Turk. J. Math. Comput. Sci.
16(2)(2024) 386–399
© MatDer
DOI : 10.47000/tjmcs.1310722

D_i-Darboux Slant Helices on Surface

Akin Alkan[1,*], Hüseyin Kocayiğit[2], Tuba Ağirman Aydin[3]

[1] *Gördes Vocational School, Manisa Celal Bayar University, 45750, Manisa, Turkey.*
[2] *Department of Mathematics, Faculty of Science and Arts, Manisa Celal Bayar University, 45140, Manisa, Turkey.*
[3] *Department of Mathematics, Faculty of Education, Bayburt University, 69000, Bayburt, Turkey.*

Received: 07-06-2023 • Accepted: 30-10-2024

Abstract. In this study, we consider D_i-Darboux slant helices which are new surface curves on an oriented surface. We give some characterizations for such curves according to the Darboux frame, OD-frame, ND-frame, RD-frame, and obtain axes of the D_i-Darboux slant helices. Moreover, the position vectors of the D_i-Darboux slant helices are obtained.

2020 AMS Classification: 53A04

Keywords: D_o-Darboux slant helix, D_n-Darboux slant helix, D_r-Darboux slant helix, helix, isophote curve, relatively normal-slant helix.

1. Introduction

In differential geometry, the curves play an important role. The special curves that differ according to their structural features offer an important studying area [3, 4]. One of the most intensively studied special curves are helices. The helices are widely used not only in geometry, but also in nature and science. A curve in E^3 is a general helix if and only if $\frac{\tau}{\kappa}$ is constant, where τ and $\kappa \neq 0$ are torsion and curvature of curve, respectively [5]. Izumiya and Takeuchi defined the slant helix as the principal normal vector of the curve makes a constant angle with a constant straight line. They characterized these curves with $\frac{\kappa^2}{(\kappa^2+\tau^2)^{\frac{3}{2}}}\left(\frac{\tau}{\kappa}\right)'(s)$ is a constant function [8]. The position vector of the slant helix was given by Ali depending on the curvature of the curve [1]. Kula and Yaylı have introduced the spherical indicatrix of a slant helix and they have shown that these spherical images are spherical helix [9]. Later, Zıplar et al. have defined Darboux helices and they have characterized such curve as $\frac{1}{\frac{\kappa^2}{(\kappa^2+\tau^2)^{\frac{3}{2}}}\left(\frac{\tau}{\kappa}\right)'}(s)$ is constant. They have provided that a curve is a Darboux helix if and only if it is a slant helix [13].

For curves on a surface, the Darboux frame $\{T, V, U\}$ is defined, where T and U are unit tangent vector and unit surface normal along curve, respectively, and $V = U \times T$ is a unit vector field along the curve on surface. A curve on the surface is called a helix if the vector T makes a constant angle with a constant (fixed) direction. Puig-Pey et al. have found a new method for obtaining a general helix on the surface [12]. Similarly, a surface curve is called relatively normal-slant helix (resp. isophote curve) if the vector V (resp. vector U) makes a constant angle with a constant (fixed) direction. The relatively normal-slant helix have defined and characterized by Macit and Düldül. They

*Corresponding Author
Email addresses: akin.alkan@cbu.edu.tr (A. Alkan), huseyin.kocayigit@cbu.edu.tr (H. Kocayiğit), tubaaydin@bayburt.edu.tr (T. Ağırman Aydın)

have also given the position vector of the relatively normal-slant helix [10]. The characterizations of isophote curve have been introduced by Doğan and Yaylı. They have also given the axis of this curve [6].

In a recent paper, Hananoi et al. have defined vector fields D_o, D_n, and D_r which are called the osculating Darboux vector field, the normal Darboux vector field and the rectifying Darboux vector field, respectively, along a curve on the surface [7]. Considering this definition, Önder has defined three new curves on the surface and he has called these curves as the D_i -Darboux slant helices, where the indices $i \in \{o, n, r\}$ represent the osculator, normal, and rectifying planes of the curve on surface, respectively [11].

This paper aims to investigate the D_i -Darboux slant helices that have been defined but not examined. We give characterizations of the D_i -Darboux slant helices according to curvatures of the Darboux frame, OD-frame, ND-frame, and RD-frame. We give the relationships between the D_i -Darboux slant helices and special surface curves (helix, relatively normal-slant helix, and isophote curve). Finally, we give the methods for obtaining the D_i-Darboux slant helices on the surface.

2. Preliminaries

Let M is an oriented surface in E^3 and $\alpha : I \to M$ be a unit speed curve on the surface M. The Frenet frame $\{T, N, B\}$ is well-defined along the curve α and derivative formulas of the Frenet frame are given by;

$$T' = \kappa N,\ N' = -\kappa T + \tau B,\ B' = -\tau N,$$

where T is the unit tangent vector, N is the principal normal vector, B is the binormal vector; κ and τ are the curvature and the torsion of α, respectively. If we denote the Darboux frame along the α by $\{T, V, U\}$, derivative formulas of the Darboux frame are given by

$$T' = k_g V + k_n U,\ V' = -k_g T + \tau_g U,\ U' = -k_n T - \tau_g V,$$

where T is the unit tangent of α, U is the unit normal of M along the α, and $V = U \times T$. The functions k_g, k_n, and τ_g are the geodesic curvature, normal curvature, geodesic torsion of the α, respectively. Let ω denotes the angle between the surface normal vector U and binormal vector B. Then, we have

$$\begin{cases} \kappa^2 = k_g^2 + k_n^2, \\ k_g = \kappa \cos\omega, \\ k_n = \kappa \sin\omega, \\ \tau_g = \tau - \omega'. \end{cases} \tag{2.1}$$

The relationships between Darboux frame and Frenet frame of α are

$$\begin{aligned} T &= T, \\ N &= \cos\omega V + \sin\omega U, \\ B &= -\sin\omega V + \cos\omega U, \end{aligned}$$

and

$$\begin{cases} V = \cos\omega N - \sin\omega B, \\ U = \sin\omega N + \cos\omega B. \end{cases} \tag{2.2}$$

The vector fields $D_o(s), D_n(s), D_r(s)$ along α given by

$$\begin{aligned} D_o &= \tau_g T - k_n V, \\ D_n &= -k_n V + k_g U, \\ D_r &= \tau_g T + k_g U \end{aligned}$$

are called the osculating Darboux vector field, the normal Darboux vector field, and the rectifying Darboux vector field along α , respectively [7].

Recently, Alkan et al. have defined three new frames for a surface curve. They have called these frame as osculating Darboux frame (OD-frame), normal Darboux frame (ND-frame) and rectifying Darboux Frame (RD-frame), respectively [2]. If we denote the osculating Darboux frame (OD-frame) along the α by $\{\widetilde{D}_o, U, Y_o\}$, derivative formulas of the OD-frame are given by;

$$\begin{cases} \widetilde{D}_o' = -\delta_o Y_o, \\ U' = \mu_o Y_o, \\ Y_o' = \delta_o \widetilde{D}_o - \mu_o U, \end{cases} \tag{2.3}$$

where $\widetilde{D}_o = \frac{D_o}{\|D_o\|}$, U is the unit normal of M along the α and $Y_o = \widetilde{D}_o \times U$. $\mu_o = \sqrt{k_n^2 + \tau_g^2}$ and $\delta_o = \frac{k_n^2}{k_n^2+\tau_g^2}\left(\frac{\tau_g}{k_n}\right)' + k_g$ are curvatures of α according to the OD-frame. If we denote the normal Darboux frame (ND-frame) along the α by $\{\widetilde{D}_n, T, Y_n\}$, derivative formulas of the ND-frame are given by;

$$\begin{aligned} \widetilde{D}_n' &= -\delta_n Y_n, \\ T' &= \mu_n Y_n, \\ Y_n' &= \delta_n \widetilde{D}_n - \mu_n T, \end{aligned}$$

where $\widetilde{D}_n = \frac{D_n}{\|D_n\|}$, T is the unit tangent vector of α and $Y_n = \widetilde{D}_n \times T$. $\mu_n = \sqrt{k_n^2 + k_g^2}$ and $\delta_n = \frac{k_g^2}{k_n^2+k_g^2}\left(\frac{k_n}{k_g}\right)' + \tau_g$ are curvatures of α according to the ND-frame. The rectifying Darboux Frame (RD-frame) along the α is denoted by $\{\widetilde{D}_r, V, Y_r\}$ whose derivative formulas are given by;

$$\begin{aligned} \widetilde{D}_r' &= -\delta_r Y_r, \\ V' &= \mu_r Y_r, \\ Y_r' &= \delta_r \widetilde{D}_r - \mu_r V, \end{aligned}$$

where $\widetilde{D}_r = \frac{D_r}{\|D_r\|}$, $V = U \times T$ and $Y_r = \widetilde{D}_r \times V$. $\mu_r = \sqrt{k_g^2 + \tau_g^2}$ and $\delta_r = \frac{k_g^2}{k_g^2+\tau_g^2}\left(\frac{\tau_g}{k_g}\right)' - k_n$ are curvatures of α according to the ND-frame [2].

Theorem 2.1. *Let α be a unit speed curve with $\kappa \neq 0$. Then, α is a general helix iff $\frac{\tau}{\kappa}(s)$ is a constant function [5].*

Theorem 2.2. *Let α be a unit speed curve with $\kappa \neq 0$. Then, α is a slant helix iff $\frac{\kappa^2}{(\kappa^2+\tau^2)^{\frac{3}{2}}}\left(\frac{\tau}{\kappa}\right)'(s)$ is a constant function [8].*

Theorem 2.3. *Let α be a unit speed curve with $\kappa \neq 0$,*
i) Then, α is a Darboux helix iff $\dfrac{1}{\frac{\kappa^2}{(\kappa^2+\tau^2)^{\frac{3}{2}}}\left(\frac{\tau}{\kappa}\right)'}(s)$ is a constant function.
ii)Then, α is a Darboux helix iff α is a slant helix [13].

3. D_o-Darboux Slant Helices

In this section, we introduce D_o-Darboux slant helix which is a new kind of surface curve.

Let M is an oriented surface in E^3, and $\alpha : I \to M$ be a unit speed curve on surface M. Let $\{T, V, U\}$ be Darboux frame, k_g, k_n, τ_g be the curvatures and $D_o(s) = \tau_g(s)T(s) - k_n(s)V(s)$ be the osculating Darboux vector of α. The concept of the D_o-Darboux slant helix on the surface is defined as follows.

Definition 3.1. Let M is an oriented surface in E^3 and $\alpha : I \to M$ be a unit speed curve on surface M. Then, α is called D_o-Darboux slant helix if there exists a constant angle θ between the Darboux vector field D_o (or equivalently, unit Darboux vector field $\widetilde{D}_o = \frac{D_o}{\|D_o\|}$) and a fixed(constant) unit direction d_o , i.e., $\langle D_o, d_o \rangle = \cos\theta$ is constant [11].

Theorem 3.2. *Let M be an oriented surface in E^3 and $\alpha : I \to M$ be a unit speed curve with OD-frame $\{\widetilde{D}_o, U, Y_o\}$ on surface M . Then, α is a D_o-Darboux slant helix iff $\frac{\mu_o}{\delta_o}(s)$ is a constant function, where $\delta_o \neq 0$.*

Proof. Let α is a D_o-Darboux slant helix with OD-frame $\{\tilde{D}_o, U, Y_o\}$. From Definition 3.1, $\left\langle \tilde{D}_o, d_o \right\rangle = \cos\theta$. Differentiating last equation, we have $\left\langle \tilde{D}'_o, d_o \right\rangle = 0$ and using (2.3), $-\delta_o \langle Y_o, d_o\rangle = 0$. Since $\delta_o \neq 0$, we get $\langle Y_o, d_o\rangle = 0$, i.e., $d_o \in sp\{\tilde{D}_o, U\}$. So, we can write

$$d_o = \cos\theta \tilde{D}_o + \sin\theta U. \tag{3.1}$$

If we differentiate (3.1), since d_o is a constant vector, we obtain

$$Y_o\left(\mu_o \sin\theta - \delta_o \cos\theta\right) = 0.$$

Hence,

$$\frac{\mu_o}{\delta_o} = \frac{\cos\theta}{\sin\theta} = constant.$$

Conversely, let $\frac{\mu_o}{\delta_o} = \frac{\cos\theta}{\sin\theta} = constant$. So we get $\mu_o \sin\theta = \delta_o \cos\theta$. We assume that

$$d_o = \cos\theta \tilde{D}_o + \sin\theta U.$$

Differentiating d_o, and using (2.3) we have $d'_o = 0$. Furthermore, $\left\langle \tilde{D}_o, d_o \right\rangle = \cos\theta$. Therefore, α is a D_o-Darboux slant helix. □

Corollary 3.3. *Let M be an oriented surface in E^3, and $\alpha : I \to M$ be a unit speed curve on surface M. Then, α is a D_o-Darboux slant helix iff*

$$\sigma_o(s) = \left(\frac{\sqrt{k_n^2 + \tau_g^2}}{\frac{k_n^2}{k_n^2+\tau_g^2}\left(\frac{\tau_g}{k_n}\right)' + k_g} \right)(s) \tag{3.2}$$

is a constant function.

Corollary 3.4. *Axis of the D_o-Darboux slant helix according to OD-frame is given by*

$$d_o = \cos\theta \tilde{D}_o + \sin\theta U.$$

Corollary 3.5. *Axis of the D_o-Darboux slant helix according to Darboux frame is*

$$d_o = \frac{\tau_g}{\sqrt{k_n^2 + \tau_g^2}} \cos\theta T - \frac{k_n}{\sqrt{k_n^2 + \tau_g^2}} \cos\theta V + \sin\theta U. \tag{3.3}$$

Corollary 3.6. *α is a D_o-Darboux slant helix if and only if α is a isophote curve.*

Proof. By the inner product of both sides of (3.3) with U, we get $\langle U, d_o\rangle = \sin\theta = constant$. So, α is an isophote curve. □

Corollary 3.7. *Let α be a unit speed D_o-Darboux slant helix on surface M. Then,*
i) α is a geodesic curve on M with $k_n \neq 0$ iff α is a Darboux helix with the axis

$$d_o = \frac{\tau}{\sqrt{\tau^2 + \kappa^2}} \cos\theta T \mp \sin\theta N + \frac{\kappa}{\sqrt{\tau^2 + \kappa^2}} \cos\theta B.$$

.
ii) α is an asymptotic curve on M with $k_g \neq 0$ iff α is a general helix with the axis $d_o = \cos\theta T \mp \sin\theta B$.
iii) Let α be a line of curvature on M. Then, α is a plane curve.

Proof. i) Since α is a geodesic, we have $k_g = 0$ and so from (2.1) it follows $k_n = \pm\kappa$ and $\tau_g = \tau$. From (3.2), we get

$$\sigma_o = \left(\frac{1}{\frac{\kappa^2}{(\kappa^2+\tau^2)^{\frac{3}{2}}}\left(\frac{\tau}{\kappa}\right)'} \right)$$

is a constant function. Then, by Theorem 2.3, α is a Darboux helix and α is a slant helix. Using (2.2) in (3.3), we get

$$d_o = \frac{\tau_g}{\sqrt{k_n^2+\tau_g^2}}\cos\theta T + \left(-\frac{k_n k_g}{\kappa\sqrt{k_n^2+\tau_g^2}}\cos\theta + \sin\theta\frac{k_n}{\kappa}\right)N + \left(\cos\theta\frac{k_n^2}{\kappa\sqrt{k_n^2+\tau_g^2}} + \sin\theta\frac{k_g}{\kappa}\right)B. \tag{3.4}$$

Since $k_g = 0$, $k_n = \pm\kappa$, and $\tau_g = \tau$, the axis of the Darboux helix is obtained as $d_o = \frac{\tau}{\sqrt{\tau^2+\kappa^2}}\cos\theta T \mp \sin\theta N + \frac{\kappa}{\sqrt{\tau^2+\kappa^2}}\cos\theta B$.

Conversely, let α be a Darboux helix with the axis

$$d_o = \frac{\tau}{\sqrt{\tau^2+\kappa^2}}\cos\theta T \mp \sin\theta N + \frac{\kappa}{\sqrt{\tau^2+\kappa^2}}\cos\theta B.$$

From (3.3), we have $k_g = 0$.

ii) Since α is an asymptotic curve, we have $k_n = 0$ and from (2.1) it follows $k_g = \pm\kappa$ and $\tau_g = \tau$. By substituting k_g and τ_g in (3.2), we obtain $\sigma_o = \mp\frac{\tau}{\kappa}$ is a constant function. So, α is a general helix. Using $k_n = 0$, $k_g = \pm\kappa$ and $\tau_g = \tau$ in (3.4), we obtain $d_o = \cos\theta T \mp \sin\theta B$.

Conversely, let α be a helix with the axis $d_o = \cos\theta T \mp \sin\theta B$. From (3.3), we have $k_n = 0$.

iii) Since α is a line of curvature on M, we have $\tau_g = 0$. From (3.2) we have $\sigma_o = \frac{k_n}{k_g} = constant$. Using (2.1), we get

$$\sigma_o = \frac{k_n}{k_g} = \frac{\kappa\sin\xi}{\kappa\cos\xi} = \tan\xi,$$

where ξ is the angle between binormal vector B and surface normal U. So ξ is constant. Since $\tau_g = \tau - \xi'$, we obtain $\tau = 0$, i.e, the α is a plane curve. □

3.1. **The Position Vector of the D_o-Darboux Slant Helix on a Surface.** In this section, we introduce the methods for finding the D_o-Darboux Slant helix on a given surface. We discuss the methods separately for parametric and implicit surfaces.

3.1.1. *The D_o-Darboux Slant Helix on a Parametric Surface.* Let M be a regular oriented surface in E^3 with given the parametrization $X = X(u, v)$. Let $\alpha(s) = X(u(s), v(s))$ be a unit speed D_o-Darboux slant helix on M with axis d_o, constant angle θ, and Darboux frame $\{T, V, U\}$. For obtaining α, we find $u(s)$ and $v(s)$. Since

$$\gamma' = T = X_u\frac{du}{ds} + X_v\frac{dv}{ds},\ U = \frac{X_u\times X_v}{\|X_u\times X_v\|},\ V = U\times T,$$

we get

$$V = \frac{1}{\|X_u\times X_v\|}\left[(EX_v - FX_u)\frac{du}{ds} + (FX_v - GX_u)\frac{dv}{ds}\right],$$

where $E = \langle X_u, X_u\rangle$, $F = \langle X_u, X_v\rangle$ and $G = \langle X_v, X_v\rangle$ are the first fundamental form coefficients of the surface along the α [10]. Since α is a D_o-Darboux slant helix, we have

$$\left\langle \frac{\tau_g}{\sqrt{k_n^2+\tau_g^2}}T - \frac{k_n}{\sqrt{k_n^2+\tau_g^2}}V, d_o\right\rangle = \cos\theta. \tag{3.5}$$

From (3.5), we have

$$\begin{aligned}&\left[\left(\frac{\tau_g}{\sqrt{k_n^2+\tau_g^2}}\|X_u\times X_v\| + \frac{k_n}{\sqrt{k_n^2+\tau_g^2}}F\right)\langle X_u, d_o\rangle - \frac{k_n}{\sqrt{k_n^2+\tau_g^2}}E\langle X_v, d_o\rangle\right]\frac{du}{ds}\\ &+\left[\left(\frac{\tau_g}{\sqrt{k_n^2+\tau_g^2}}\|X_u\times X_v\| - \frac{k_n}{\sqrt{k_n^2+\tau_g^2}}F\right)\langle X_v, d_o\rangle + \frac{k_n}{\sqrt{k_n^2+\tau_g^2}}G\langle X_u, d_o\rangle\right]\frac{dv}{ds} = \cos\theta.\end{aligned} \tag{3.6}$$

On the other hand, since $\langle T, T\rangle = 1$, we obtain

$$E\left(\frac{du}{ds}\right)^2 + 2F\frac{du}{ds}\frac{dv}{ds} + G\left(\frac{dv}{ds}\right)^2 = 1. \tag{3.7}$$

If (3.6) and (3.7) are solved together, we get

$$\begin{cases} \frac{du}{ds} = \frac{-2\|X_u\times X_v\|\cos\theta\left[K\left(EG-F^2\right)\langle X_v,d_o\rangle+\Gamma\|X_u\times X_v\|(F\langle X_v,d_o\rangle-G\langle X_u,d_o\rangle)\right]\pm\sqrt{\Delta}}{2A\left(EG-F^2\right)}, \\ \frac{dv}{ds} = \frac{-2\|X_u\times X_v\|\cos\theta\left[-K\left(EG-F^2\right)\langle X_u,d_o\rangle+\Gamma\|X_u\times X_v\|(F\langle X_u,d_o\rangle-E\langle X_v,d_o\rangle)\right]\pm\sqrt{\Delta^*}}{2A\left(EG-F^2\right)}, \end{cases} \tag{3.8}$$

where

$$K = \frac{k_n}{\sqrt{k_n^2+\tau_g^2}}, \quad \Gamma = \frac{\tau_g}{\sqrt{k_n^2+\tau_g^2}},$$

$$A = E\langle X_v, d_o\rangle^2 - 2F\langle X_u, d_o\rangle\langle X_v, d_o\rangle + G\langle X_u, d_o\rangle^2,$$

$$\Delta = 4\|X_u\times X_v\|^2\cos^2\theta\left\{\begin{array}{c}\left[K\left(EG-F^2\right)\langle X_v,d_o\rangle+\Gamma\|X_u\times X_v\|(F\langle X_v,d_o\rangle-G\langle X_u,d_o\rangle)\right]^2\\ -AG\left(EG-F^2\right)\end{array}\right\}$$
$$+4A\left(EG-F^2\right)\left[(\Gamma\|X_u\times X_v\|-KF)\langle X_v,d_o\rangle+KG\langle X_u,d_o\rangle\right]^2,$$

$$\Delta^* = 4\|X_u\times X_v\|^2\cos^2\theta\left\{\begin{array}{c}\left[\Gamma\|X_u\times X_v\|(F\langle X_u,d_o\rangle-E\langle X_v,d_o\rangle)-K\left(EG-F^2\right)\langle X_u,d_o\rangle\right]^2\\ -AE\left(EG-F^2\right)\end{array}\right\}$$
$$+4A\left(EG-F^2\right)\left[(\Gamma\|X_u\times X_v\|+KF)\langle X_u,d_o\rangle-KE\langle X_v,d_o\rangle\right]^2.$$

If we solve the system (3.8) together with the initial point

$$\begin{cases} u(0) = u_0 \\ v(0) = v_0, \end{cases}$$

we obtain the desired D_o-Darboux slant helix on M by substituting $u(s), v(s)$ into $X(u,v)$.

3.1.2. *The D_o-Darboux Slant Helix on an Implicit Surface.* Let M be a surface given in implicit form by $f(x,y,z) = 0$.

Let us now find the D_o-Darboux slant helix $\alpha(s)$ which makes the given constant angle with the given axis $d_o = (a,b,c)$ and lying on M.

Let $\alpha(s) = (x(s), y(s), z(s))$ and $\{T, V, U\}$ be its Darboux frame. We need to find $x(s), y(s), z(s)$ to obtain $\alpha(s)$. The vectors of Darboux frame of α

$$T = \alpha' = \left(\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}\right),$$
$$U = \frac{\nabla f}{\|\nabla f\|} = \frac{1}{\|\nabla f\|}\left(f_x, f_y, f_z\right),$$
$$V = U\times T = \frac{1}{\|\nabla f\|}\left(f_y\frac{dz}{ds} - f_z\frac{dy}{ds}, f_z\frac{dx}{ds} - f_x\frac{dz}{ds}, f_x\frac{dy}{ds} - f_y\frac{dx}{ds}\right),$$

[10]. If T and V vectors are written at $\tilde{D}_o$ vector, we get

$$\tilde{D}_o = \Gamma\left(\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}\right) - K\frac{1}{\|\nabla f\|}\left(f_y\frac{dz}{ds} - f_z\frac{dy}{ds}, f_z\frac{dx}{ds} - f_x\frac{dz}{ds}, f_x\frac{dy}{ds} - f_y\frac{dx}{ds}\right),$$

where $K = \frac{k_n}{\sqrt{k_n^2+\tau_g^2}}$, $\Gamma = \frac{\tau_g}{\sqrt{k_n^2+\tau_g^2}}$. Since $\left\langle \tilde{D}_o, d_o\right\rangle = \cos\theta$

$$\left(\Gamma a\|\nabla f\| - K(bf_z - cf_y)\right)\frac{dx}{ds} + \left(\Gamma b\|\nabla f\| - K(cf_x - af_z)\right)\frac{dy}{ds} + \left(\Gamma c\|\nabla f\| - K(af_y - bf_x)\right)\frac{dz}{ds} = \|\nabla f\|\cos\theta. \tag{3.9}$$

On the other hand, since $\langle T, U\rangle = 0$

$$f_x\frac{dx}{ds} + f_y\frac{dy}{ds} + f_z\frac{dz}{ds} = 0. \tag{3.10}$$

If the value of $\frac{dy}{ds}$ in (3.10) is written in (3.9), we get

$$\frac{dx}{ds} = \frac{1}{\Omega}\left\{\|\nabla f\| f_y \cos\theta - \left(\Gamma \|\nabla f\| \left(cf_y - bf_z\right) - K\left[f_y\left(af_y - bf_x\right) - f_z\left(cf_x - af_z\right)\right]\right)\frac{dz}{ds}\right\}, \tag{3.11}$$

and if the value of $\frac{dx}{ds}$ in (3.10) is written in (3.9), we obtain

$$\frac{dy}{ds} = \frac{1}{\Omega}\left\{\left(\Gamma \|\nabla f\| \left(cf_x - af_z\right) - K\left[f_x\left(af_y - bf_x\right) - f_z\left(bf_z - cf_y\right)\right]\right)\frac{dz}{ds} - \|\nabla f\| f_x \cos\theta\right\}, \tag{3.12}$$

where $\Omega = \Gamma \|\nabla f\| \left(af_y - bf_x\right) - K\left[f_y\left(bf_z - cf_y\right) - f_x\left(cf_x - af_z\right)\right] \neq 0$. Since $\langle T, T\rangle = 1$,

$$\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 + \left(\frac{dz}{ds}\right)^2 = 1. \tag{3.13}$$

By substituting (3.11) and (3.12) in (3.13), we obtain

$$q_1\left(\frac{dz}{ds}\right)^2 + q_2\left(\frac{dz}{ds}\right) + q_3 = 0,$$

where

$$q_1 = \frac{1}{\Omega^2}\left[\begin{array}{c} K^2\left(\left(af_y - bf_x\right)^2\left(f_x^2 + f_y^2 + 2f_z^2\right) + f_z^2\left((cf_x - af_z)^2 + (bf_z - cf_y)^2\right)\right) \\ -2\Gamma K \|\nabla f\| \left(\left(b^2 - a^2\right) f_x f_y f_z + c\left(f_x^2 + f_y^2\right)\left(af_y - bf_x\right) + abf_z\left(f_x^2 - f_y^2\right)\right) \\ + (\Gamma \|\nabla f\|)^2\left(c^2\left(f_x^2 + f_y^2\right) + \left(a^2 + b^2\right) f_z^2 - 2cf_z\left(bf_y + af_x\right)\right) \end{array}\right] + 1,$$

$$q_2 = -2\|\nabla f\| \cos\theta \frac{1}{\Omega^2}\left[\Gamma \|\nabla f\| \left(c\left(f_x^2 + f_y^2\right) - f_z\left(af_x + bf_y\right)\right) - K\left[\left(af_y - bf_x\right)\left(f_x^2 + f_y^2 + f_z^2\right)\right]\right],$$

$$q_3 = \frac{1}{\Omega^2}\|\nabla f\|^2 \cos^2\theta\left(f_x^2 + f_y^2\right) - 1.$$

From this equation, we have

$$\frac{dz}{ds} = \frac{-q_2 \pm \sqrt{q_2^2 - 4q_1 q_3}}{2q_1}. \tag{3.14}$$

If (3.14) is written in (3.11) and (3.12), the first-order differential equation system is obtained. Thus, together with the initial point

$$x(0) = x_0,$$
$$y(0) = y_0,$$
$$z(0) = z_0\,,$$

we have an initial value problem. The solution of this problem gives us the D_o-Darboux slant helix on M.

Example 3.8. Let us consider the surface M given by the parametrization $X(u, v) = (u, u\sin v, u\cos v)$ and curve $\alpha : I \to M$ defined by the parametric form $\alpha(s) = \left(\frac{s}{2}, \frac{s}{2}\sin\left(\sqrt{2}\ln\frac{s}{2}\right), \frac{s}{2}\cos\left(\sqrt{2}\ln\frac{s}{2}\right)\right)$(Figure 1). The vectors of the Darboux frame and curvatures of α are computed as follows,

$$T = \left(\frac{1}{2}, \frac{1}{2}\sin\left(\sqrt{2}\ln\frac{s}{2}\right) + \frac{1}{\sqrt{2}}\cos\left(\sqrt{2}\ln\frac{s}{2}\right), \frac{1}{2}\cos\left(\sqrt{2}\ln\frac{s}{2}\right) - \frac{1}{\sqrt{2}}\sin\left(\sqrt{2}\ln\frac{s}{2}\right)\right),$$

$$V = \left(-\frac{1}{2}, \frac{1}{\sqrt{2}}\cos\left(\sqrt{2}\ln\frac{s}{2}\right) - \frac{1}{2}\sin\left(\sqrt{2}\ln\frac{s}{2}\right), -\frac{1}{\sqrt{2}}\sin\left(\sqrt{2}\ln\frac{s}{2}\right) - \frac{1}{2}\cos\left(\sqrt{2}\ln\frac{s}{2}\right)\right),$$

$$U = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\sin\left(\sqrt{2}\ln\frac{s}{2}\right), \frac{1}{\sqrt{2}}\cos\left(\sqrt{2}\ln\frac{s}{2}\right)\right),$$

$$k_n = -\frac{1}{\sqrt{2}s},\ k_g = \frac{1}{s},\ \tau_g = -\frac{1}{\sqrt{2}s}.$$

Hence, from (3.2), we obtain $\sigma(s) = 1 = constant$. Therefore, α is a D_o-Darboux slant helix on M.

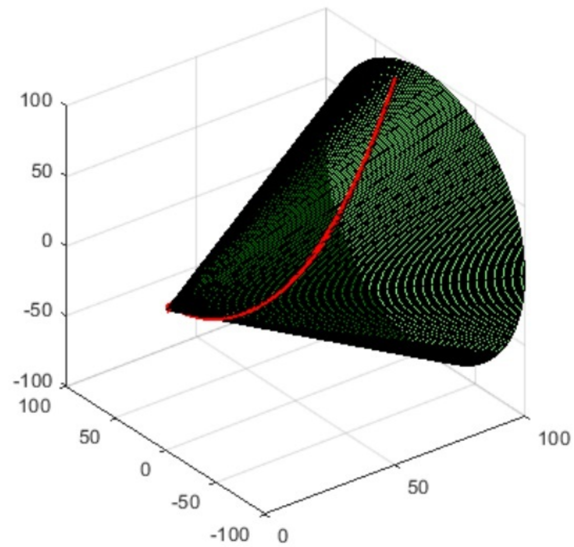

FIGURE 1. The curve α on the surface M

4. D_n-DARBOUX SLANT HELIX

In this section, we introduce the D_n-Darboux slant helices which a new kind of surface curve. Since theorems given in this section are proven similar to theorems in section 3, they will be given without proof.

Let M is a oriented surface in E^3 and $\beta : I \to M$ be a unit speed curve on the surface M. Let $\{T, V, U\}$ be the Darboux frame, k_g, k_n, τ_g be the curvatures and $D_n(s) = -k_n(s)V(s) + k_g(s)U(s)$ the normal Darboux vector of β. The concept of the D_n-Darboux slant helix on the surface is defined as follows.

Definition 4.1. Let M is an oriented surface in E^3, and $\beta : I \to M$ be a unit speed curve with on the surface M. Then, β is called D_n-Darboux slant helix if there exists a constant angle φ between the Darboux vector field D_n (or equivalently, unit Darboux vector field $\widetilde{D}_n = \frac{D_n}{\|D_n\|}$) and a fixed(constant) unit direction d_n , i.e., $\langle D_n, d_n \rangle = \cos\varphi$ is constant [11].

Theorem 4.2. *Let M be an oriented surface in E^3, and $\beta : I \to M$ be a unit speed curve on surface M with the ND-frame $\{\widetilde{D}_n, T, Y_n\}$. Then, β is a D_n-Darboux slant helix iff $\frac{\mu_n}{\delta_n}(s)$ is a contant function, where $\delta_n \neq 0$.*

Corollary 4.3. *Let M be an oriented surface in E^3, and $\beta : I \to M$ be a unit speed curve on the surface M. Then, β is a D_n-Darboux Slant helix iff*

$$\sigma_n(s) = \left(\frac{\sqrt{k_n^2 + k_g^2}}{\frac{k_g^2}{k_n^2+k_g^2} \left(\frac{k_n}{k_g}\right)' + \tau_g} \right)(s) \tag{4.1}$$

is a constant function.

Corollary 4.4. *Axis of the D_n-Darboux slant helix is given by*

$$d_n = \cos\varphi \widetilde{D}_n + \sin\varphi T.$$

Corollary 4.5. *Axis of the D_n-Darboux slant helix according to the Darboux frame $\{T, V, U\}$ is*

$$d_n = \sin\varphi T - \frac{k_n}{\sqrt{k_n^2 + k_g^2}} \cos\varphi V + \frac{k_g}{\sqrt{k_n^2 + k_g^2}} \cos\varphi U.$$

Corollary 4.6. *β is a D_n-Darboux slant helix if and only if β is a general helix.*

Corollary 4.7. *Let β be a unit speed D_n-Darboux slant helix on surface M. Then,*

i) β is a geodesic curve on the M with $k_n \neq 0$ iff β is a general helix with the axis $d_n = \sin\varphi T + \cos\varphi B$.

ii) β is an asymptotic curve on the M with $k_g \neq 0$ iff β is a general helix with the axis $d_n = \sin\varphi T + \cos\varphi B$.

iii) Let β be a line of curvature on the M. Then, $\frac{k_g^2}{(k_n^2+k_g^2)^{\frac{3}{2}}}\left(\frac{k_n}{k_g}\right)'$ is constant.

4.1. **The Position Vector of the D_n-Darboux Slant Helix on a Surface.** In this section, we introduce the methods for finding the D_n-Darboux slant helix on a given surface. We discuss the methods separately for parametric and implicit surfaces.

4.1.1. *The D_n-Darboux Slant Helix on a Parametric Surface.* Let M be a regular oriented surface in E^3 with given the parametrization $X = X(u, v)$. Let $\beta(s) = X(u(s), v(s))$ be a unit speed D_n-Darboux slant helix on M with axis d_n, constant angle φ, and Darboux frame $\{T, V, U\}$. For obtaining β, we find $u(s)$ and $v(s)$. In a way similar to those in chapter 4, we obtain

$$\begin{cases} \frac{du}{ds} = \frac{-2P\langle X_v,d_n\rangle\left(EG-F^2\right)(\|X_u\times X_v\|\cos\varphi-Q\langle X_u\times X_v,d_n\rangle)\pm\sqrt{\Delta_n}}{2A_nP^2\left(EG-F^2\right)}, \\ \frac{dv}{ds} = \frac{2P\langle X_u,d_n\rangle\left(EG-F^2\right)(\|X_u\times X_v\|\cos\varphi-Q\langle X_u\times X_v,d_n\rangle)\pm\sqrt{\Delta_n^*}}{2A_nP^2\left(EG-F^2\right)}, \end{cases} \tag{4.2}$$

where

$$\begin{aligned} A_n &= E\langle X_v, d_n\rangle^2 - 2F\langle X_u, d_n\rangle\langle X_v, d_n\rangle + G\langle X_u, d_n\rangle^2, \\ \Delta_n &= 4P^2(\|X_u\times X_v\|\cos\varphi - Q\langle X_u\times X_v, d_n\rangle)^2\left(EG-F^2\right)\left[\langle X_v, d_n\rangle^2\left(EG-F^2\right) - A_nG\right] \\ &\quad + 4A_n\left(EG-F^2\right)P^4\left[-F\langle X_v, d_n\rangle + G\langle X_u, d_n\rangle\right]^2, \\ \Delta_n^* &= 4P^2\left(EG-F^2\right)(\|X_u\times X_v\|\cos\varphi - Q\langle X_u\times X_v, d_n\rangle)^2\left[\langle X_u, d_n\rangle^2\left(EG-F^2\right) - A_nE\right] \\ &\quad + 4A_n\left(EG-F^2\right)P^4\left[-E\langle X_v, d_n\rangle + F\langle X_u, d_n\rangle\right]^2, \\ P &= \frac{k_n(s)}{\sqrt{k_n^2(s)+k_g^2(s)}},\ Q = \frac{k_g(s)}{\sqrt{k_n^2(s)+k_g^2(s)}}. \end{aligned}$$

If we solve the system (4.2) together with the initial point

$$\begin{cases} u(0) = u_0 \\ v(0) = v_0, \end{cases}$$

we obtain the desired D_n-Darboux slant helix on M by substituting $u(s), v(s)$ into $X(u, v)$.

4.1.2. *The D_n-Darboux Slant Helix on an Implicit Surface.* Let M be a surface given in implicit form by $f(x, y, z) = 0$.

Let us now find the D_n-Darboux slant helix $\beta(s)$ which makes the given constant angle with the given axis $d_n = (a, b, c)$ and lying on M.

Let $\beta(s) = (x(s), y(s), z(s))$ and $\{T, V, U\}$ be its Darboux frame field. We need to find $x(s), y(s), z(s)$ to obtain $\beta(s)$. In a way similar to those in chapter 4, we obtain

$$\begin{cases} \frac{dx}{ds} = \frac{1}{\Omega_n}\left[-\frac{1}{P}f_y\left(\|\nabla f\|\cos\varphi - Q\left(af_x+bf_y+cf_z\right)\right) + \left[f_y\left(-af_y+bf_x\right) - f_z\left(af_z-cf_x\right)\right]\frac{dz}{ds}\right], \\ \frac{dy}{ds} = \frac{1}{\Omega_n}\left[\frac{1}{P}f_x\left(\|\nabla f\|\cos\varphi - Q\left(af_x+bf_y+cf_z\right)\right) - \left[f_x\left(bf_x-af_y\right) - f_z\left(cf_y-bf_z\right)\right]\frac{dz}{ds}\right], \end{cases} \tag{4.3}$$

where $\Omega_n = f_x\left(af_z - cf_x\right) - f_y\left(cf_y - bf_z\right) \neq 0$. Substituting (4.3) into (3.13) gives us a quadratic equation with respect to $\frac{dz}{ds}$ as

$$q_1\left(\frac{dz}{ds}\right)^2 + q_2\left(\frac{dz}{ds}\right) + q_3 = 0,$$

where

$$q_1 = \frac{1}{\Omega_n^2}\left[\left(bf_x - af_y\right)^2\left(f_x^2 + f_y^2 + 2f_z^2\right) + f_z^2\left((af_z - cf_x)^2 + (cf_y - bf_z)^2\right)\right] + 1,$$
$$q_2 = -2\left(\|\nabla f\|\cos\varphi - Q\left(af_x + bf_y + cf_z\right)\right)\frac{1}{P}\frac{1}{\Omega_n^2}\left[\left(bf_x - af_y\right)\left(f_x^2 + f_y^2 + f_z^2\right)\right],$$
$$q_3 = \frac{1}{\Omega_n^2}\frac{1}{P^2}\left(\|\nabla f\|\cos\varphi - Q\left(af_x + bf_y + cf_z\right)\right)^2\left(f_x^2 + f_y^2\right) - 1,$$
$$P = \frac{k_n(s)}{\sqrt{k_n^2(s) + k_g^2(s)}},\ Q = \frac{k_g(s)}{\sqrt{k_n^2(s) + k_g^2(s)}}.$$

From this equation, we get

$$q_1\left(\frac{dz}{ds}\right)^2 + q_2\left(\frac{dz}{ds}\right) + q_3 = 0. \tag{4.4}$$

If (4.4) written in (4.3), we obtain a first-order differential equation system.Thus, together with the initial point

$$x(0) = x_0,$$
$$y(0) = y_0,$$
$$z(0) = z_0,$$

we have an initial value problem. The solution of this problem gives us the D_n-Darboux slant helix on M.

Example 4.8. Let us consider the surface M given by the parametrization

$$X(u,v) = \begin{pmatrix} \frac{u}{3\sqrt{2}}\cos\left(\sqrt{2}\ln u\right) + \frac{u}{3}\sin\left(\sqrt{2}\ln u\right) + \frac{v}{\sqrt{2}}\cos\left(\sqrt{2}\ln u\right), \\ \frac{u}{3\sqrt{2}}\sin\left(\sqrt{2}\ln u\right) - \frac{u}{3}\cos\left(\sqrt{2}\ln u\right) + \frac{v}{\sqrt{2}}\sin\left(\sqrt{2}\ln u\right), \\ \frac{u}{\sqrt{2}} + \frac{v}{\sqrt{2}} \end{pmatrix}$$

and curve $\beta : I \to M$ defined by the parametric form

$$\beta(s) = \left(\frac{s}{3\sqrt{2}}\cos\left(\sqrt{2}\ln s\right) + \frac{s}{3}\sin\left(\sqrt{2}\ln s\right), \frac{s}{3\sqrt{2}}\sin\left(\sqrt{2}\ln s\right) - \frac{s}{3}\cos\left(\sqrt{2}\ln s\right), \frac{s}{\sqrt{2}}\right),$$

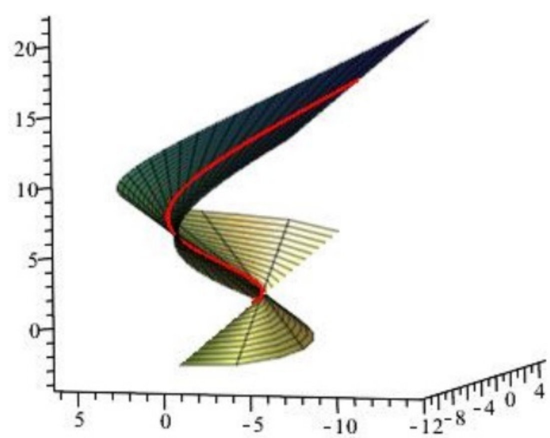

Figure 2. The curve β on the surface M

where $s > 0$ (Figure 2). The vectors of the Darboux frame and curvatures of β are computed as follows,

$$T = \frac{1}{\sqrt{2}}\left(\cos\left(\sqrt{2}\ln s\right), \sin\left(\sqrt{2}\ln s\right), 1\right),$$

$$V = \left(\sin\left(\sqrt{2}\ln s\right), -\cos\left(\sqrt{2}\ln s\right), 0\right),$$

$$U = \frac{1}{\sqrt{2}}\left(\cos\left(\sqrt{2}\ln s\right), \sin\left(\sqrt{2}\ln s\right), -1\right),$$

$$k_n = 0,\ k_g = -\frac{1}{s},\ \tau_g = \frac{1}{s}.$$

Hence from (4.1), we obtain $\sigma_n(s) = 1 = constant$. Therefore, β is a D_n-Darboux slant helix on M.

5. D_r-Darboux Slant Helix

In this section, we introduce D_r-Darboux slant helices which a new kind of surface curve . Since theorems given in this section are proven similar to theorems in section 3, they will be given without proof.

Let M is an oriented surface in E^3, and $\gamma : I \to M$ be a unit speed curve on surface M. Let be $\{T, V, U\}$ Darboux frame, k_g, k_n, τ_g are the curvatures and $D_r(s) = \tau_g(s)T(s) + k_g(s)U(s)$ rectifying Darboux vector of γ. The concept of D_r-Darboux slant helix on the surface is defined as follows.

Definition 5.1. Let M is an oriented surface in E^3, and $\gamma : I \to R \subset M$ be a unit speed curve with on surface M. Then, γ is called D_r-Darboux slant helix if there exists a constant angle ψ between the Darboux vector field D_r (or equivalently, unit Darboux vector field $\tilde{D}_r = \frac{D_r}{\|D_r\|}$) and a fixed(constant) unit direction d_r , i.e., $\langle D_r, d_r\rangle = \cos\psi$ is constant [11].

Theorem 5.2. *Let M be an oriented surface in E^3 and $\gamma : I \to M$ be a unit speed curve on surface M with RD-frame $\{\tilde{D}_r, V, Y_r\}$. Then, γ is a D_r-Darboux slant helix iff $\frac{\mu_r}{\delta_r}(s)$ is a constant function, where $\delta_r \neq 0$.*

Corollary 5.3. *Let M be an oriented surface in E^3, and $\gamma : I \to M$ be a unit speed curve on surface M. Then, γ is a D_r-Darboux Slant helix iff*

$$\sigma_r(s) = \left(\frac{\sqrt{k_g^2 + \tau_g^2}}{\frac{k_g^2}{k_g^2+\tau_g^2}\left(\frac{\tau_g}{k_g}\right)' - k_n}\right)(s) \tag{5.1}$$

is a constant function.

Corollary 5.4. *Axis of D_r-Darboux slant helix is given by*

$$d_r = \cos\psi \tilde{D}_r + \sin\psi V.$$

Corollary 5.5. *Axis of D_r-Darboux slant helix according to $\{T, V, U\}$ Darboux frame is*

$$d_r = \cos\psi\frac{\tau_g}{\sqrt{\tau_g^2 + k_g^2}}T + \sin\psi V + \cos\psi\frac{k_g}{\sqrt{\tau_g^2 + k_g^2}}U.$$

Corollary 5.6. *γ is a D_r-Darboux slant helix if and only if γ is a relatively normal-slant helix.*

Corollary 5.7. *Let γ be a unit speed D_r-Darboux slant helix on surface M. Then,*

i) γ is a geodesic curve on M with $k_n \neq 0$ iff γ is a general helix with the axis $d_r = \cos\psi T \pm \sin\psi B$.
ii) γ is an asymptotic curve on M with $k_g \neq 0$ iff γ is a Darboux helix with the axis

$$d_r = \cos\psi\frac{\tau}{\sqrt{\tau^2 + \kappa^2}}T \mp \sin\psi N + \cos\psi\frac{\kappa}{\sqrt{\tau^2 + \kappa^2}}B.$$

.

iii) Let γ be a line of curvature on M . Then, γ is a planar curve.

5.1. **The Position Vector of the D_r-Darboux Slant Helix on a Surface.** In this section, we introduce the methods for finding the D_r-Darboux Slant helix on a given surface. We discuss the methods separately for parametric and implicit surfaces.

5.1.1. *The D_r-Darboux Slant Helix on a Parametric Surface.* Let M be a regular oriented surface in E^3 with given the parametrization $X = X(u,v)$. Let $\gamma(s) = X(u(s), v(s))$ be a unit speed D_r-Darboux slant helix on M with axis d_r, constant angle ψ, and Darboux frame $\{T, V, U\}$. For obtaining γ, we find $u(s)$ and $v(s)$. In a way similar to those in chapter 4, we obtain

$$\begin{cases} \frac{du}{ds} = \frac{-2S\,\|X_u\times X_v\|(\|X_u\times X_v\|\cos\psi - R\langle X_u\times X_v, d_r\rangle)(F\langle X_v,d_r\rangle - G\langle X_u,d_r\rangle)\pm\sqrt{\Delta_r}}{2A_r S^2\|X_u\times X_v\|^2}, \\ \frac{dv}{ds} = \frac{-2S\,\|X_u\times X_v\|(\|X_u\times X_v\|\cos\psi - R\langle X_u\times X_v, d_r\rangle)(F\langle X_u,d_r\rangle - E\langle X_v,d_r\rangle)\pm\sqrt{\Delta_r^*}}{2A_r S^2\|X_u\times X_v\|^2}, \end{cases} \tag{5.2}$$

where

$$\begin{aligned} A_r &= E\langle X_v, d_r\rangle^2 - 2F\langle X_u, d_r\rangle\langle X_v, d_r\rangle + G\langle X_u, d_r\rangle^2, \\ \Delta_r &= 4S^2\|X_u\times X_v\|^2(\|X_u\times X_v\|\cos\psi - R\langle X_u\times X_v, d_r\rangle)^2\left[(F\langle X_v, d_r\rangle - G\langle X_u, d_r\rangle)^2 - A_r G\right] \\ &\quad + 4S^4\|X_u\times X_v\|^4\langle X_v, d_r\rangle^2 A_r \\ \Delta_r^* &= 4S^2\|X_u\times X_v\|^2(\|X_u\times X_v\|\cos\psi - R\langle X_u\times X_v, d_r\rangle)^2\left[(F\langle X_u, d_r\rangle - E\langle X_v, d_r\rangle)^2 - A_r E\right] \\ &\quad + 4S^4\|X_u\times X_v\|^4\langle X_u, d_r\rangle^2 A_r, \\ S &= \frac{\tau_g}{\sqrt{k_g^2+\tau_g^2}},\ \ R = \frac{k_g}{\sqrt{k_g^2+\tau_g^2}}. \end{aligned}$$

If we solve the system (5.2) together with the initial point

$$\begin{cases} u(0) = u_0 \\ v(0) = v_0, \end{cases}$$

we obtain the desired D_r-Darboux slant helix on M by substituting $u(s), v(s)$ into $X(u,v)$.

5.1.2. *The D_r-Darboux Slant Helix on an Implicit Surface.* Let M be a surface given in implicit form by $f(x,y,z) = 0$.

Let us now find the D_r-Darboux slant helix $\gamma(s)$ which makes the given constant angle with the given axis $d_r = (a,b,c)$ and lying on M.

Let $\gamma(s) = (x(s), y(s), z(s))$ and $\{T, V, U\}$ be its Darboux frame field. We need to find $x(s), y(s), z(s)$ to obtain $\gamma(s)$. In a way similar to those in chapter 4, we obtain

$$\begin{cases} \frac{dx}{ds} = \frac{1}{\Omega_r}\left[\frac{1}{S\|\nabla f\|} f_y\left(\|\nabla f\|\cos\psi - R\left(af_x + bf_y + cf_z\right)\right) + \left(bf_z - cf_y\right)\frac{dz}{ds}\right], \\ \frac{dy}{ds} = \frac{1}{\Omega_r}\left[(cf_x - af_z)\frac{dz}{ds} - \frac{1}{S\|\nabla f\|} f_x\left(\|\nabla f\|\cos\psi - R\left(af_x + bf_y + cf_z\right)\right)\right], \end{cases} \tag{5.3}$$

where $\Omega_r = af_y - bf_x \neq 0$. Substituting (5.3) into (3.13) give us a quadratic equation with respect to $\frac{dz}{ds}$ as

$$q_1\left(\frac{dz}{ds}\right)^2 + q_2\left(\frac{dz}{ds}\right) + q_3 = 0,$$

where

$$\begin{aligned} q_1 &= \frac{1}{\Omega_r^2}\left[\ (a^2+b^2)f_z^2 + c^2\left(f_x^2 + f_y^2\right) - 2cf_z(af_x + bf_y)\ \right] + 1, \\ q_2 &= 2\frac{1}{S\,\|\nabla f\|}\left(\|\nabla f\|\cos\psi - Q\left(af_x + bf_y + cf_z\right)\right)\frac{1}{\Omega_r^2}\left[f_z(af_x + bf_y) - c\left(f_x^2 + f_y^2\right)\right], \\ q_3 &= \frac{1}{\Omega_r^2}\frac{1}{S^2\|\nabla f\|^2}\left(\|\nabla f\|\cos\psi - Q\left(af_x + bf_y + cf_z\right)\right)^2\left(f_x^2 + f_y^2\right) - 1. \end{aligned}$$

From this equation, we get

$$q_1\left(\frac{dz}{ds}\right)^2 + q_2\left(\frac{dz}{ds}\right) + q_3 = 0. \tag{5.4}$$

If (5.4) is written in (5.3) , we obtain a first-order differential equation system. Thus, together with the initial point

$$x(0) = x_0,$$
$$y(0) = y_0,$$
$$z(0) = z_0$$

we have an initial value problem. The solution of this problem gives us the D_r-Darboux slant helix on the M.

Example 5.8. Let us consider the surface M given by the parametrization $X(u, v) = (v, \sin u, \cos u)$ and curve $\gamma : I \to M$ defined by the parametric form $\gamma(s) = \left(\frac{s}{\sqrt{2}}, \sin\frac{s}{\sqrt{2}}, \cos\frac{s}{\sqrt{2}}\right)$ (Figure 3). The vectors of the Darboux frame and curvatures of γ are computed as follows,

$$T = \frac{1}{\sqrt{2}}\left(1, \cos\frac{s}{\sqrt{2}}, -\sin\frac{s}{\sqrt{2}}\right),$$
$$V = \frac{1}{\sqrt{2}}\left(1, -\cos\frac{s}{\sqrt{2}}, \sin\frac{s}{\sqrt{2}}\right),$$
$$U = \left(0, -\sin\frac{s}{\sqrt{2}}, -\cos\frac{s}{\sqrt{2}}\right),$$

$$k_n = \frac{1}{2},\ k_g = 0,\ \tau_g = -\frac{1}{2}.$$

Hence, from (5.1), we obtain $\sigma_r(s) = -1 = constant$. Therefore, γ is a D_r-Darboux slant helix on M.

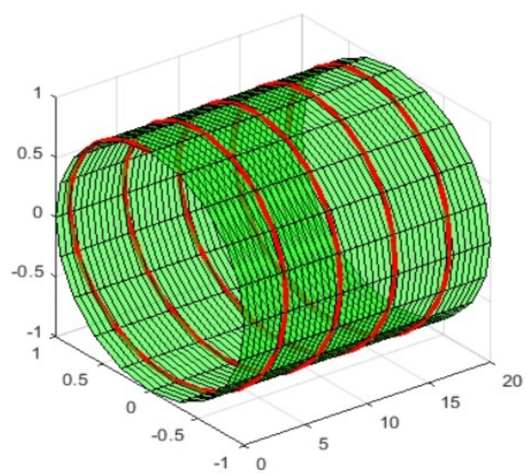

Figure 3. The curve γ on the surface M

6. Conclusion

In this study, some new types of surface curves called D_i- slant helices have been defined on a surface. The investigation of these special curves has been carried out by considering newly defined orthonormal frames for curves lying on a surface. The definitions made here and the characterizations obtained as a result can be applied to the surface curves in other spaces such as Minkowski space. Additionally, this work is important in terms of defining and studying different types of curves on a surface.

Acknowledgement

This article was prepared from Akın ALKAN's PhD thesis conducted at Manisa Celal Bayar University, Institute of Natural and Applied Sciences.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

Authors Contribution Statement

All authors jointly worked on the results and they have read and agreed to the published version of the manuscript.

References

[1] Ali, A.T., *Position vectors of slant helices in Euclidean space E^3*, J. Egyptian Math. Soc., **20**(2012) 1–6.
[2] Alkan, A., Kocayiğit, H., Ağırman Aydın, T., *New moving frames for the curves lying on a surface*, Sigma Journal of Engineering and Natural Sciences, **42**(4)(2024), 2023–2029.
[3] Babaarslan, M., Tandogan, Y.A., Yayli, Y., *A note on Bertrand curves and constant slope surfaces according to Darboux frame*, Journal of Advanced Mathematical Studies, **5**(1)(2012), 87–97.
[4] Babaarslan, M., Yayli, Y., *On helices and Bertrand curves in Euclidean 3-space*, Mathematical and Computational Applications, **18**(1)(2013), 1–11.
[5] Barros, M., *General helices and a theorem of Lancret*, Proc Amer Math Soc., **125**(5)(1997), 1503–1509.
[6] Doğan, F., Yaylı Y., *On isophote curves and their characterizations*, Turk J. Math., **39**(2015), 650–664.
[7] Hananoi, S., Ito, N., Izumiya, S., *Spherical Darboux images of curves on surfaces*, Beitr Algebra Geom., **56**(2015), 575–585.
[8] Izumiya, S., Takeuchi, N., *New special curves and developable surfaces*, Turk J. Math., **28**(2004), 153–163.
[9] Kula, L., Yaylı, Y., *On slant helix and its spherical indicatrix*, Applied Mathematics and Computation, **169**(1)(2005), 600–607.
[10] Macit, N., Düldül, M., *Relatively normal-slant helices lying on a surface and their characterizations*, Hacettepe Journal of Mathematics and Statistics, **46**(3)(2017), 397–408.
[11] Önder, M., *Helices associated to helical curves, relatively normal-slant helices and isophote curves*, arXiv preprint arXiv:2201.09684., (2022).
[12] Puig-Pey, J., Gálvez, A., Iglesias, A., *Helical curves on surfaces for computer-aided geometric design and manufacturing*, in: Computational Science and Its Applications-ICCSA Part II, 771-778. in: Lecture Notes in Comput Sci Vol. 3044, Springer, Berlin, (2004).
[13] Zıplar, E., Şenol, A., Yaylı, Y., *On Darboux helices in Euclidean 3-space*, Global Journal of Science Frontier Research Mathematics and Decision Sciences, **12**(3)(2012), 73–80.