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Research Article

Construction of a new generalization for n-polynomial convexity with their certain inequalities

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Abstract

In this paper, we first construct a new generalization of n-polynomial convex function. That is, this study is a generalization of the definition of "n-polynomial convexity" previously found in the literature. By making use of this construction, we derive certain inequalities for this new generalization and show that the first derivative in absolute value corresponds to a new class of n-polynomial convexity. Also, we see that the obtained results in the paper while comparing with Hölder, Hölder-İşcan and power-mean, improved-power-mean integral inequalities show that the results give a better approach than the others. Finally, we conclude our paper with applications containing some means.

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1. Preliminaries and fundamentals

Let $S : I \to \mathbb{R}$ be a convex function. Then the inequalities

$$S\left(\frac{\lambda+\delta}{2}\right) \leq \frac{1}{\delta-\lambda}\int_{\lambda}^{\delta} S(z)dz \leq \frac{S(\lambda)+S(\delta)}{2} \tag{1.1}$$

hold for all $\lambda, \delta \in I$ with $\lambda < \delta$. This double inequality is well known as the Hermite-Hadamard (H-H) inequality [6]. Some refinements of the H-H inequality for convex functions have been obtained [5, 16].

Additionally, readers can refer to [1, 3, 8, 11, 15] and the references in these papers to learn about different convexity classes and how to find the Hermite-Hadamard integral inequalities of these classes.

In [13], Tekin et al. gave the following definition and related H-H integral inequalities as follow:

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Definition 1.1 ([13]). Let $n \in \mathbb{N}$. A non-negative function $S : I \subset \mathbb{R} \to \mathbb{R}$ is called n-polynomial convex if for every $\lambda, \delta \in I$ and $t \in [0, 1]$,

$$S(t\lambda + (1-t)\delta) \leq \frac{1}{n}\sum_{s=1}^{n}\left[1-(1-t)^{s}\right]S(\lambda) + \frac{1}{n}\sum_{s=1}^{n}\left[1-t^{s}\right]S(\delta). \tag{1.2}$$

Theorem 1.2 ([13]). *Let $S : [\lambda, \delta] \to \mathbb{R}$ be a n-polynomial convex function. If $\lambda < \delta$ and $S \in L[\lambda, \delta]$, then the following H-H type inequalities hold:*

$$\frac{1}{2}\left(\frac{n}{n+2^{-n}-1}\right)S\left(\frac{\lambda+\delta}{2}\right) \leq \frac{1}{\delta-\lambda}\int_{\lambda}^{\delta}S(z)dz \leq \left(\frac{S(\lambda)+S(\delta)}{n}\right)\sum_{s=1}^{n}\frac{s}{s+1}. \tag{1.3}$$

The following inequality is known as Hölder-İşcan integral inequality:

Theorem 1.3 ([7]). *Let $p > 1$ and $\frac{1}{p}+\frac{1}{q}=1$. If K, L are real functions defined on $[\lambda, \delta]$ and if $|K|^p$, $|L|^q$ are integrable on interval $[\lambda, \delta]$ then*

$$\begin{aligned}&\int_{\lambda}^{\delta}|K(z)L(z)|\,dz\\ \leq\ &\frac{1}{\delta-\lambda}\left\{\left(\int_{\lambda}^{\delta}(\delta-z)\,|K(z)|^{p}\,dz\right)^{\frac{1}{p}}\left(\int_{\lambda}^{\delta}(\delta-z)\,|L(z)|^{q}\,dz\right)^{\frac{1}{q}}\right.\\ &\left.+\left(\int_{\lambda}^{\delta}(z-\lambda)\,|K(z)|^{p}\,dz\right)^{\frac{1}{p}}\left(\int_{\lambda}^{\delta}(z-\lambda)\,|L(z)|^{q}\,dz\right)^{\frac{1}{q}}\right\}.\end{aligned} \tag{1.4}$$

The following inequality is known as improved power-mean integral inequality:

Theorem 1.4 ([10]). *Let $q \geq 1$. If K, L are real functions defined on $[\lambda, \delta]$ and if $|K|$, $|K|\,|L|^q$ are integrable on interval $[\lambda, \delta]$ then*

$$\begin{aligned}&\int_{\lambda}^{\delta}|K(z)L(z)|\,dz \qquad (1.5)\\ \leq\ &\frac{1}{\delta-\lambda}\left\{\left(\int_{\lambda}^{\delta}(\delta-z)\,|K(z)|\,dz\right)^{1-\frac{1}{q}}\left(\int_{\lambda}^{\delta}(\delta-z)\,|K(z)|\,|L(z)|^{q}\,dz\right)^{\frac{1}{q}}\right.\\ &\left.+\left(\int_{\lambda}^{\delta}(z-\lambda)\,|K(z)|\,dz\right)^{1-\frac{1}{q}}\left(\int_{\lambda}^{\delta}(z-\lambda)\,|K(z)|\,|L(z)|^{q}\,dz\right)^{\frac{1}{q}}\right\}.\end{aligned}$$

Motivated by [13], we construct Definition 2.1 which seems to be a new generalization of n-polynomial convex function. By making use of this definition, we derive certain inequalities for this new generalization and show that the first derivative in absolute value corresponds to a new class of n-polynomial convexity. Also, we see that the obtained results in the paper while comparing with Hölder, Hölder-İşcan and power-mean, improved-power-mean integral inequalities show that the results give a better approach than the others. Finally, we conclude our paper with applications containing some means.

2. The construction of generalized n-polynomial convex functions

In this section, we introduce a new concept, which is called generalized n-polynomial convexity and then we give some algebraic properties for the generalized n-polynomial convex functions.

Definition 2.1. Let be $n \in \mathbb{N}$ and $a_i \geq 0$ $\left(i = \overline{1, n}\right)$ such that $\sum_{i=1}^{n} a_i > 0$. A non-negative $S : I \subset \mathbb{R} \to \mathbb{R}$ is called generalized n-polynomial convex function (or we can also called as $(a_1, a_2, ..., a_n)$-polynomial convex) if for every $\lambda, \delta \in I$ and $t \in [0,1]$,

$$S(t\lambda + (1-t)\delta) \leq \frac{\sum_{i=1}^{n} a_i \left(1-(1-t)^i\right)}{\sum_{i=1}^{n} a_i} S(\lambda) + \frac{\sum_{i=1}^{n} a_i \left(1-t^i\right)}{\sum_{i=1}^{n} a_i} S(\delta), \quad \sum_{i=1}^{n} a_i > 0. \quad (2.1)$$

The class of all generalized n-polynomial convex functions on I is denoted by $GPOLC(I)$. Every generalized n-polynomial convex function is a h-convex with

$$h(t) = \frac{\sum_{i=1}^{n} a_i \left(1-(1-t)^i\right)}{\sum_{i=1}^{n} a_i}.$$

Therefore, if $f, g \in GPOLC(I)$, then

i.) $f+g \in GPOLC(I)$ *and for* $c \in \mathbb{R}$ $(c \geq 0)$ $cf \in GPOLC(I)$ *(see [14], Proposition 9).*

ii.) if the functions f, g be a similarly ordered on interval I, then $fg \in GPOLC(I)$ *.(see [14], Proposition 10).*

Also, if $f : I \to J$ is a convex and $g \in GPOLC(J)$ and nondecreasing, then $g \circ f \in GPOLC(I)$ *(see [14], Theorem 15).*

Researchers can look at [2, 9, 14] for studies about h-convexity.

Remark 2.2. If we take $n = 1$ in (2.1), then the generalized 1-polynomial convexity reduces to the clasical convexity.

Remark 2.3. If we take $a_i = 1$ $\left(i = \overline{1,n}\right)$ in (2.1), then the generalized n-polynomial convexity reduces to the n-polynomial convexity.

More generally, we can give the following remark together with proof:

Remark 2.4. Every nonnegative convex function is also a generalized n-polynomial convex.. Indeed, this case is clear from the following inequalities

$$\sum_{i=1}^{n} a_i (1-t)^i \leq \sum_{i=1}^{n} a_i (1-t) \text{ and } \sum_{i=1}^{n} a_i (1-t) \leq \sum_{i=1}^{n} a_i \left(1-t^i\right)$$

for all $t \in [0,1]$ and $n \in \mathbb{N}$.

Example 2.5. $f : (0,\infty) \to \mathbb{R}$, $f(x) = x^m$, $m \in (-\infty, 0) \cup [1, \infty)$, is a generalized n-polynomial convex function.

Theorem 2.6. *Let* $\delta > \lambda > 0$ *and* $K_\alpha : [\lambda, \delta] \to \mathbb{R}$ *be an arbitrary family of generalized* n*-polynomial convex and let* $K(x) = \sup_\alpha K_\alpha(x)$*. If* $J = \{u \in [\lambda, \delta] : K(u) < \infty\}$ *is nonempty, then* J *is an interval and* K *is a generalized* n*-polynomial convex function on* J*.*

Proof. Let $t \in [0,1]$ and $x, y \in J$ be arbitrary. Then

$$\begin{aligned} & K(t\lambda + (1-t)y) \\ = & \sup_\alpha K_\alpha (t\lambda + (1-t)\delta) \\ \leq & \sup_\alpha \left[\frac{\sum_{i=1}^{n} a_i \left(1-(1-t)^i\right)}{\sum_{i=1}^{n} a_i} K_\alpha(\lambda) + \frac{\sum_{i=1}^{n} a_i \left(1-t^i\right)}{\sum_{i=1}^{n} a_i} K_\alpha(\delta) \right] \\ \leq & \frac{\sum_{i=1}^{n} a_i \left(1-(1-t)^i\right)}{\sum_{i=1}^{n} a_i} \sup_\alpha K_\alpha(\lambda) + \frac{\sum_{i=1}^{n} a_i \left(1-t^i\right)}{\sum_{i=1}^{n} a_i} \sup_\alpha K_\alpha(\delta) \\ = & \frac{\sum_{i=1}^{n} a_i \left(1-(1-t)^i\right)}{\sum_{i=1}^{n} a_i} K(\lambda) + \frac{\sum_{i=1}^{n} a_i \left(1-t^i\right)}{\sum_{i=1}^{n} a_i} K(\delta) < \infty. \end{aligned}$$

□

3. H-H inequality for generalized n-polynomial convex functions

In this section, we will establish some inequalities of H-H type for the generalized n-polynomial convex functions. We will denote by $L[\lambda,\delta]$ the space of (Lebesgue) integrable functions on $[\lambda,\delta]$.

Theorem 3.1. *Let $S:[\lambda,\delta]\to\mathbb{R}$ be a generalized n-polynomial convex function. If $\lambda<\delta$ and $S\in L[\lambda,\delta]$, then the following H-H type inequalities hold:*

$$\begin{aligned} &\frac{1}{2}\left(\frac{\sum_{i=1}^{n} a_i}{\sum_{i=1}^{n} a_i\left[1-\left(\frac{1}{2}\right)^i\right]}\right) S\left(\frac{\lambda+\delta}{2}\right) \\ \leq\ & \frac{1}{\delta-\lambda}\int_\lambda^\delta S(z)dz \leq \left[\frac{S(\lambda)+S(\delta)}{\sum_{i=1}^{n} a_i}\right]\sum_{i=1}^{n} a_i\left(\frac{i}{i+1}\right). \end{aligned} \tag{3.1}$$

Proof. Here, we will use the property of the generalized n-polynomial convex function of S. So, we have

$$\begin{aligned} & S\left(\frac{\lambda+\delta}{2}\right) \\ =\ & S\left(\frac{1}{2}\left[t\lambda+(1-t)\delta\right]+\frac{1}{2}\left[(1-t)\lambda+t\delta\right]\right) \\ \leq\ & \frac{\sum_{i=1}^{n} a_i\left[1-\left(1-\frac{1}{2}\right)^i\right]}{\sum_{i=1}^{n} a_i} S\left(t\lambda+(1-t)\delta\right)+\frac{\sum_{i=1}^{n} a_i\left[1-\left(\frac{1}{2}\right)^i\right]}{\sum_{i=1}^{n} a_i} S\left((1-t)\lambda+t\delta\right) \\ =\ & \frac{\sum_{i=1}^{n} a_i\left[1-\left(\frac{1}{2}\right)^i\right]}{\sum_{i=1}^{n} a_i}\left[S\left(t\lambda+(1-t)\delta\right)+S\left((1-t)\lambda+t\delta\right)\right]. \end{aligned}$$

By taking integral in the last inequality with respect to $t\in[0,1]$, we deduce that

$$S\left(\frac{\lambda+\delta}{2}\right)\leq\frac{2}{\delta-\lambda}\frac{\sum_{i=1}^{n} a_i\left[1-\left(\frac{1}{2}\right)^i\right]}{\sum_{i=1}^{n} a_i}\int_\lambda^\delta S(z)dz.$$

By using the property of the generalized n-polynomial convex function S, if the variable is changed as $z=t\lambda+(1-t)\delta$, then

$$\begin{aligned} & \frac{1}{\delta-\lambda}\int_\lambda^\delta S(z)dz \\ =\ & \int_0^1 S\left(t\lambda+(1-t)\delta\right)dt \\ \leq\ & \int_0^1\left[\frac{\sum_{i=1}^{n} a_i\left(1-(1-t)^i\right)}{\sum_{i=1}^{n} a_i}S(\lambda)+\frac{\sum_{i=1}^{n} a_i\left(1-t^i\right)}{\sum_{i=1}^{n} a_i}S(\delta)\right]dt \\ =\ & \frac{S(\lambda)}{\sum_{i=1}^{n} a_i}\sum_{i=1}^{n} a_i\int_0^1\left[1-(1-t)^i\right]dt+\frac{S(\delta)}{\sum_{i=1}^{n} a_i}\sum_{i=1}^{n} a_i\int_0^1\left[1-t^i\right]dt \\ =\ & \left[\frac{S(\lambda)+S(\delta)}{\sum_{i=1}^{n} a_i}\right]\sum_{i=1}^{n} a_i\left(\frac{i}{i+1}\right), \end{aligned}$$

where

$$\int_0^1\left[1-(1-t)^i\right]dt=\int_0^1\left[1-t^i\right]=\frac{i}{i+1}.$$

□

Remark 3.2. In case of $n=1$, the inequality (3.1) coincides with the the inequality (1.1).

Remark 3.3. In case of $a_i = 1$ $\left(i=\overline{1,n}\right)$ the inequality (3.1) coincides with the the inequality (1.3)

4. New inequalities for generalized n-polynomial convex functions

In this section, we will establish new estimates that refine H-H inequality for functions whose first derivative in absolute value is a generalized n-polynomial convex function. Dragomir and Agarwal [4] used the following lemma:

Lemma 4.1 ([4]). *Let $S: I^\circ \to \mathbb{R}$ be a differentiable mapping on I°, $\lambda, \delta \in I^\circ$ with $\lambda < \delta$. If $S' \in L[\lambda, \delta]$, then the following identity holds:*

$$\frac{S(\lambda)+S(\delta)}{2} - \frac{1}{\delta-\lambda}\int_\lambda^\delta S(z)dz = \frac{\delta-\lambda}{2}\int_0^1 (1-2t)S'(t\lambda+(1-t)\delta)\,dt. \tag{4.1}$$

Theorem 4.2. *Let be $S: I \to \mathbb{R}$ be a differentiable function on I°, $\lambda, \delta \in I^\circ$ with $\lambda < \delta$ and assume that $S' \in L[\lambda, \delta]$. If $|S'|$ is a generalized n-polynomial convex function on interval $[\lambda, \delta]$, then the following inequality holds for $t \in [0,1]$.*

$$\begin{aligned} &\left|\frac{S(\lambda)+S(\delta)}{2} - \frac{1}{\delta-\lambda}\int_\lambda^\delta S(z)dz\right| \qquad (4.2)\\ &\leq \frac{\delta-\lambda}{\sum_{i=1}^n a_i}\sum_{i=1}^n a_i\left[\frac{(i^2+i+2)\,2^i-2}{(i+1)(i+2)2^{i+1}}\right] A\left(\left|S'(\lambda)\right|, \left|S'(\delta)\right|\right), \end{aligned}$$

where $A(u,v)=(u+v)/2$ is the arithmetic mean.

Proof. From Lemma 4.1 and

$$\left|S'(t\lambda+(1-t)\delta)\right| \leq \frac{\sum_{i=1}^n a_i\left(1-(1-t)^i\right)}{\sum_{i=1}^n a_i}\left|S'(\lambda)\right| + \frac{\sum_{i=1}^n a_i\left(1-t^i\right)}{\sum_{i=1}^n a_i}\left|S'(\delta)\right|,$$

we get

$$\begin{aligned} &\left|\frac{S(\lambda)+S(\delta)}{2} - \frac{1}{\delta-\lambda}\int_\lambda^\delta S(z)dz\right| \\ &\leq \frac{\delta-\lambda}{2}\int_0^1 |1-2t|\left|S'(t\lambda+(1-t)\delta)\right| dt \\ &\leq \frac{\delta-\lambda}{2\sum_{i=1}^n a_i}\left(\begin{array}{c} \left|S'(\lambda)\right|\sum_{i=1}^n a_i\int_0^1|1-2t|\left(1-(1-t)^i\right)dt \\ +\left|S'(\delta)\right|\sum_{i=1}^n a_i\int_0^1|1-2t|\left(1-t^i\right)dt \end{array}\right) \\ &= \frac{\delta-\lambda}{\sum_{i=1}^n a_i}\sum_{s=1}^n a_i\left[\frac{(i^2+i+2)\,2^i-2}{(i+1)(i+2)2^{i+1}}\right] A\left(\left|S'(\lambda)\right|, \left|S'(\delta)\right|\right), \end{aligned}$$

where

$$\int_0^1 |1-2t|\left[1-(1-t)^i\right]dt = \int_0^1|1-2t|\left[1-t^i\right]dt = \frac{(i^2+i+2)\,2^i-2}{(i+1)(i+2)2^{i+1}}.$$

□

Corollary 4.3. *Taking $n=1$ in (4.2), we have the following inequality:*

$$\left|\frac{S(\lambda)+S(\delta)}{2} - \frac{1}{\delta-\lambda}\int_\lambda^\delta S(z)dz\right| \leq \frac{\delta-\lambda}{4}A\left(\left|S'(\lambda)\right|, \left|S'(\delta)\right|\right). \tag{4.3}$$

This coincides with the inequality in [4, Theorem 2.2].

Corollary 4.4. *Taking $a_i = 1$ $(i = \overline{1,n})$ in (4.2), we have the following inequality:*

$$\left|\frac{S(\lambda)+S(\delta)}{2} - \frac{1}{\delta-\lambda}\int_\lambda^\delta S(z)dz\right| \le \frac{\delta-\lambda}{n}\sum_{i=1}^{n}\left[\frac{(i^2+i+2)\,2^i-2}{(i+1)(i+2)2^{i+1}}\right] A\left(\left|S'(\lambda)\right|,\left|S'(\delta)\right|\right). \tag{4.4}$$

This coincides with the inequality in [13, Theorem 5].

Theorem 4.5. *Let $S : I \to \mathbb{R}$ be a differentiable function on I°, $\lambda, \delta \in I^\circ$ with $\lambda < \delta$, $q > 1, \frac{1}{p} + \frac{1}{q} = 1$ and assume that $S' \in L[\lambda, \delta]$. If $|S'|^q$ is a generalized n-polynomial convex function on interval $[\lambda, \delta]$, then the following inequality holds*

$$\begin{aligned}&\left|\frac{S(\lambda)+S(\delta)}{2} - \frac{1}{\delta-\lambda}\int_\lambda^\delta S(z)dz\right| \qquad (4.5)\\ &\le \frac{\delta-\lambda}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{2}{\sum_{i=1}^n a_i}\sum_{i=1}^{n} a_i\frac{i}{i+1}\right)^{\frac{1}{q}} A^{\frac{1}{q}}\left(\left|S'(\lambda)\right|^q,\left|S'(\delta)\right|^q\right),\end{aligned}$$

where A is the arithmetic mean.

Proof. Using Lemma 4.1, Hölder's integral inequality and the following inequality

$$\left|S'(t\lambda+(1-t)\delta)\right|^q \le \frac{\sum_{i=1}^n a_i\left(1-(1-t)^i\right)}{\sum_{i=1}^n a_i}\left|S'(\lambda)\right|^q + \frac{\sum_{i=1}^n a_i\left(1-t^i\right)}{\sum_{i=1}^n a_i}\left|S'(\delta)\right|^q, \tag{4.6}$$

which is the generalized n-polynomial convex function of $|S'|^q$, we get

$$\begin{aligned}&\left|\frac{S(\lambda)+S(\delta)}{2} - \frac{1}{\delta-\lambda}\int_\lambda^\delta S(z)dz\right|\\ &\le \frac{\delta-\lambda}{2}\left(\int_0^1|1-2t|^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1\left|S'(t\lambda+(1-t)\delta)\right|^q dt\right)^{\frac{1}{q}}\\ &\le \frac{\delta-\lambda}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\begin{array}{c}\frac{|S'(\lambda)|^q}{\sum_{i=1}^n a_i}\int_0^1\sum_{i=1}^n a_i\left(1-(1-t)^i\right)dt\\ +\frac{|S'(\delta)|^q}{\sum_{i=1}^n a_i}\int_0^1\sum_{i=1}^n a_i\left(1-t^i\right)dt\end{array}\right)^{\frac{1}{q}}\\ &= \frac{\delta-\lambda}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{2}{\sum_{i=1}^n a_i}\sum_{i=1}^{n} a_i\frac{i}{i+1}\right)^{\frac{1}{q}} A^{\frac{1}{q}}\left(\left|S'(\lambda)\right|^q,\left|S'(\delta)\right|^q\right),\end{aligned}$$

where

$$\begin{aligned}\int_0^1|1-2t|^p\,dt &= \frac{1}{p+1},\\ \int_0^1\left[1-(1-t)^i\right]dt &= \int_0^1\left[1-t^i\right]dt = \frac{i}{i+1}.\end{aligned}$$

□

Corollary 4.6. *If we take $n = 1$ in (4.5), we get the following inequality:*

$$\left|\frac{S(\lambda)+S(\delta)}{2} - \frac{1}{\delta-\lambda}\int_\lambda^\delta S(z)dz\right| \le \frac{\delta-\lambda}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}} A^{\frac{1}{q}}\left(\left|S'(\lambda)\right|^q,\left|S'(\delta)\right|^q\right). \tag{4.7}$$

This coincides with the inequality in [4, Theorem 2.3].

Corollary 4.7. *If we take $a_i = 1$ $\left(i = \overline{1,n}\right)$ in (4.5), we get the following inequality:*

$$\left|\frac{S(\lambda)+S(\delta)}{2} - \frac{1}{\delta-\lambda}\int_{\lambda}^{\delta} S(z)dz\right| \tag{4.8}$$

$$\leq \frac{\delta-\lambda}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{2}{n}\sum_{i=1}^{n}\frac{i}{i+1}\right)^{\frac{1}{q}} A^{\frac{1}{q}}\left(\left|S'(\lambda)\right|^q, \left|S'(\delta)\right|^q\right).$$

This coincides with the inequality in [13, Theorem 6].

Theorem 4.8. *Let $S : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on interval I°, $\lambda, \delta \in I^\circ$ with $\lambda < \delta$, $q \geq 1$ and assume that $S' \in L[\lambda,\delta]$. If $|S'|^q$ is a generalized n-polynomial convex function on $[a,b]$, then the following inequality holds*

$$\left|\frac{S(\lambda)+S(\delta)}{2} - \frac{1}{\delta-\lambda}\int_{\lambda}^{\delta} S(z)dz\right| \tag{4.9}$$

$$\leq \frac{\delta-\lambda}{2}\left(\frac{1}{2}\right)^{1-\frac{2}{q}}\left(\frac{1}{\sum_{i=1}^{n} a_i}\sum_{i=1}^{n} a_i\frac{(i^2+i+2)\,2^i-2}{(i+1)(i+2)2^{i+1}}\right)^{\frac{1}{q}} A^{\frac{1}{q}}\left(\left|S'(\lambda)\right|^q, \left|S'(\delta)\right|^q\right),$$

where A is the arithmetic mean.

Proof. Firstly, let $q > 1$. By using the Lemma 4.1, Hölder inequality and the property of the generalized n-polynomial convex function of $|S'|^q$, we obtain

$$\left|\frac{S(\lambda)+S(\delta)}{2} - \frac{1}{\delta-\lambda}\int_{\lambda}^{\delta} S(z)dz\right|$$

$$\leq \frac{\delta-\lambda}{2}\left(\int_0^1 |1-2t|\,dt\right)^{1-\frac{1}{q}}\left(\int_0^1 |1-2t|\left|S'(t\lambda+(1-t)\delta)\right|^q dt\right)^{\frac{1}{q}}$$

$$\leq \frac{\delta-\lambda}{2}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left[\frac{|S'(\lambda)|^q}{\sum_{i=1}^{n} a_i}\sum_{i=1}^{n} a_i\int_0^1 |1-2t|\left[1-(1-t)^i\right]dt\right.$$

$$\left.+\frac{|S'(\delta)|^q}{\sum_{i=1}^{n} a_i}\sum_{i=1}^{n} a_i\int_0^1 |1-2t|\left[1-t^s\right]dt\right]^{\frac{1}{q}}$$

$$= \frac{\delta-\lambda}{2}\left(\frac{1}{2}\right)^{1-\frac{2}{q}}\left(\frac{1}{\sum_{i=1}^{n} a_i}\sum_{i=1}^{n} a_i\frac{(i^2+i+2)\,2^i-2}{(i+1)(i+2)2^{i+1}}\right)^{\frac{1}{q}} A^{\frac{1}{q}}\left(\left|S'(\lambda)\right|^q, \left|S'(\delta)\right|^q\right).$$

For $q = 1$, the proof of the Theorem 4.2 is followed step by step. □

Corollary 4.9. *Under the assumption of Theorem 4.8 with $q = 1$, we have the conclusion of Theorem 4.2.*

Corollary 4.10. *Taking $n = 1$ in (4.9), we get the following inequality:*

$$\left|\frac{S(\lambda)+S(\delta)}{2} - \frac{1}{\delta-\lambda}\int_{\lambda}^{\delta} S(z)dz\right| \leq \frac{\delta-\lambda}{4}A^{\frac{1}{q}}\left(\left|S'(\lambda)\right|^q, \left|S'(\delta)\right|^q\right). \tag{4.10}$$

The obtaining inequality for $q = 1$ coincides with in [12, Theorem 1].

Corollary 4.11. *If we take $a_i = 1$ $\left(i = \overline{1,n}\right)$ in the inequality (4.9), we get the following inequality:*

$$\left|\frac{S(\lambda)+S(\delta)}{2} - \frac{1}{\delta-\lambda}\int_{\lambda}^{\delta} S(z)dz\right| \tag{4.11}$$

$$\leq \frac{\delta-\lambda}{2}\left(\frac{1}{2}\right)^{1-\frac{2}{q}}\left(\frac{1}{n}\sum_{i=1}^{n}\frac{(i^2+i+2)\,2^i-2}{(i+1)(i+2)2^{i+1}}\right)^{\frac{1}{q}} A^{\frac{1}{q}}\left(\left|S'(\lambda)\right|^q, \left|S'(\delta)\right|^q\right).$$

This inequality coincides with the inequality in [13, Theorem 7].

Now, we will prove the Theorem 4.5 by using Hölder-İşcan inequality. Then we will show that the result we have obtained in this theorem gives a better approach than that obtained in the Theorem 4.5.

Theorem 4.12. *Let $S : I \to \mathbb{R}$ be a differentiable function on I°, $\lambda, \delta \in I^\circ$ with $\lambda < \delta$, $q > 1, \frac{1}{p} + \frac{1}{q} = 1$ and assume that $S' \in L[\lambda, \delta]$. If $|S'|^q$ is a generalized n-polynomial convex on interval $[\lambda, \delta]$, then the following inequality holds*

$$\left|\frac{S(\lambda)+S(\delta)}{2} - \frac{1}{\delta-\lambda}\int_\lambda^\delta S(z)dz\right| \tag{4.12}$$
$$\leq \frac{\delta-\lambda}{2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\left(\frac{|S'(\lambda)|^q}{\sum_{i=1}^n a_i}\sum_{i=1}^n \frac{ia_i}{2(i+2)} + \frac{|S'(\delta)|^q}{\sum_{i=1}^n a_i}\sum_{i=1}^n \frac{i(i+3)a_i}{2(i+1)(i+2)}\right)^{\frac{1}{q}}$$
$$+\frac{\delta-\lambda}{2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\left(\frac{|S'(\lambda)|^q}{\sum_{i=1}^n a_i}\sum_{i=1}^n \frac{i(i+3)a_i}{2(i+1)(i+2)} + \frac{|S'(\delta)|^q}{\sum_{i=1}^n a_i}\sum_{i=1}^n \frac{ia_i}{2(i+2)}\right)^{\frac{1}{q}}.$$

Proof. If we use Lemma 4.1, Hölder-İşcan inequality and the inequality

$$|S'(t\lambda+(1-t)\delta)|^q \leq \frac{\sum_{i=1}^n a_i\left(1-(1-t)^i\right)}{\sum_{i=1}^n a_i}|S'(\lambda)|^q + \frac{\sum_{i=1}^n a_i\left(1-t^i\right)}{\sum_{i=1}^n a_i}|S'(\delta)|^q,$$

which is the generalized n-polynomial convex function of the function $|S'|^q$, we get

$$\left|\frac{S(\lambda)+S(\delta)}{2} - \frac{1}{\delta-\lambda}\int_\lambda^\delta S(z)dz\right|$$
$$\leq \frac{\delta-\lambda}{2}\left(\int_0^1 (1-t)|1-2t|^p dt\right)^{\frac{1}{p}}\left(\int_0^1 (1-t)|S'(t\lambda+(1-t)\delta)|^q dt\right)^{\frac{1}{q}}$$
$$+\frac{\delta-\lambda}{2}\left(\int_0^1 t|1-2t|^p dt\right)^{\frac{1}{p}}\left(\int_0^1 t|S'(t\lambda+(1-t)\delta)|^q dt\right)^{\frac{1}{q}}$$
$$\leq \frac{\delta-\lambda}{2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\left(\frac{|S'(\lambda)|^q}{\sum_{i=1}^n a_i}\sum_{i=1}^n a_i\int_0^1 (1-t)\left[1-(1-t)^i\right]dt\right.$$
$$\left.+\frac{|S'(\delta)|^q}{\sum_{i=1}^n a_i}\sum_{i=1}^n a_i\int_0^1 (1-t)\left[1-t^i\right]dt\right)^{\frac{1}{q}}$$
$$+\frac{\delta-\lambda}{2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\left(\frac{|S'(\lambda)|^q}{\sum_{i=1}^n a_i}\sum_{i=1}^n a_i\int_0^1 t\left[1-(1-t)^i\right]dt\right.$$
$$\left.+\frac{|S'(\delta)|^q}{\sum_{i=1}^n a_i}\sum_{i=1}^n a_i\int_0^1 t\left[1-t^i\right]dt\right)^{\frac{1}{q}}$$
$$= \frac{b-a}{2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\left(\frac{|f'(a)|^q}{\sum_{i=1}^n a_i}\sum_{i=1}^n \frac{ia_i}{2(i+2)} + \frac{|f'(b)|^q}{\sum_{i=1}^n a_i}\sum_{i=1}^n \frac{i(i+3)a_i}{2(i+1)(i+2)}\right)^{\frac{1}{q}}$$
$$+\frac{b-a}{2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\left(\frac{|f'(a)|^q}{\sum_{i=1}^n a_i}\sum_{i=1}^n \frac{i(i+3)a_i}{2(i+1)(i+2)} + \frac{|f'(b)|^q}{\sum_{i=1}^n a_i}\sum_{i=1}^n \frac{ia_i}{2(i+2)}\right)^{\frac{1}{q}},$$

where

$$\begin{aligned}
\int_0^1 (1-t)\,|1-2t|^p\,dt &= \int_0^1 t\,|1-2t|^p\,dt = \frac{1}{2\,(p+1)}, \\
\int_0^1 (1-t)\left[1-(1-t)^i\right]dt &= \int_0^1 t\left[1-t^i\right]dt = \frac{i}{2(i+2)}, \\
\int_0^1 (1-t)\left[1-t^i\right]dt &= \int_0^1 t\left[1-(1-t)^i\right]dt = \frac{i(i+3)}{2(i+1)(i+2)}.
\end{aligned}$$

□

Corollary 4.13. *Taking $n=1$ in (4.12), we get the following:*

$$\begin{aligned}
&\left|\frac{S(\lambda)+S(\delta)}{2}-\frac{1}{\delta-\lambda}\int_\lambda^\delta S(z)dz\right| \qquad (4.13)\\
\leq\ & \frac{\delta-\lambda}{4}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[\left(\frac{|S'(\lambda)|^q+2\,|S'(\delta)|^q}{3}\right)^{\frac{1}{q}}+\left(\frac{2\,|S'(\lambda)|^q+|S'(\delta)|^q}{3}\right)^{\frac{1}{q}}\right].
\end{aligned}$$

This coincides with the inequality in [7, Theorem 3.2].

Corollary 4.14. *Taking $a_i=1$ $\left(i=\overline{1,n}\right)$ in (4.12), we get the following:*

$$\begin{aligned}
&\left|\frac{S(\lambda)+S(\delta)}{2}-\frac{1}{\delta-\lambda}\int_\lambda^\delta S(z)dz\right| \qquad (4.14)\\
\leq\ & \frac{\delta-\lambda}{2}\left(\frac{1}{2\,(p+1)}\right)^{\frac{1}{p}}\left(\frac{|S'(\lambda)|^q}{n}\sum_{i=1}^n\frac{i}{2(i+2)}+\frac{|S'(\delta)|^q}{n}\sum_{i=1}^n\frac{i(i+3)}{2(i+1)(i+2)}\right)^{\frac{1}{q}}\\
&+\frac{\delta-\lambda}{2}\left(\frac{1}{2\,(p+1)}\right)^{\frac{1}{p}}\left(\frac{|S'(\lambda)|^q}{n}\sum_{i=1}^n\frac{i(i+3)}{2(i+1)(i+2)}+\frac{|S'(\delta)|^q}{n}\sum_{i=1}^n\frac{i}{2(i+2)}\right)^{\frac{1}{q}}.
\end{aligned}$$

This coincides with the inequality in [13, Theorem 8].

Remark 4.15. (4.12) gives better results than the inequality (4.5). Let us show that

$$\begin{aligned}
&\frac{\delta-\lambda}{2}\left(\frac{1}{2\,(p+1)}\right)^{\frac{1}{p}}\left(\frac{|S'(\lambda)|^q}{\sum_{i=1}^n a_i}\sum_{i=1}^n\frac{ia_i}{2(i+2)}+\frac{|S'(\delta)|^q}{\sum_{i=1}^n a_i}\sum_{i=1}^n\frac{i(i+3)a_i}{2(i+1)(i+2)}\right)^{\frac{1}{q}}\\
&+\frac{\delta-\lambda}{2}\left(\frac{1}{2\,(p+1)}\right)^{\frac{1}{p}}\left(\frac{|S'(\lambda)|^q}{\sum_{i=1}^n a_i}\sum_{i=1}^n\frac{i(i+3)a_i}{2(i+1)(i+2)}+\frac{|S'(\delta)|^q}{\sum_{i=1}^n a_i}\sum_{i=1}^n\frac{ia_i}{2(i+2)}\right)^{\frac{1}{q}}\\
\leq\ & \frac{\delta-\lambda}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{2}{\sum_{i=1}^n a_i}\sum_{i=1}^n a_i\frac{i}{i+1}\right)^{\frac{1}{q}}A^{\frac{1}{q}}\left(|S'(\lambda)|^q,|S'(\delta)|^q\right).
\end{aligned}$$

Using concavity of the function $h:[0,\infty)\to\mathbb{R}$, $h(x)=x^{\lambda}, 0<\lambda\leq 1$ by sample calculation we get

$$\begin{aligned}
& \frac{\delta-\lambda}{2}\left(\frac{1}{2\,(p+1)}\right)^{\frac{1}{p}}\left(\frac{|S'(\lambda)|^q}{\sum_{i=1}^n a_i}\sum_{i=1}^n\frac{ia_i}{2(i+2)}+\frac{|S'(\delta)|^q}{\sum_{i=1}^n a_i}\sum_{i=1}^n\frac{i(i+3)a_i}{2(i+1)(i+2)}\right)^{\frac{1}{q}} \\
& +\frac{\delta-\lambda}{2}\left(\frac{1}{2\,(p+1)}\right)^{\frac{1}{p}}\left(\frac{|S'(\lambda)|^q}{\sum_{i=1}^n a_i}\sum_{i=1}^n\frac{i(i+3)a_i}{2(i+1)(i+2)}+\frac{|S'(\delta)|^q}{\sum_{i=1}^n a_i}\sum_{i=1}^n\frac{ia_i}{2(i+2)}\right)^{\frac{1}{q}} \\
\leq\ & \frac{\delta-\lambda}{2}\left(\frac{1}{2\,(p+1)}\right)^{\frac{1}{p}}2\left[\frac{1}{2}\frac{|S'(\lambda)|^q}{\sum_{i=1}^n a_i}\sum_{i=1}^n\frac{i}{i+1}+\frac{1}{2}\frac{|S'(\delta)|^q}{\sum_{i=1}^n a_i}\sum_{i=1}^n\frac{i}{i+1}\right]^{\frac{1}{q}} \\
=\ & \frac{\delta-\lambda}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{2}{\sum_{i=1}^n a_i}\sum_{i=1}^n\frac{i}{i+1}\right)^{\frac{1}{q}}A^{\frac{1}{q}}\left(|S'(\lambda)|^q,|S'(\delta)|^q\right),
\end{aligned}$$

which is the required.

Theorem 4.16. *Let $S:I\subseteq\mathbb{R}\to\mathbb{R}$ be a differentiable function on interval I°, $\lambda,\delta\in I^{\circ}$ with $\lambda<\delta$,$q\geq 1$ and assume that $S'\in L[\lambda,\delta]$. If $|S'|^q$ is a generalized n-polynomial convex on the interval $[\lambda,\delta]$, then the following holds*

$$\begin{aligned}
& \left|\frac{S(\lambda)+S(\delta)}{2}-\frac{1}{\delta-\lambda}\int_{\lambda}^{\delta}S(z)dz\right| \qquad (4.15)\\
\leq\ & \frac{\delta-\lambda}{2}\left(\frac{1}{2}\right)^{2-\frac{2}{q}}\left(\frac{|S'(\lambda)|^q}{\sum_{i=1}^n a_i}\sum_{i=1}^n a_iM_1(i)+\frac{|S'(\delta)|^q}{\sum_{i=1}^n a_i}\sum_{i=1}^n a_iM_2(i)\right)^{\frac{1}{q}} \\
& +\frac{\delta-\lambda}{2}\left(\frac{1}{2}\right)^{2-\frac{2}{q}}\left(\frac{|S'(\lambda)|^q}{\sum_{i=1}^n a_i}\sum_{i=1}^n a_iM_2(i)+\frac{|S'(\delta)|^q}{\sum_{i=1}^n a_i}\sum_{i=1}^n a_iM_1(i)\right)^{\frac{1}{q}},
\end{aligned}$$

where

$$\begin{aligned}
M_1(i)\ :\ &=\int_0^1(1-t)\,|1-2t|\left[1-(1-t)^i\right]dt=\int_0^1 t\,|1-2t|\left[1-t^i\right]dt \\
&=\frac{(i^2+i+2)\,2^i-2}{2^{i+2}(i+2)(i+3)},
\end{aligned}$$

and

$$\begin{aligned}
M_2(i)\ :\ &=\int_0^1 t\,|1-2t|\left[1-(1-t)^i\right]dt=\int_0^1(1-t)\,|1-2t|\left[1-t^i\right]dt \\
&=\frac{(i+5)\left[(i^2+i+2)\,2^i-2\right]}{2^{i+2}(i+1)(i+2)(i+3)}.
\end{aligned}$$

Proof. Firstly, let $q>1$. By using the Lemma 4.1, improved power-mean inequality and the property of the generalized n-polynomial convexity of the function $|f'|^q$, we obtain

$$\begin{aligned}
&\left|\frac{S(\lambda)+S(\delta)}{2}-\frac{1}{\delta-\lambda}\int_\lambda^\delta S(z)dz\right| \\
\leq\ & \frac{\delta-\lambda}{2}\left(\int_0^1(1-t)\left|1-2t\right|dt\right)^{1-\frac{1}{q}}\left(\int_0^1(1-t)\left|1-2t\right|\left|S'\left(t\lambda+(1-t)\delta\right)\right|^q dt\right)^{\frac{1}{q}} \\
&+\frac{\delta-\lambda}{2}\left(\int_0^1 t\left|1-2t\right|dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t\left|1-2t\right|\left|S'\left(t\lambda+(1-t)\delta\right)\right|^q dt\right)^{\frac{1}{q}} \\
\leq\ & \frac{\delta-\lambda}{2}\left(\frac{1}{2}\right)^{2-\frac{2}{q}}\left(\frac{\left|S'\left(\lambda\right)\right|^q}{\sum_{i=1}^n a_i}\sum_{i=1}^n a_iM_1(i)+\frac{\left|S'\left(\delta\right)\right|^q}{\sum_{i=1}^n a_i}\sum_{i=1}^n a_iM_2(i)\right)^{\frac{1}{q}} \\
&+\frac{\delta-\lambda}{2}\left(\frac{1}{2}\right)^{2-\frac{2}{q}}\left(\frac{\left|S'\left(\lambda\right)\right|^q}{\sum_{i=1}^n a_i}\sum_{i=1}^n a_iM_2(i)+\frac{\left|S'\left(\delta\right)\right|^q}{\sum_{i=1}^n a_i}\sum_{i=1}^n a_iM_1(i)\right)^{\frac{1}{q}}.
\end{aligned}$$

For $q=1$, the proof of the Theorem 4.2 is followed step by step. □

Corollary 4.17. *Taking $n=1$ in (4.15), we get the following inequality:*

$$\begin{aligned}
&\left|\frac{S(\lambda)+S(\delta)}{2}-\frac{1}{\delta-\lambda}\int_\lambda^\delta S(z)dz\right| \qquad (4.16)\\
\leq\ & \frac{\delta-\lambda}{8}\left[\left(\frac{\left|S'\left(\lambda\right)\right|^q+3\left|S'\left(\delta\right)\right|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{3\left|S'\left(\lambda\right)\right|^q+\left|S'\left(\delta\right)\right|^q}{4}\right)^{\frac{1}{q}}\right].
\end{aligned}$$

This coincides with the inequality in [13, Corollary 6].

Corollary 4.18. *If we take $a_i=1$ $(i=\overline{1,n})$ in the inequality (4.15), we get the following inequality:*

$$\begin{aligned}
&\left|\frac{S(\lambda)+S(\delta)}{2}-\frac{1}{\delta-\lambda}\int_\lambda^\delta S(z)dz\right| \qquad (4.17)\\
\leq\ & \frac{\delta-\lambda}{2}\left(\frac{1}{2}\right)^{2-\frac{2}{q}}\left(\frac{\left|S'\left(\lambda\right)\right|^q}{n_i}\sum_{i=1}^n M_1(i)+\frac{\left|S'\left(\delta\right)\right|^q}{n}\sum_{i=1}^n M_2(i)\right)^{\frac{1}{q}} \\
&+\frac{\delta-\lambda}{2}\left(\frac{1}{2}\right)^{2-\frac{2}{q}}\left(\frac{\left|S'\left(\lambda\right)\right|^q}{n}\sum_{i=1}^n M_2(i)+\frac{\left|S'\left(\delta\right)\right|^q}{n}\sum_{i=1}^n M_1(i)\right)^{\frac{1}{q}}.
\end{aligned}$$

This coincides with the inequality in [13, Theorem 9].

Remark 4.19. (4.15) gives better result than (4.9). If we use the concavity of the function $h:[0,\infty)\to\mathbb{R}$, $h(x)=x^\lambda, 0<\lambda\leq 1$, we get

$$\begin{aligned}
&\frac{\delta-\lambda}{2}\left(\frac{1}{2}\right)^{2-\frac{2}{q}}\left(\frac{\left|S'\left(\lambda\right)\right|^q}{\sum_{i=1}^n a_i}\sum_{i=1}^n a_iM_1(i)+\frac{\left|S'\left(\delta\right)\right|^q}{\sum_{i=1}^n a_i}\sum_{i=1}^n a_iM_2(i)\right)^{\frac{1}{q}} \\
&+\frac{\delta-\lambda}{2}\left(\frac{1}{2}\right)^{2-\frac{2}{q}}\left(\frac{\left|S'\left(\lambda\right)\right|^q}{\sum_{i=1}^n a_i}\sum_{i=1}^n a_iM_2(i)+\frac{\left|S'\left(\delta\right)\right|^q}{\sum_{i=1}^n a_i}\sum_{i=1}^n a_iM_1(i)\right)^{\frac{1}{q}} \\
\leq\ & \frac{\delta-\lambda}{2}\left(\frac{1}{2}\right)^{1-\frac{2}{q}}\left(\frac{1}{\sum_{i=1}^n a_i}\sum_{i=1}^n a_i\frac{\left(i^2+i+2\right)2^i-2}{(i+1)(i+2)2^{i+1}}\right)^{\frac{1}{q}}A^{\frac{1}{q}}\left(\left|S'\left(\lambda\right)\right|^q,\left|S'\left(\delta\right)\right|^q\right),
\end{aligned}$$

where

$$M_1(i) + M_2(i) = \frac{(i^2+i+2)\,2^i - 2}{(i+1)(i+2)2^{i+1}},$$

which completes the proof of remark.

5. Applications for special means

Throughout this section, for shortness, the following notations will be used.

1. The arithmetic mean

$$A := A(\lambda,\delta) = \frac{\lambda+\delta}{2}, \quad \lambda,\delta \geq 0.$$

2. The geometric mean

$$G := G(\lambda,\delta) = \sqrt{\lambda\delta}, \quad \lambda,\delta \geq 0.$$

3. The harmonic mean

$$H := H(\lambda,\delta) = \frac{2\lambda\delta}{\lambda+\delta}, \quad \lambda,\delta > 0.$$

4. The logarithmic mean

$$L := L(\lambda,\delta) = \begin{cases} \frac{\delta-\lambda}{\ln\delta-\ln\lambda}, & \lambda\neq\delta \\ \lambda, & \lambda=\delta \end{cases} \; ; \quad \lambda,\delta > 0.$$

5. The p-logaritmic mean

$$L_p := L_p(\lambda,\delta) = \begin{cases} \left(\frac{\delta^{p+1}-\lambda^{p+1}}{(p+1)(\delta-\lambda)}\right)^{\frac{1}{p}}, & \lambda\neq\delta, p\in\mathbb{R}\setminus\{-1,0\} \\ \lambda, & \lambda=\delta \end{cases} \; ; \quad \lambda,\delta > 0.$$

6.The identric mean

$$I := I(\lambda,\delta) = \frac{1}{e}\left(\frac{\delta^\delta}{\lambda^\lambda}\right)^{\frac{1}{\delta-\lambda}}, \quad \lambda,\delta > 0.$$

It is known that L_p is monotonically increasing over $p\in\mathbb{R}$, denoting $L_0 = I$ and $L_{-1} = L$.

Proposition 5.1. *Let $\lambda,\delta \in [0,\infty)$ with $\lambda<\delta$ and $m \in (-\infty,0)\cup[1,\infty)\setminus\{-1\}$. Then, the following inequalities are obtained:*

$$\frac{1}{2}\left(\frac{\sum_{i=1}^n a_i}{\sum_{i=1}^n a_i\left[1-\left(\frac{1}{2}\right)^i\right]}\right)A^m(\lambda,\delta) \leq L_m^m(\lambda,\delta) \leq A(\lambda^m,\delta^m)\frac{2}{\sum_{i=1}^n a_i}\sum_{i=1}^n a_i\left(\frac{i}{i+1}\right).$$

Proof. The assertion follows from the inequalities (3.1) for the function

$$f(x) = x^m, \quad x\in[0,\infty).$$

□

Proposition 5.2. *Let $\lambda,\delta \in (0,\infty)$ with $\lambda<\delta$. Then, the following inequalities are obtained:*

$$\frac{1}{2}\left(\frac{\sum_{i=1}^n a_i}{\sum_{i=1}^n a_i\left[1-\left(\frac{1}{2}\right)^i\right]}\right)A^{-1}(\lambda,\delta) \leq L^{-1}(\lambda,\delta) \leq H^{-1}(\lambda,\delta)\frac{2}{\sum_{i=1}^n a_i}\sum_{i=1}^n a_i\left(\frac{i}{i+1}\right).$$

Proof. The assertion follows from the inequalities (3.1) for the function

$$f(x) = x^{-1}, \quad x\in(0,\infty).$$

□

Conclusion 1. In this paper, we give the generalization of the definition of n-polynomial convexity, which will appear for the first time in the literature and study some algebraic properties of this definition. We proved some new Hermite-Hadamard type integral inequalities for the generalized n-polynomial convex functions using an identity together with Hölder's integral inequality. Different types of integral inequalities can be obtained using this new definition.

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