left(\frac{A^{\otimes n}}{\ker(\mu_{A,n})^{m+1}}\right) & \longrightarrow & 0\\
\downarrow & & \downarrow & & \downarrow{\scriptstyle ev^{\otimes n}_{(\Lambda V,d)}} & & \downarrow{\scriptstyle H^{nN}(\theta)} & & \\
H^{nN-1}\left(\frac{(\Lambda V)^{\otimes n}}{(ker\mu_n)^{m+1}}\right) & \longrightarrow & H^{nN}((ker\mu_n)^{m+1}) & \longrightarrow & (H^N(\Lambda V))^{\otimes n} & \xrightarrow[H^{nN}(\rho_m)]{} & H^{nN}\left(\frac{(\Lambda V)^{\otimes n}}{(ker\mu_n)^{m+1}}\right) & \longrightarrow & 0.
\end{array}$$

Now, since $ev_{(\Lambda V,d)} = ev_{C^*(X,\mathbb{Q})} \neq 0$, this is also the case for the horizontal arrow $ev^{\otimes n}_{(\Lambda V,d)}$. Thus, if $H^{nN}(\rho_m)$ is injective then $H^{nN}(\Gamma_m)$ is also injective. It results that: $HTC_n^{\mathcal{E}xt}(X,\mathbb{Q}) \leq HTC_n(X)$.

Next, let m denote the smallest integer such that $H^{nN}(\Gamma_m)$ is injective or, equivalently, $f^{\otimes n}$ is a cocycle in $A^{\otimes n}$ and $f^{\otimes n} \in \ker(\mu_{A,n})^m \backslash \ker(\mu_{A,n})^{m+1}$ (see Remark below). Moreover, since $(\mathcal{A}^N)^{\otimes n}$ is one dimensional, $H^{nN}(\Gamma_m)$ is indeed a bijection. Hence, $H^{nN}(\rho_m)$ is injective if and only if $H^{nN}(\theta)$ is injective. But, $ev^{\otimes n}_{(\Lambda V,d)}$ being non-zero, the commutativity of the right diagram implies that this is equivalent to $f(1)^{\otimes n} \in (\ker \mu_n)^m \backslash (\ker \mu_n)^{m+1}$. Notice that $\mu_n(f(1)^{\otimes n})$ is a cocycle in $(\Lambda V)^{\otimes n}$ and $ev^{\otimes n}_{(\Lambda V,d)}[f^{\otimes n}] = [f(1)]^{\otimes n} \neq 0$. It results that $HTC_n(X) = HTC_n^{\mathcal{E}xt}(X,\mathbb{Q}) =: m \Leftrightarrow f(1)^{\otimes n} \in (\ker \mu_n)^m \backslash (\ker \mu_n)^{m+1}$. □

Remark 4.6. An equivalent definition of $HTC_n(X)$, when X is a Poincaré duality space, reads as follows: It is the smallest integer $m \geq 0$ such that some cocycle ω representing the fundamental class of $(\Lambda V, d)^{\otimes n}$, can be written as a product of m elements of $ker(\mu_n)$ (not necessarily cocycles). Similarly, for any Gorenstein space X, $HTC_n^{\mathcal{E}xt}(X,\mathbb{Q})$ is the smallest integer m such that some cocycle representing the fundamental class of $A^{\otimes n}$, namely $\Omega = [f]^{\otimes n}$ where $[f]$ designates the generating element of $\mathcal{A}^N$, can be written as a product of length m of elements in $ker(\mu_{A,n})$. Therefore, in order to determine $HTC_n(X)$ we may, using the precedent theorem, calculate $m = HTC_n^{\mathcal{E}xt}(X,\mathbb{Q})$, which is quite simpler since $\mathcal{A}^*$ is one dimensional, and afterwards deal with the obstruction to have the equality.

Now, if $\dim V < \infty$ and $\dim H(\Lambda V, d) = \infty$ then, by [18, Theorem A], we have $(\Lambda V, d)$ is a Gorenstein but not a Poincaré duality algebra. Moreover $ev_{(\Lambda V,d)} = 0$. Hence, in this case, to compare the invariants $HTC_n^{\mathcal{E}xt}(X,\mathbb{Q})$ and $HTC_n(X)$, we determine them separately.

5. Use of the Adams-Hilton model

Let R be a principal ideal domain containing $\frac{1}{2}$, and let $\rho(R)$ denote the least non invertible prime (or ∞) in R. $CW_r(R)$ is the sub-category of finite r-connected CW-complexes X $(r \geq 1)$ satisfying $\dim(X) \leq r\rho(R)$.

In his attempt to extend Sullivan's theory to arbitrary rings ([14], see also [2]), S. Halperin associated to every X in $CW_r(R)$ an appropriate differential graded Lie algebra (L, ∂) and showed that its Cartan-Eilenberg-Chevally complex $C^*(L, \partial)$ is, on one hand, linked with the cochains algebra $C^*(X; R)$ by a series of quasi-isomorphisms ([14, p. 274]) and, on the other hand, is quasi-isomorphic to a free commutative differential graded algebra $(\Lambda W, d)$ [14, §7]. $(\Lambda W, d)$ is then called *a free commutative model* of X or a *minimal Sullivan model* of X.

Now, if $\mathbb{K}$ is a field with odd characteristic (containing $\frac{1}{2}$) and X is a q-connected $(q \geq 1)$ finite CW-complex such that $\dim X \leq q \cdot char(\mathbb{K})$, i.e. $X \in CW_q(\mathbb{K})$, it has a minimal Sullivan model $(\Lambda W, d)$ [14, Theorem 7.1].

The Adams-Hilton model ([1]) of X over $\mathbb{K}$ is a chain algebra quasi-isomorphism $\theta_X : (TV, d) \xrightarrow{\simeq} C_*(\Omega X; \mathbb{K})$, i.e. $H_*(\theta_X)$ is an isomorphism of graded algebras. Here V satisfies

$H_{i-1}(V, d_1) \cong H_i(X; \mathbb{K})$ and $d_1 : V \to V$ is the linear part of d. (TV, d) is called a *free model* of X.

Therefore, using the isomorphism (2.1) we have successively:

$$\mathcal{E}xt_{C^*(X;\mathbb{K})}(\mathbb{K}, C^*(X;\mathbb{K})) \overset{\cong}{\to} \mathcal{E}xt_{(\Lambda W,d)}(\mathbb{K}, (\Lambda W, d)).$$

and

$$\mathcal{E}xt_{(TV,d)}(\mathbb{K}, (TV, d)) \overset{\cong}{\to} \mathcal{E}xt_{C_*(\Omega X;\mathbb{K})}(\mathbb{K}, C_*(\Omega X;\mathbb{K})).$$

Now, combining these two models via the isomorphism of graded $\mathbb{K}$-vector spaces [11, Theorem 2.1] yields

$$\mathcal{E}xt_{C^*(X;\mathbb{K})}(\mathbb{K}, C^*(X;\mathbb{K})) \overset{\cong}{\to} \mathcal{E}xt_{C_*(\Omega X;\mathbb{K})}(\mathbb{K}, C_*(\Omega X;\mathbb{K})). \tag{5.1}$$

gives the isomorphism of graded $\mathbb{K}$-vector spaces:

$$\mathcal{E}xt_{(\Lambda W,d)}(\mathbb{K}, (\Lambda W, d)) \cong \mathcal{E}xt_{(TV,d)}(\mathbb{K}, (TV, d)).$$

Argument used in the rational case allows us to conclude that $\mathcal{E}xt_{(\Lambda W,d)}(\mathbb{K}, (\Lambda W, d))$ has the structure of a graded commutative algebra with unit. The latter isomorphism serves to endow $\mathcal{E}xt_{(TV,d)}(\mathbb{K}, (TV, d))$ with the same structure. It results the following

Proposition 5.1. *Let $\mathbb{K}$ be a field with odd characteristic and $X \in CW_q(\mathbb{K})$. Then, the graded vector spaces $\mathcal{E}xt_{C^*(X;\mathbb{K})}(\mathbb{K}, C^*(X;\mathbb{K}))$ and $\mathcal{E}xt_{C_*(\Omega X;\mathbb{K})}(\mathbb{K}, C_*(\Omega X;\mathbb{K}))$ have isomorphic graded commutative algebra structures with unit. In particular, the Adams-Hilton model can be used to make this structure explicit.*

Recall that $\mathcal{E}xt_{(TV,d)}(\mathbb{K}, (TV, d))$ is, as in the rational case, obtained in terms of the acyclic closure of $\mathbb{K}$ of the form $(TV \otimes (\mathbb{K} \oplus sV), \delta)$, where the differential δ satisfies $\delta s + sd = id$, d being the differential of TV. That is, for any element $z \otimes sv$ of $TV \otimes (\mathbb{K} \oplus sV)$, we have

$$\delta(z \otimes sv) = dz \otimes sv + (-1)^{|z|} zv \otimes 1 - (-1)^{|z|} z \otimes sdv.$$

Notice that any element f in $Hom^p_{(TV,d)}((TV \otimes (\mathbb{K} \oplus sV), \delta), (TV, d))$ is entirely determined by its image of $1 \otimes (\mathbb{K} \oplus sV)$ since $TV \otimes (\mathbb{K} \oplus sV)$ is a left (TV, d)-module acting on the first factor. Thus we have

$$\begin{aligned}(D(g))(1 \otimes sv) &= d \circ f(1 \otimes sv) - (-1)^p f \circ \delta(1 \otimes sv)\\ &= df(1 \otimes sv) - (-1)^{p(|v|+1)} vf(1) + (-1)^p f(1 \otimes sdv).\end{aligned}$$

Therefore,

(a) An element g in $Hom^{p-1}_{(TV,d)}((TV \otimes (\mathbb{K} \oplus sV), \delta), (TV, d))$ is in $Im(D)$ if and only if $g = D(f)$ for some f in $Hom^p_{(TV,d)}((TV \otimes (\mathbb{K} \oplus sV), \delta), (TV, d))$, i.e.

$$g(1 \otimes sv) = df(1 \otimes sv) - (-1)^{p(|v|+1)} vf(1) + (-1)^p f(1 \otimes sdv).$$

Consequently:

$$g \in Im(D) \Leftrightarrow g(1 \otimes sv) = df(1 \otimes sv) - (-1)^{p(|v|+1)} vf(1) + (-1)^p f(1 \otimes sdv) \text{ for some } f. \tag{5.2}$$

(b) An element $f \in Hom^p_{(TV,d)}((TV \otimes (\mathbb{K} \oplus sV), \delta), (TV, d))$ is in $Ker(D)$ if and only if $D(f) = 0$, that is, $df(1 \otimes sv) = (-1)^{p(|v|+1)} vf(1) - (-1)^p f(1 \otimes sdv)$. Consequently:

$$f \in Ker(D) \Leftrightarrow df(1 \otimes sv) = (-1)^{p(|v|+1)} vf(1) - (-1)^p f(1 \otimes sdv). \tag{5.3}$$

Now, since $\deg(d) = -1$, $\mathcal{A}_* = \left(Hom_{(TV,d)}((TV \otimes (\mathbb{K} \oplus sV), \delta), (TV, d)), D\right)$ is a dga_* in the sense of [11]. Using the standard convention $\mathcal{A}^{-q} = \mathcal{A}_q$, for all $q \in \mathbb{Z}$, we obtain a dga^* whose cohomology at $-p$ is the $\mathbb{K}$-module:

$$\mathcal{E}xt^{-p}_{(TV,d)}(\mathbb{Q}, (TV, d)) = H_p\left(Hom_{(TV,d)}((TV \otimes (\mathbb{K} \oplus sV), \delta), (TV, d)), D\right).$$

More explicitly, if $f \in Hom_{(TV,d)}(TV \otimes (\mathbb{K} \oplus sV), \Lambda V)$ is a cycle of (homological) degree p, it defines a cohomological class $[f] \in \mathcal{E}xt^{-p}_{(TV,d)}(\mathbb{Q}, (TV, d))$ of degree $-p$.

5.1. Case of a suspension

Assume that $\mathbb{K} = \mathbb{Q}$ and let X be a simply connected space, and $Y = \Sigma X$ its suspension. The morphism of graded modules $\sigma_* : H_*(X; \mathbb{Q}) \longrightarrow H_*(\Omega\Sigma X; \mathbb{Q})$ induced by the adjoint $\sigma : X \to \Omega\Sigma X$ of $id_{\Sigma X}$ extends to a morphism of graded algebras $T(\sigma_*) : TH_*(X; \mathbb{Q}) \longrightarrow H_*(\Omega\Sigma X; \mathbb{Q})$. In fact, this is an isomorphism of graded algebras since $H_*(X; \mathbb{Q})$ is a free graded $\mathbb{Q}$-module. Therefore ([3])

$$(TH_*(X; \mathbb{Q}), 0) \xrightarrow{\simeq} C_*(\Omega\Sigma X; \mathbb{Q})$$

is an Adams-Hilton model of $Y = \Sigma X$. It results that

$$\mathcal{E}xt_{C_*(\Omega\Sigma X;\mathbb{Q})}(\mathbb{Q}, C_*(\Omega\Sigma X; \mathbb{Q})) \cong \mathrm{Ext}_{TV}(\mathbb{Q}, TV),$$

where $V = H_*(X; \mathbb{Q})$. Now if $(\Lambda W, d)$ is a Sullivan model of $Y = \Sigma X$, using (5.1) we obtain a commutative diagram

$$\begin{array}{ccc}
\mathrm{Ext}_{TV}(\mathbb{Q}, TV) \otimes_{\mathbb{Q}} \mathrm{Ext}_{TV}(\mathbb{Q}, TV) & \xrightarrow{\mu_{\mathcal{A}}} & \mathrm{Ext}_{TV}(\mathbb{Q}, TV) \\
\cong \downarrow & & \downarrow \cong \\
\mathcal{E}xt_{(\Lambda W,d)}(\mathbb{Q}, (\Lambda W, d)) \otimes_{\mathbb{Q}} \mathcal{E}xt_{(\Lambda W,d)}(\mathbb{Q}, (\Lambda W, d)) & \xrightarrow[\mu_{\mathcal{A}}]{} & \mathcal{E}xt_{(\Lambda W,d)}(\mathbb{Q}, (\Lambda W, d)).
\end{array}$$

This permits the use of Adams-Hilton models to explicitly describe the algebra structure on $\mathcal{A}$, since it restricts to ordinary Ext which loosen the calculations. Notice that $\Omega(\Sigma X)$ is weakly equivalent to the James space $J(X)$ and, referring to [12, Example 7], that there exists a minimal Sullivan model for ΣX of the form $(\Lambda Z, d)$ with quadratic differential, i.e. such that $d(Z) \subset \Lambda^2 Z$.

5.2. When X is a 2-cell CW complex

Let $\mathbb{K}$ any field containing $\frac{1}{2}$.

In this subsection, we showcase another use of the Adams-Hiton models to help picture the algebra structure on $\mathcal{A}$. Let then $X = S^q \cup_\varphi e^{q+1}$, $q \geq 2$, be the space where the cell e^{q+1} is attached by a map φ of degree r. The Adams-Hilton model of X has the form (TV, d), where V is a $\mathbb{K}$-vector space generated by a and a' with $deg(a) = q - 1$, $deg(a') = q$, $\quad da = 0$ and $\quad da' = -ra$.

Let us go back to where we left off at the beginning of this section and apply, in this case, the obtained formulas (5.2) and (5.3).

$$g \in Im(D) \Leftrightarrow \begin{cases} g(1) = df(1), \\ g(1 \otimes sa) = df(1 \otimes sa) - (-1)^{pq} a f(1), & \text{(for some } f\text{)} \\ g(1 \otimes sa') = df(1 \otimes sa') - (-1)^p r f(1 \otimes sa) - (-1)^{p(q+1)} a' f(1), \end{cases} \tag{5.4}$$

and

$$f \in Ker(D) \Leftrightarrow \begin{cases} df(1) = 0, \\ df(1 \otimes sa) = (-1)^{pq} a f(1), \\ df(1 \otimes sa') = (-1)^p r f(1 \otimes sa) + (-1)^{p(q+1)} a' f(1). \end{cases} \tag{5.5}$$

Recall that to any pointed topological space X, it is associated in [11] an invariant called the *formal dimension* of X (with respect to a field $\mathbb{K}$) defined as follows:

$$fd(X, \mathbb{K}) = sup\{r \in \mathbb{Z} \mid [\mathcal{E}xt^p_{C^*(X;\mathbb{K})}(\mathbb{K}, C^*(X; \mathbb{K}))]^r \neq 0\},$$

or $fd(X,\mathbb{K}) = -\infty$ if such integer does not exist. In particular [11, Proposition 5.1], if $H^*(X;\mathbb{K})$ is finite dimensional,

$$fd(X,\mathbb{K}) = sup\{r \in \mathbb{Z} \mid H^r(X;\mathbb{K}) \neq 0\}.$$

Notice that, using cellular homology, we see that $H_*(X,\mathbb{Z}) = H_0(X,\mathbb{Z}) \oplus H_q(X,\mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}/r\mathbb{Z}$. We should then discuss two cases:

(i) If $char(\mathbb{K}) = 0$ or co-prime with r, we have $H^*(X,\mathbb{K}) = H^0(X,\mathbb{K}) \cong \mathbb{K}$. In this case, $H^*(X,\mathbb{K})$ has formal dimension $fd(X) = 0$, thus, it is a Poincaré duality space. Moreover, since it has finite dimensional cohomology, it is also a Gorenstein space [11, Theorem 3.1].

(ii) If $char(\mathbb{K})$ divides r then, $H^*(X,\mathbb{K}) = H^0(X,\mathbb{K}) \oplus H^q(X,\mathbb{K}) \oplus H^{q+1}(X,\mathbb{K}) \cong \mathbb{K}\oplus\mathbb{K}\oplus\mathbb{K}$. Thus, since $q \geq 2$, X is neither a Poincaré duality space nor a Gorenstein space [7, Theorem 1]. In this case, $fd(X) = q+1$, so that $\mathcal{E}xt^k_{(TV,d)}(\mathbb{K},(TV,d)) = 0, \forall k > q+1$.

Example: In this example, we specify the case where $q = 2$, i.e. $X = S^2 \cup_\varphi e^3$. Thus $V = \mathbb{K}a \oplus \mathbb{K}a'$ with $|a| = 1$ and $|a'| = 2$. We give below an explicit computation of it to illustrate the use of Adams-Hilton models.

i. *Assume that* $char(\mathbb{K}) = 0$ *or co-prime with* r (we specialize in the case where $\mathbb{K} = \mathbb{Q}$).

Let f be a cycle of degree 0, we have $df(1) = 0$, then $f(1)$ is necessarily a scalar $f(1) = \gamma$. The second equation in (5.5) implies that $df(sa) = \gamma a$, therefore $f(sa) = -\frac{\gamma}{r}a' + \gamma' a^2$. The last equation in (5.5) gives, after a simple simplification, $df(sa') = r\gamma' a^2$, then $f(sa') = -\gamma'_1 a' \cdot a + \gamma'_2 a \cdot a' + \gamma'' a^3$,

$$Ker(D) = \frac{\mathbb{Q} \oplus \mathbb{Q}a' \oplus \mathbb{Q}a^2 \oplus \mathbb{Q}a' \cdot a \oplus \mathbb{Q}a \cdot a' \oplus \mathbb{Q}a^3}{< x_1 = -rx_2; x_3 = -x_4 + x_5 >}.$$

Now let g be an arbitrary element of degree 1:

$$\begin{cases} g(1) = \alpha_1 a \\ g(sa) = \alpha_2 a' \cdot a + \alpha_3 a \cdot a' + \alpha_4 a^3 \\ g(sa') = \alpha_5 a'^2 + \alpha_6 a' \cdot a^2 + \alpha_7 a \cdot a' \cdot a + \alpha_8 a^2 \cdot a' + \alpha_9 a^4, \end{cases}$$

by (5.4) we have

$$\begin{cases} D(g)(1) = 0 \\ D(g)(sa) = (-\alpha_1 - r\alpha_2 + r\alpha_3)a^2 \\ D(g)(sa') = (\alpha_1 + r\alpha_2 - r\alpha_5)a' \cdot a + (r\alpha_3 - r\alpha_5)a \cdot a' + (r\alpha_4 - r\alpha_6 + r\alpha_7 - r\alpha_8)a^3, \end{cases}$$

therefore

$$Im(D) = \frac{\mathbb{Q}a^2 \oplus \mathbb{Q}a' \cdot a \oplus \mathbb{Q}a \cdot a' \oplus \mathbb{Q}a^3}{< x_1 = -x_2 + x_3 >}.$$

We consequently obtain $\mathcal{E}xt^0_{(TV,d)}(\mathbb{Q},(TV,d)) = \mathbb{Q}$.

Applying the same process for $i \neq 0$, we recover the previously stated fact $\mathcal{E}xt^i_{(TV,d)}(\mathbb{Q},(TV,d)) = 0$.

ii. *Assume that* $char(\mathbb{K})$ *divides* r, so that $r = 0$ (mod $char(\mathbb{K})$) and $da = da' = 0$.

Recall that in general, we have:

$$(D(f))(1 \otimes sa) = df(1 \otimes sa) - (-1)^{pq} af(1) = 0,$$

and

$$(D(f))(1 \otimes sa') = df(1 \otimes sa') - (-1)^p rf(1 \otimes sa) - (-1)^{p(q+1)} a' f(1) = 0.$$

These become respectively in this case:

$$(D(f))(1 \otimes sa) = -(-1)^{pq} af(1) = 0, \text{ and } (D(f))(1 \otimes sa') = -(-1)^{p(q+1)} a' f(1) = 0.$$

Notice that in this case, for an element f to be in $ker(D)$, it is necessarily that $f(1) = 0$.

Let f be a cycle of degree 0, then we have $f(1)=0$, consequently $df(sa)=df(sa')=0$, which implies that $f(sa)=\gamma_1 a'+\gamma_2 a^2$ and $f(sa')=\gamma_3 a'\cdot a+\gamma_4 a\cdot a'+\gamma_5 a^3$. Therefore

$$Ker(D)=\mathbb{K}a^2\oplus\mathbb{K}a'\oplus\mathbb{K}a'\cdot a\oplus\mathbb{K}a\cdot a'\oplus\mathbb{K}a^3.$$

Now let g be an arbitrary element of degree 1:

$$\begin{cases} g(1)=\alpha_1 a\\ g(sa)=\alpha_2 a'\cdot a+\alpha_3 a\cdot a'+\alpha_4 a^3\\ g(sa')=\alpha_5 a'^2+\alpha_6 a'\cdot a^2+\alpha_7 a\cdot a'\cdot a+\alpha_8 a^2\cdot a'+\alpha_9 a^4,\end{cases}$$

hence

$$\begin{cases} D(g)(1)=0\\ D(g)(sa)=-ag(1)=-\alpha_1 a^2\\ D(g)(sa')=a'g(1)=\alpha_1 a'\cdot a,\end{cases}$$

therefore

$$Im(D)=\frac{\mathbb{K}a^2\oplus\mathbb{K}a'\cdot a}{<x_1=-x_2>}\cong\mathbb{K}(a'\cdot a-a^2).$$

we obtain $\mathcal{E}xt^0_{(TV,d)}(\mathbb{K},(TV,d))\cong\mathbb{K}a^2\oplus\mathbb{K}a'\oplus\mathbb{K}a\cdot a'\oplus\mathbb{K}a^3\cong\mathbb{K}^4$.

An application of the same argument yields:

$$\mathcal{E}xt^{-1}_{(TV,d)}(\mathbb{K},(TV,d))\cong\mathbb{K}a^3\oplus\mathbb{K}a\cdot a'\oplus\mathbb{K}a'\cdot a\oplus\mathbb{K}a\cdot a'\cdot a\oplus\mathbb{K}a^2\cdot a'\oplus\mathbb{K}a^4\cong\mathbb{K}^6,$$

$$\begin{aligned}\mathcal{E}xt^{-2}_{(TV,d)}(\mathbb{K},(TV,d)) &\cong \mathbb{K}a^4\oplus\mathbb{K}a^2\cdot a'\oplus\mathbb{K}a\cdot a'\cdot a\oplus\mathbb{K}a'\cdot a^2\oplus\mathbb{K}a'^2\oplus\\ &\quad\ \mathbb{K}a\cdot a'^2\oplus\mathbb{K}a^3\cdot a'\oplus\mathbb{K}a^2\cdot a'\cdot a\oplus\mathbb{K}a\cdot a'\cdot a^2\oplus\mathbb{K}a^5\\ &\cong \mathbb{K}^{10}.\end{aligned}$$

From the previous cases, it is obvious that $\mathcal{E}xt^{-i}_{(TV,d)}(\mathbb{K},(TV,d))\neq 0,\ \forall i\geq 0$. Since $fd(X,\mathbb{K})=3$, we have $\mathcal{E}xt^{i}_{(TV,d)}(\mathbb{K},(TV,d))=0,\ \forall i\geq 4$. It remains then to calculate the cases $i=-1,-2,-3$ which are given successively by applying the same process as follows: $\mathcal{E}xt^{1}_{(TV,d)}(\mathbb{K},(TV,d))\cong\mathbb{K}a'\oplus\mathbb{K}a^2$, $\mathcal{E}xt^{2}_{(TV,d)}(\mathbb{K},(TV,d))\cong\mathbb{K}\oplus\mathbb{K}a$ and $\mathcal{E}xt^{3}_{(TV,d)}(\mathbb{K},(TV,d))\cong\mathbb{K}$.

Now notice that for $q>2$, Adams-Hilton model for $X=S^q\cup_\varphi e^{q+1}$ has two generators a and a' of degrees respectively $q-1$ and q, thus the degree of a' does not double that of a, so the previous example is somewhat a special case. However the computational process still holds, and we have, for the case **i**, $\mathcal{E}xt^{0}_{(TV,d)}(\mathbb{K},(TV,d))=\mathbb{K}$ and $\mathcal{E}xt^{i}_{(TV,d)}(\mathbb{K},(TV,d))=0$ for $i\neq 0$. Whereas for the case **ii**, computation process holds but the results differ, since we might have $\mathcal{E}xt^{-i}_{(TV,d)}(\mathbb{K},(TV,d))=0$ for finitely many $i\geq-(q+1)$ (e.g. for $q=7$ we have $\mathcal{E}xt^{7}_{(TV,d)}(\mathbb{K},(TV,d))=0$), on the other hand, we always have $\mathcal{E}xt^{-i}_{(TV,d)}(\mathbb{K},(TV,d))=0,\ \forall i<-(q+1)$ since $fd(X)=q+1$.

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References

[1] J. F. Adams and P. J. Hilton, *On the chain algebra of a loop space*, Comment. Math. Helv. **30**, 305-330, 1955.

[2] D. Anick, *Hopf algebras up to homotopy,* J. Amer. Math. Soc. **2**, 417-453, 1989.

[3] R. Bott and H. Samelson, *On the Pontryagin product in spaces of paths,* Comment. Math. Helv. **27** (1), 320-337, 1953.

[4] J. G. Carrasquel-Vera, *Computations in rational sectional category*, Bull. Belg. Math. Soc. Simon Stevin **22** (3), 455-469, 2015.

[5] J. G. Carrasquel, *Rational methods applied to sectional category and topological complexity*, Contemp. Math. **702**, 2018.

[6] M. Farber, *Topological Complexity of Motion Planning*, Discrete Comput. Geom. **29**, 211-221, 2003.

[7] Y. Félix and S. Halperin, *A note on Gorenstein spaces*, J. Pure Appl. Algebra **223**, 4937-4953, 2019.

[8] Y. Félix and S. Halperin, *Rational LS-category and its applications*, Trans. Am. Math. Soc. **273** (1), 1-37, 1982.

[9] Y. Félix, S. Halperin, C. Jacobson, C. Löfwall and J. C. Thomas, *The radical of the homotopy Lie algebra*, Amer. J. Math. **110**, 301-322, 1988.

[10] Y. Félix, S Halperin and J. C. Thomas, *LS-catégorie et suite spectrale de Milnor-Moore*, Bull. Soc. Math. France **111**, 89-96, 1983.

[11] Y. Félix, S. Halperin and J. C. Thomas, *Gorenstein spaces.* Adv. Math. **71**, 92-112, 1988.

[12] Y. Félix, S. Halperin and J. C. Thomas, *Rational Homotopy Theory*, Graduate Texts in Mathematics **215**, Springer Verlag, 2000.

[13] H. Gammelin, Gorenstein spaces with nonzero evaluation map, Trans. Amer. Math. Soc. **351** (8), 3433-3440, 1999.

[14] S. Halperin, *Universal enveloping algebras and loop space homology*, J. Pure Appl. Algebra **83**, 237-282, 1992.

[15] S. Hamoun, Y. Rami and L. Vandembroucq, *On the rational topological complexity of coformal elliptic spaces*, J. Pure Appl. Algebra **227** (7), 107318, 2023.

[16] L. Lechuga and A. Murillo, *Complexity in rational homotopy*, Topology **39**, 89-94, 2000.

[17] L. Lusternik and L. Shnirelmann, *Méthodes Topologiques dans les problèmes variationnels*, Hermann, Paris, 1934.

[18] A. Murillo, *The evaluation map of some Gorenstein algebras*, J. Pure. Appl. Algebra **91**, 209-218, 1994.

[19] Y. B. Rudyak, *On higher analogs of topological complexity*, Topology Appl. **157**, 916-920, 2010.

[20] M. Spivak, *Spaces satisfying Poincaré duality*, Topology **6**, 77-102, 1967.