LOCAL ENTROPY FUNCTION OF DYNAMICAL SYSTEM

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Abstract

In this work, we first,define the entropy function of the topological dynamical system and investigate basic properties of this function without going into details. Let (X,**A,**T) be a probability measure space and consider **P** = { $p_1, p_2, \ldots, p_n$} a finite measurable partition of all sub-sets of topological dynamical system (X,T).Then,the quantity H_μ (**P**) = $\sum_{i=1}^{n} z(\mu(p_i))$ is called the entropy function of finite measurable partition **P**.Where

$$z(t)=\begin{cases} -t\log t & if \quad 0<t\leq 1 \\ 0 & if \quad t=0 \end{cases}$$

is a non-negative,continuous and strictly concave function.In this paper,all logarithms will be taken to be the natural base ″e ″.

After that,we give the definition of the local entropy function of the topological dynamical system. Let **P** be a measurable partition of topological dynamical system (X,T) with H_μ(**P**)<∞, and ε>0.If diam(**P**) < ε,then the quantity L_μ (T) = h_μ (T) - h_μ (T,**P**) is called a local entropy function of topological dynamical system (X,T) . In conclusion, Let (X,T) and (Y,S) be two topological dynamical system. If T×S is a transformation defined on the product space (X×Y,T×S) with (T×S)(x , y) = (Tx,Sy) for all (x,y) X × Y.Then $L_{\mu\times\mu_1}$ (T×S) = $L_{\mu,D}$(T) + L_{μ_1} (S) .and, we prove some fundamental properties of this function.

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Key Words:*Probability measure space, probability measure- preserving transformation, topological dynamical system, partition, generator, factor,ergodic measure,entropy function, local entropy function.*

1.INTRODUCTION

Several searchers examined some results relating to the entropy function of the topological dynamical system .For more properties of this function, see, [2] [3] [4] [5]

In this work,we shall define the local entropy function of the topological dynamical system starting from these studies and prove some results of this function.

The purpose of this paper is to prove some fundamental properties of the local entropy function of the topological dynamical system and prove some important properties of this function.

2. ENTROPY FUNCTION

Let (X,**A**,μ ,T) be a topological dynamical system.
Where X is a compact metric space,**A** is a Borel σ-algebra defined on X,μ is Radon probability measure defined on the topological measurable space (X,**A**) and T : X → X is a Radon probability measure- preserving homeomorphism.One writes briefly (X,T) instead of (X,**A**,μ,T).Let M(X,T) be the set of all T –invariant probability measure defined on the topological measurable space (X,**A**) with the weak*- toplogy.For details,see,[6]

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[3]Denker M.,Grıllenberger C. and SIGMUND K.,Ergodic theory and compact spaces,Lecture Notes in Math.,No:527, Springer-Verlag, Berlin,1976.
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Definition II.1. Let **(**X,**A,**T) be a probability measure space and consider **P** = { p_1,p_2,…,p_n} a finite measurable partition of all sub-sets of topological dynamical system (X,T).Then,the quantity H_μ **(P)** = $\sum_{i=1}^{n} z(\mu(p_i))$ is called the entropy function of finite measurable partition **P**.Where

$$z(t)=\begin{cases} -t\log t & if \quad 0<t\leq 1 \\ 0 & if \quad t=0 \end{cases}$$

is a non-negative,continuous and strictly concave function.In this paper,all logarithms will be taken to be the natural base ″e ″.

The entropy function H_μ **(P)** is a positive,monotone and sub-additive function.For more properties of this function, see, [6] and [7]

L**emma II.2.**Let $(a_n)_{n\geq 1}$ be a sequence of real numbers such which is positive and sub-additive .Then $\lim\limits_{n\to\infty} \frac{1}{n} a_n$ exists and is equal to $\inf\limits_{n\in N} \frac{1}{n} a_n$.

Proof. See,Lemma II.3 of [7]

Theorem II.3. If **P** is a finite measurable partition of topological dynamical system
(X,T) with H_μ**(P)** ≤ ∞ , then, $\lim\limits_{n\to\infty}\frac{1}{n}H_\mu\left(\bigvee_{i=0}^{n-1} T^{-i}P\right)$ exists and is equal to the infimum.**Proof.** See,Proposition II.4 of [8]

Definition II.4. i) Let **P** be a finite measurable partition of topological dynamical system

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(X,T) with $H_\mu(\mathbf{P}) < \infty$. Then,the limit function $h_\mu(T,\mathbf{P}) = \lim_{n\to\infty}\frac{1}{n} H_\mu\left(\overset{n-1}{\underset{i=0}{V}} T^{-i}P\right)$ is called the entropy function of T with respect to the finite measurable partition **P**.

i) The quatity

$h_\mu(T) = \sup\{ h_\mu(T,\mathbf{P}) : \mathbf{P}$ is a finite measurable partition of (X,T) with $H_\mu(\mathbf{P})<\infty \}$ is called the entropy function of topological dynamicl system (X,T).Where the supremum is taken over all measurable partitions of topological dynamical system (X,T) with the finite entropy function.

Proposition II.5. Let **P** and **Q** be two finite measurable partitions of topological dynamical system (X,T) with $H_\mu(\mathbf{P})\leq\infty$ and $H_\mu(\mathbf{Q})\leq\infty$.Then,

i) $h_\mu(T,\mathbf{P})\geq 0$.
ii) $h_\mu(T,\mathbf{P}) \leq H_\mu(\mathbf{P})$.
iii) If $\mathbf{P} \subset \mathbf{Q}$,then $h_\mu(T,\mathbf{P}) \leq h_\mu(T,\mathbf{Q})$.
ıv) $h_\mu(T,\mathbf{P} \vee \mathbf{Q}) \leq h_\mu(T,\mathbf{P}) + h_\mu(T,\mathbf{Q})$.
v) If T is an automorphism, then $h_\mu(T,\mathbf{P}) = h_\mu(T^{-1},\mathbf{P})$.
vı) $h_\mu(T,\mathbf{P}) = \lim_{n\to\infty}\frac{1}{n} H_\mu\left(\overset{n-1}{\underset{i=0}{V}} T^{-i}P\right)$.

vı) Let $(\mathbf{P}_n)_{n \in N}$ be a sequence of partitions of the topological dynamical system (X,T) with $H_\mu(\mathbf{P}_n)\leq\infty$ all $n \in N$ .If $\mathbf{P}_n \to \mathbf{P}$ for $n \to \infty$, then, $h_\mu(T, \mathbf{P}) = \lim_{n\to\infty} h_\mu(T, \mathbf{P}_n)$.

Proof. See, [9],[10] and [11]

Lemma II.6. If **P** is a finite measurable partition of (X,T) with $H_\mu(\mathbf{P})<\infty$, then,the function $h_\mu(T, \mathbf{P})$ is a continuous function.

Proof. See, [12]and [13]

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[10]Rokhlın V.,A., Lectures on the entropy theory of measure preserving- transformations,Russian Math. Surveys ,22,(5),(1967),1-52

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[12] Denker M.,Grıllenberger C. and Sıgmund K.,
Ergodic theory and compact spaces,Lecture Notes in Math.,No:527, Springer-Verlag, Berlin,1976.

Definition II.7. Let **P** be a finite measurable partition of (X,T) with $H_\mu(\mathbf{P})<\infty$.

If $\overset{+\infty}{\underset{i=-\infty}{V}} T^i$ **P** ≡ **A** ,then the partition **P** is called a generator of the σ-algebra **A** for T.

Theorem II.8. (Kolmogorov-Sinai).Let **P** be a generator of (X,T) with $H_\mu(\mathbf{P})<\infty$.
Then, $h_\mu (T) = h_\mu (T, \mathbf{P})$.
Proof. See, [9]and [11]

Definition II.9. Let (X,T) and (Y,S) be two topological dynamical systems.We say that(Y,S) is a factor of (X,T), if there exist A∈**A** and B∈ **B** such that

i) $\mu(A) = 1$ and $\mu_1(B) = 1$.

ii) There exists a measure- preserving transformation $\varphi : A \to B$ with $\varphi(T(x)) = S(\varphi(x))$
for all $x \in X$.

Proposition II.10. Let **P** be a finite measurable partition of (X,T) with $H_\mu(\mathbf{P})<\infty$.Then,

i) $h_\mu(T) \geq 0$.
ii) For $k > 0$, $h_\mu(T^k,\mathbf{P}) = k \, . \, h_\mu(T,\mathbf{P})$.
iii) For $k > 0$, $h_\mu(T^k) = k \, . \, h_\mu(T,)$.

ıv) If T is an invertible transformation and $k \in Z$, $h_\mu(T^k,\mathbf{P}) = |k| \, . \, h_\mu(T,\mathbf{P})$.
v) If T is an invertible transformation and $k \in Z$, $h_\mu(T^k) = |k| \, . \, h_\mu(T)$.
vı) If **P** is a finite measurable partition of (X,T) with $H_\mu(\mathbf{P})<\infty$ and $\overset{+\infty}{\underset{i=0}{V}} T^i$ **P** ≡ **A,**then $h_\mu (T) = 0$.[14]

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[13] Rokhlın V.,A., Lectures on the entropy theory of measure preserving- transformations,Russian Math. Surveys ,22,(5),(1967),1-52

vii) If (Y,S) is a factor of (X,T),then h_{μ_1} (S,**P**) ≤ h_μ (T,**P**) and h_{μ_1} (S) ≤ h_μ (T). **viii**) Let (X,T) and (Y,S) be two topological dynamical systems .Then, $h_{\mu x \mu_1}$ (T×S) = h_μ (T) + h_{μ_1} (S) (2-28).Where T×S is a transformation defined on the product space (X×Y,T×S) with (T×S)(x , y) = (Tx,Sy) for all (x,y) ∈ X × Y. **ix**) If $(\mathbf{P}_n)_{n \geq 1}$ is a generating sequence of measurable partitions of topologicale dynamical system (X,T), then,h_μ(T) = $\lim_{n\to\infty}$ h_μ (T,$\mathbf{P}_n$).

Proof. See, [15] and [16]

Definition II.11. Let **P** be a finite measurable partition of (X,T) with $H_\mu(\mathbf{P})<\infty$.If the measure of boundary of **P** is null (i.e. $\mu(\partial \mathbf{P}) = 0$) , then the partition **P** is called μ-continuous.

Proposition II.12. Let $\mu,\mu_n \in$ M (X,T).Then the following statements are equivalent.

i) If f is a continuous funtion on X, then $\lim_{n\to\infty} \int f.d\mu_n = \int f.d\mu$.

ii) If K is a closed subset of X , then $\limsup_{n\to\infty} \mu_n(K) \leq \mu(K)$.

iii) If A is a open subset of X , then $\liminf_{n\to\infty} \mu_n(A) \geq \mu(A)$.

iv) If C is a μ-continuous measurable subset of X , then $\lim_{n\to\infty} \mu_n(C) = \mu(C)$.

v) The sequence $(\mu_n)_{n\in N}$ of measures converges weakly to the measure μ.

Proof. See, [17].

Lemma II.13. Let $(\mathbf{A}_i)_{1 \leq i \leq s}$ be a sequence of open covers of (X,T).There exists a measurable partition **P** = { $p_1,p_2,\ldots,p_n$} of (X,T) such that $p_i \subset \mathbf{A}_i$, for i = 1,2,…,s.

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[17] Denker M.,Grıllenberger C. and SIGMUND K.,Ergodic theory and compact spaces,Lecture Notes in Math.,No:527, Springer-Verlag, Berlin,1976.

Proof. See, [17]

Proposition II.14. The function $h_{\circ}(T) : M(X,T) \rightarrow R^{+}$, $\mu \rightarrow h_{\mu}(T)$ is an upper semicontinuous function.

Proof. See, [17]and [18]

Proposition II.15. Let **P** be a measurable partition of topological dynamical system (X,T) with H_{μ} (**P**)<∞ and E(X,T) be a set of ergodic measures of M (X,T) .

If $\mu = \int_{E(X,T)} m.d\delta(m)$ is an ergodic decomposition,then,

i) $h_{\mu}(T, \mathbf{P}) = \int_{E(X,T)} h_m(T,\mathbf{P})\, d\delta(m)$

ii) $h_{\mu}(T) = \int_{E(X,T)} h_m(T)\, d\delta(m)$

Proof. See, [19].

3. LOCAL ENTROPY FUNCTION

Definition III.1. Let **P** be a measurable partition of topological dynamical system (X,T) with $H_{\mu}(\mathbf{P})<\infty$, and ε>0.If diam(**P**) < ε,then the quantity $L_{\mu}(T) = h_{\mu}(T) - h_{\mu}(T,\mathbf{P})$ is called a local entropy function of topological dynamical system (X,T)

Proposition III.2.i) $L_{\mu}(T) \geq 0$.

ii) For k > 0 , $L_{\mu}(T^k) = k \cdot L_{\mu}(T)$.

iii) If T is an invertible transformation and $k \in Z$, $L_{\mu}(T^k) = |k| \cdot L_{\mu}(T)$.

Proof. i) As we write from the Definition II.4 (ii), Proposition II.5 (i) and Proposition II-10 (i) ;

$0 \leq h_{\mu}(T, \mathbf{P}) \leq h_{\mu}(T)$ (III-1).

Hence,we obtain the result from the Definition III.1 ;

$L_{\mu}(T) \geq 0$ (III-2).

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[18] Walters P.,An introduction to ergodic theory,Springer-Verlag,New York,1982.

[19] Walters P.,An introduction to ergodic theory,Springer-Verlag,New York,1982.

i) Let **P** be a measurable partition of topological dynamical system (X,T) with $H_\mu(\mathbf{P})<\infty$. Therefore,we have the following equalities from the Proposition II-10 (ii)
and (iii) ; for k > 0, $h_\mu(T^k,\mathbf{P}) = k \,.\, h_\mu(T,\mathbf{P})$ (III-3) and
for k > 0, $h_\mu(T^k) = k \,.\, h_\mu(T,)$ (III-4).

Thus, we can also write the following inequality;

for k > 0, $h_\mu(T^k) - h_\mu(T^k,\mathbf{P}) = k \,.\, h_\mu(T) - k \,.\, h_\mu(T,\mathbf{P}) = k \,.\, (h_\mu(T) - h_\mu(T,\mathbf{P}))$ (III-5).
Hence, the result follows from the Definition III.1 ;

for k > 0 , $L_\mu (T^k) = k \,.\, L_\mu (T)$ (III-6).

iii) Let **P** be a measurable partition of topological dynamical system (X,T) with $H_\mu(\mathbf{P})<\infty$.If T is an invertible transformation and $k \in Z$, then we write the following equalities from the Proposition II-10 (ıv) and (v) ;
$h_\mu(T^k,\mathbf{P}) = |k| \,.\, h_\mu(T,\mathbf{P})$ (III-7) and $h_\mu(T^k) = |k| \,.\, h_\mu(T)$ (III-8).
Therefore, we have;
$h_\mu(T^k) - h_\mu(T^k,\mathbf{P}) = |k| \,.\, h_\mu(T) - |k| \,.\, h_\mu(T,\mathbf{P}) = |k| \,.\, (h_\mu(T) - h_\mu(T,\mathbf{P}))$ (III-9).
Hence, we obtain the result from the Definition III.1 ;
$L_\mu(T^k) = |k| \,.\, L_\mu(T)$ (III-10).

Corollary III.3. If the partition **P** is a measurable generator of the topological dynamical system (X,T) with $H_\mu(\mathbf{P})<\infty$, then $L_\mu (T) = 0$.

Proof. If the partition **P** is a measurable generator of the topological dynamical system (X,T) with $H_\mu(\mathbf{P})<\infty$, then we have the following equality from the Theorem II.8;
$h_\mu (T) = h_\mu (T, \mathbf{P})$ (III-11).
Thus, the result follows from the Definition III.1 ;
$L_\mu (T) = h_\mu (T) - h_\mu (T,\mathbf{P}) = 0$ (III-12).

Proposition III.4. I f the topological dynamical system (Y,S) is a factor of (X,T) and
$\mu \in$ M (X,T),then

$$L_{\mu_1}\text{(S)} \leq L_{\mu}\text{(T)} + h_{\mu}(\text{T},\mathbf{P}) - h_{\mu_1}(\text{S},\mathbf{Q}) .$$

Proof. We write the following inequality from the Proposition II.10 (vii);

h_{μ_1} (S) ≤ h_{μ}(T) (III-13).

Let **P** be a finite measurable partition of topological dynamical system (X,T) with H_{μ}(**P**)<∞.As h(T,**P**) ≥ 0 from the Proposition II-5 (i),we have the following inequality;

h_{μ_1} (S) - h_{μ} (T,**P**) ≤ h_{μ} (T) - h_{μ} (T,**P**) (III-14).

Therefore,we obtain the following inequality from the Definition III.1;

h_{μ_1} (S) ≤ L_{μ} (T) - h_{μ} (T,**P**) (III-15).

Let **Q** be a finite measurable partition of topological dynamical system (Y,S) with H_{μ_1}(**Q**) <∞.As h_{μ_1} (S,**Q**) ≥ 0 from the Proposition II-5 (i),we can also write the following inequality;

h_{μ_1} (S) - h_{μ_1} (S,**Q**) ≤ L_{μ}(T) + h_{μ}(T,**P**) - h_{μ_1} (S,**Q**) (III-16).

Hence,we obtain the result from the Definition III.1;

L_{μ_1} (S) ≤ L_{μ}(T) + h_{μ} (T,**P**) - h_{μ_1} (S,**Q**) (III-17) **.**

Proposition III.5. Let (X,T) and (Y,S) be two topological dynamical system.If T×S is a transformation defined on the product space (X×Y,T×S) with (T×S)(x , y) = (Tx,Sy) for all (x,y) X × Y.Then $L_{\mu\times\mu_1}$ (T×S) = $L_{\mu,D}$(T) + L_{μ_1}(S)

Proof. Let $(\mathbf{P}_n)_{n \geq 1}$ (resp. $(\mathbf{P'}_n)_{n \geq 1}$) be an increasing sequence of partition of the topological dynamical system (X,T) with $H_{\mu}(\mathbf{P}_n)$<∞ for all n ∈ N (resp. topological dynamical system (Y,S) with $H_{\mu_1}(\mathbf{P'}_n)$<∞ for all n ∈ N) which generates **A** (resp.**B**).Each $\mathbf{P}_n$ induces a partition $\mathbf{Q}_n$ of the product space (X×Y,T×S).The atoms of $\mathbf{Q}_n$ being of the form **A**×Y.Where **A** runs through the atoms of $\mathbf{P}_n$.Similarly $\mathbf{P'}_n$ induces a partition $\mathbf{Q'}_n$ of the product space (X×Y,T×S).It is easy $\mathbf{U}_n = \mathbf{Q}_n \times \mathbf{Q'}_n$ is an increasing sequence of the product space (X×Y,T×S) which generates **A** × **B**.Since $\mathbf{Q}_n$ and $\mathbf{Q'}_n$ are independent, one has;

$$H_{\mu\times\mu_1}\left(\overset{k-1}{\underset{i=0}{V}}(T\times S)^{-i}U_n\right)=H_{\mu_1}\left(\overset{k-1}{\underset{i=0}{V}}(T\times S)^{-i}Q_n\right)+H_{\mu_1}\left(\overset{k-1}{\underset{i=0}{V}}(T\times S)^{-i}Q'_n\right)$$

(III-18).

But clearly; $H_{\mu\times\mu_1}\left(\overset{k-1}{\underset{i=0}{V}}(T\times S)^{-i}Q_n\right)= H_{\mu}\left(\overset{k-1}{\underset{i=0}{V}}(T\times S)^{-i}P_n\right)$ (III-19)

and

$$H_{\mu\times\mu_1}\left(\overset{k-1}{\underset{i=0}{V}}(T\times S)^{-i}Q'_n\right)= H_{\mu_1}\left(\overset{k-1}{\underset{i=0}{V}}(T\times S)^{-i}P'_n\right) \quad \text{(III-20)}.$$

Thus we obtain;

$$H_{\mu\times\mu_1}\left(\overset{k-1}{\underset{i=0}{V}}(T\times S)^{-i}U_n\right) = H_{\mu}\left(\overset{k-1}{\underset{i=0}{V}}(T\times S)^{-i}P_n\right) + H_{\mu_1}\left(\overset{k-1}{\underset{i=0}{V}}(T\times S)^{-i}P'_n\right) \quad \text{(III-21)}.$$

Therefore,dividing tha last equality III-21 by k > 0 and taking the limit for $k \to \infty$,
One obtains the following equality from the Theorem II.3;

$h_{\mu\times\mu_1}$ (T×S,$\mathbf{U}_n$) = h_μ (T,$\mathbf{P_n}$)+ h_{μ_1} (S, $\mathbf{P'}_n$) (III-22).

Taking the limit for $k \to \infty$, One has therefore the result from the Proposition II.10 (vııı);

$h_{\mu\times\mu_1}$ (T×S) = h_μ (T)+ h_{μ_1} (S) (III-23).

We can also write the following equality from the Definition III.1;

$L_{\mu\times\mu_1}$ (T×S) = $h_{\mu\times\mu_1}$ (T×S) - $h_{\mu\times\mu_1}$ (T×S,$\mathbf{U}_n$) (III-24).

Therefore,we obtain from the equalities (III-22) and (III-23);

$L_{\mu\times\mu_1}$ (T×S) = h_μ (T)+ h_{μ_1} (S) - h_μ (T,$\mathbf{P_n}$)- h_{μ_1} (S, $\mathbf{P'}_n$) (III-25) and also

$L_{\mu\times\mu_1}$ (T×S) = [h_μ (T) - h_μ (T,$\mathbf{P_n}$)] + [h_{μ_1} (S) - h_{μ_1} (S, $\mathbf{P'}_n$)] (III-26).

Hence, the result follows from Proposition II.10 (ıx) and the Definition III.1;

$L_{\mu\times\mu_1}$ (T×S) = $L_{\mu,D}$(T) + L_{μ_1} (S) (III-27).

Theorem III.6. The function $L_\circ$(T) : M (X,T) → R^+ , μ → L_μ(T) is an upper semicontinuous function.

Proof. Let μ_n ∈ M (X,T) and ε > 0. If the measure μ is a weak limit of the sequence $(\mu_n)_{n\in N}$, there exists a generating measurable partition **P** = {p_l,p_2,…,p_n}such that for every i = 1,2,…,n ,p_i is μ-continuous.Therefore, we obtain from the Proposition II.12;

$\lim_{n\to\infty} \mu_n(p) = \mu(p)$ (III-28) for k ∈ N and for p∈ $\overset{n-1}{\underset{i=0}{V}} T^{-i}P$.

Thus,one has the following inequality from The Proposition II.14;

h_μ (T) ≥ $h_{\mu_n}(T)$ - 2ε (III-29) and also h_μ (T,**P**) ≥ 0 from The Proposition II.5 (i),we have

h_μ (T) - h_μ (T,**P**) ≥ $h_{\mu_n}(T)$ - h_μ (T,**P**) - 2ε (III-30).

Therefore, we obtain the following inequality from the Definition III.1;

L_μ (T) ≥ $h_{\mu_n}(T)$ - h_μ (T,**P**) - 2ε (III-31).But as $h_{\mu_n}(T,P) \geq 0$ from The Proposition II.5 (i) and the Definition III.1,we write also the following inequalies ;

L_μ(T) - $h_{\mu_n}(T,P)$ ≥ $h_{\mu_n}(T)$ - $h_{\mu_n}(T,P)$ - h_μ(T,**P**) - 2ε (III-32) and

L_μ(T) - $h_{\mu_n}(T,P)$ ≥ $L_{\mu_n}(T)$ - h_μ (T,**P**) - 2ε (III-33) .Clearly one has;

L_μ (T) ≥ $L_{\mu_n}(T) + h_{\mu_n}(T,P)$ - h_μ (T,**P**) - 2ε (III-34).Hence,the result follows from the Lemma II-6; L_μ (T) ≥ $L_{\mu_n}(T)$ - 3ε (III-35).

Theorem III.7. If m, μ_n ∈ M (X,T) and 0 < a < 1.Then $L_{a\mu+(1-a)m}$ (T) = a L_μ (T) + (1 – a) L_m (T) .

Proof. Let **P** be a finite measurable partition of topological dynamical system (X,T) with H_μ(**P**)<∞.Since the functions $h_\circ$(T,**P**) and $h_\circ$(T) : M (X,T) → R^+ are the afine functions (See,(7)), one writes the following equalities;

$h_{a\mu+(1-a)m}$ (T,**P**) = a h_μ (T,**P**) + (1-a) h_m (T,**P**) (III-36) and

$h_{a\mu+(1-a)m}$ (T) = a h_μ (T) + (1-a) h_m (T) (III-37). But clearly,

$h_{a\mu + (1-a)m}$ (T) - $h_{a\mu + (1-a)m}$ (T,**P**)=a[h_μ (T)- h_μ (T,**P**)]+(1-a)[h_m (T)- h_m (T,**P**)] (III-38)

Hence, we obtain the result from the Definition III.1;

$L_{a\mu + (1-a)m}$ (T) = a L_μ (T) + (1 – a) L_m (T) (III-39).

Proposition III.8. If $\mu \in$ M (X,T) and the measure $\mu = \int_{E(X,T)} m.d\delta(m)$ is an ergodic decomposition , then L_μ (T) = $\int_{E(X,T)}$ L_m(T) $d\delta(m)$.

Proof. If **P** is a finite measurable partition of topological dynamical system (X,T) with H_μ(**P**)< ∞ and the measure $\mu = \int_{E(X,T)} m.d\delta(m)$ is an ergodic decomposition ,we write from The Proposition II-15 (i) and (ii);

h_μ (T, **P**) = $\int_{E(X,T)}$ h_m(T,**P**) $d\delta(m)$ (III-40) and h_μ (T) = $\int_{E(X,T)}$ h_m(T) $d\delta(m)$ (III-41).

Therefore one has, h_μ (T)- h_μ (T, **P**) = $\int_{E(X,T)}$ h_m(T) $d\delta(m)$ - $\int_{E(X,T)}$ h_m(T,**P**) $d\delta(m)$ (III-42).

And also from the linearity properties of Lebesgue Integral (See,(2)),

h_μ (T)- h_μ (T, **P**) = $\int_{E(X,T)}$ [h_m(T) $d\delta(m)$ -h_m(T,**P**)] $d\delta(m)$ (III-43)

Hence, the result follows from the Definition III.1; L_μ (T) = $\int_{E(X,T)}$ L_m(T) $d\delta(m)$ (III-44) .

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