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On the Solutions of Some Equations in (p,q)-Calculus

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Abstract

In this paper, we introduce the Laplace equation in (p,q)-calculus and give the solutions of the equation using the separation method into its variables. We also give the (p,q)-calculus version of the equation of motion, which expresses the displacement of a falling field in a resistant environment. Finally, we obtain the solution of the Bernoulli's equation in (p,q)-calculus.

Keywords: *(p,q)-calculus, Falling Field Problem, Laplace Equation, Bernoulli's Equation*
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1. Introduction

Although the q-calculus theory goes back to Euler in the 18 th century, it was gained to mathematics by Jackson [13] in 1910. Later, q-calculus was investigated by many scientists. It has various applications in different mathematical fields such as number theory, quantum theory and basic hyper-geometric functions [12, 13, 15]. It has been benefited by many researchers to solve physical problems. For example, the behavior of hydrogen atoms in terms of q-calculus was studied by the authors in [9]. The solutions of some q-wave and q-heat equations investigated by Bettaibi and Mezlini [4]. For further studies, see [6, 17, 21]. Similar to advances in q-calculus, the (p,q)-calculus theory developed rapidly in many disciplines such as mathematics and physics. While q-calculus deals with a q-variable, the (p,q)-calculus deals with two independent variables p and q. The (p,q)-calculus was firstly introduced in quantum algebras by the authors in [5]. Sadjang [20], systematically established the fundamental theory of (p,q)-calculus and some (p,q)-Taylor formulas. The (p,q)-analogue of Beta and Gama functions were given by Milovanovic at. al. [18]. The readers can see [11, 16, 19, 22].
Akça et al. [1] gave the solution of some partial differential equations in terms of q-calculus. One of them is the following equation

$$u_{xx}+u_{yy}+u_{zz}=0 \tag{1.1}$$

which is known as the 3-dimensional Laplace equation. The second order partial q-derivatives of the function u; u_{xx} , u_{yy} and u_{zz} is defined in a similar way (see [1]). Ebaid et. al. [2] studied the following equation in the sense of q-calculus

$$\frac{d_q v}{d_q t}=-g-kv,(q\in(0,1)). \tag{1.2}$$

which expresses the motion of a falling field in a resistive medium. In 2020, Salih and Sami [3] defined q-calculus version of Bernoulli's equation as follows

$$D_q y+\alpha(t)y=\beta(t)y^k \tag{1.3}$$

where $\alpha(t)$ and $\beta(t)$ are differentiable functions and $k\in\mathbb{R}$ and they solved the q-Bernoulli's equation obtained from the law of protection of energy which expresses that the total energy in an isolated system will not change.
The main goal of our paper is to introduce the (p,q)-Laplace equations, (p,q)-motion equations and (p,q)-Bernoulli's equations. This paper consists of five sections. In section 2, we will give some concepts and facts which will be used. In section 3, we define the (p,q)-calculus version of the three-dimensional partial differential equation which is called as the (p,q)-Laplace's equation. To solve this equation, we give definitions of second-order partial (p,q)-derivatives. Then, we solve the equation using the method of separating variables. In section 4, we define the equation of motion, called as the falling field problem, in terms of (p,q)-calculus. Assuming that the solutions of the equation are in the form of a series, we calculate the coefficients in the series. Using the calculated coefficients, we get the instantaneous velocity equation of the body. Moreover, we give the vertical distance of the body from the obtained velocity equation. In section 5, we define Bernoulli's equation in (p,q)-calculus and give the solution of the equation using the (p,q)-derivative definition.

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2. Prelimineries

We remind some essential notions of (p,q)-calculus.
For any number n, the (p,q)-like of n is given by

$$[n]_{p,q}=\frac{p^n-q^n}{p-q}; \quad (p\neq q.)$$

The (p,q)-factorial is defined as

$$[n]_{p,q}!:=\begin{cases}[1]_{p,q}\times[2]_{p,q}\times[3]_{p,q}\times\ldots\times[n]_{p,q}, & n\geq 1\\ 1, & n=0.\end{cases}$$

The (p,q)-derivative of a function $\lambda(t)$ is given by

$$D_{p,q}\lambda(t) = \frac{\lambda(pt)-\lambda(qt)}{(p-q)t}; (t\neq 0), (p\neq q) \tag{2.1}$$

and $(D_{p,q}\lambda)(0)=\lambda^{'}(0)$ provided that $\lambda(t)$ is differentiable at 0. From (2.1) we have

$$D_{p,q}t^n=[n]_{p,q}t^{n-1}.$$

Jagannathan and Rao [14] defined the (p,q)-exponential functions as

$$e_{p,q}(t)=\sum_{n=0}^{\infty}p^{\binom{n}{r}}\frac{(t)^n}{[n]_{p,q}!}$$

and

$$E_{p,q}(t)=\sum_{n=0}^{\infty}q^{\binom{n}{r}}\frac{(t)^n}{[n]_{p,q}!}.$$

The (p,q)-derivatives of the exponential functions are given by

$$D_{p,q}e_{p,q}(t)=e_{p,q}(pt)$$

and

$$D_{p,q}E_{p,q}(t)=E_{p,q}(qt).$$

Note that the (p,q)-derivatives of exponential functions are not themselves equal, as contrary to classical calculus. Uğur et. al. [7] defined a new type of (p,q)- exponential function as

$$\tilde{e}_{p,q}(t)=\sum_{n=0}^{\infty}\frac{(t)^n}{[n]_{p,q}!}.$$

If attention is paid, it is seen that the exponential function definition made in this way is similar to the definition of the classical exponential function $e^t=\sum_{n=0}^{\infty}\frac{(t)^n}{n!}$ and the q-exponential function $e_q(t)=\sum_{n=0}^{\infty}\frac{(t)^n}{[n]_q!}$. Also, the (p,q)-exponential function defined in this way is more useful and it has the following derivative property

$$D_{p,q}\tilde{e}_{p,q}(t)=\tilde{e}_{p,q}(t).$$

It is known that there is a connection of the definitions of derivative and definite integral in classical analysis. The connection between the mentioned concepts is given by the fundamental theorem of calculus, also called the Newton's-Leibniz formula. The q-calculus version of the fundamental theorem of calculus was given by Kac and Pokman [15]. Analogously to the ordinary and the q-case, the following foundational theorem was given by Sadjang [20].

Theorem 2.1. *[20] If $\Lambda(t)$ is an anti-derivative of $\lambda(t)$ and $\Lambda(t)$ is continuous at $t=0$, we have*

$$\int_c^d\lambda(t)d_{p,q}t=\Lambda(d)-\Lambda(c)$$

where $0\leq c<d\leq\infty$.

Corollary 2.2. *[20] If $\lambda^{'}(t)$ exists in a neighborhood of $t=0$ and is continuous at $t=0$, we have*

$$\int_c^d D_{p,q}\lambda(t)d_{p,q}t=\lambda(d)-\lambda(c)$$

where $\lambda^{'}(t)$ denotes the ordinary derivative of $\lambda(t)$.

3. The (p,q)-Laplace Equation

Imagine the following second-order three dimensional partial (p,q)-differential equation ((p,q)-Laplace Equation);

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=0. \tag{3.1}$$

Following the notation and definition from Akça [1], we define the second-order partial (p,q)-derivatives of $u(x,y,z)$ function:

$$\begin{aligned}
\frac{\partial^2 u}{\partial x^2} &= \frac{\frac{u(p^2x,y,z)-u(pqx,y,z)}{(p-q)px}-\frac{u(pqx,y,z)-u(q^2x,y,z)}{(p-q)qx}}{(p-q)x}\\
\frac{\partial^2 u}{\partial y^2} &= \frac{\frac{u(x,p^2y,z)-u(x,pqy,z)}{(p-q)py}-\frac{u(x,pqy,z)-u(x,q^2y,z)}{(p-q)qy}}{(p-q)y}\\
\frac{\partial^2 u}{\partial z^2} &= \frac{\frac{u(x,y,p^2z)-u(x,y,pqz)}{(p-q)pz}-\frac{u(x,y,pqz)-u(x,y,q^2z)}{(p-q)qz}}{(p-q)z}.
\end{aligned} \tag{3.2}$$

Substituting partial (p,q)-derivatives defined at (3.2) in (3.1), we obtain

$$\begin{aligned}
&y^2z^2[qu(p^2x,y,z)-(p+q)u(pqx,y,z)+pu(q^2x,y,z)]+x^2z^2[qu(x,p^2y,z)-(p+q)u(x,pqy,z)+pu(x,q^2y,z)]+x^2y^2[qu(x,y,p^2z)\\
&-(p+q)u(x,y,pqz)+pu(x,y,q^2z)]=0.
\end{aligned} \tag{3.3}$$

By rearranging and simplifying equation (3.3), we get

$$\begin{aligned}
&q[y^2z^2u(p^2x,y,z)+x^2z^2u(x,p^2y,z)+x^2y^2u(x,y,p^2z)]-(p+q)[y^2z^2u(pqx,y,z)+x^2z^2u(x,pqy,z)+x^2y^2u(x,y,pqz)]\\
&+p[y^2z^2u(q^2x,y,z)+x^2z^2u(x,q^2y,z)+x^2y^2u(x,y,q^2z)]=0.
\end{aligned} \tag{3.4}$$

Using the method of separation of variables, we will suppose that

$$\begin{aligned}
u(x,y,z) &= \lambda(x)\mu(y)\sigma(z)\\
u(p^2x,y,z) &= \lambda(p^2x)\mu(y)\sigma(z)\\
u(x,p^2y,z) &= \lambda(x)\mu(p^2y)\sigma(z)\\
u(x,y,p^2z) &= \lambda(x)\mu(y)\sigma(p^2z)\\
u(pqx,y,z) &= \lambda(pqx)\mu(y)\sigma(z)\\
u(x,pqy,z) &= \lambda(x)\mu(pqy)\sigma(z)\\
u(x,y,pqz) &= \lambda(x)\mu(y)\sigma(pqz)\\
u(q^2x,y,z) &= \lambda(q^2x)\mu(y)\sigma(z)\\
u(x,q^2y,z) &= \lambda(x)\mu(q^2y)\sigma(z)\\
u(x,y,q^2z) &= \lambda(x)\mu(y)\sigma(q^2z).
\end{aligned}$$

If we substitute these in (3.4) then we obtain

$$\begin{aligned}
q[y^2z^2\lambda(p^2x)\mu(y)\sigma(z)+x^2z^2\lambda(x)\mu(p^2y)\sigma(z)+x^2y^2\lambda(x)\mu(y)\sigma(p^2z)]-(p+q)\\
[y^2z^2\lambda(pqx)\mu(y)\sigma(z)+x^2z^2\lambda(x)\mu(pqy)\sigma(z)+x^2y^2\lambda(x)\mu(y)\sigma(pqz)]\\
+p[y^2z^2\lambda(q^2x)\mu(y)\sigma(z)+x^2z^2\lambda(x)\mu(q^2y)\sigma(z)+x^2y^2\lambda(x)\mu(y)\sigma(q^2z)]=0.
\end{aligned}$$

Now divide each term by $\lambda(x)\mu(y)\sigma(z)$, the equality takes the form

$$q\left[\frac{\lambda(p^2x)}{x^2\lambda(x)}+\frac{\mu(p^2y)}{y^2\mu(y)}+\frac{\sigma(p^2z)}{z^2\sigma(z)}\right]-(p+q)\left[\frac{\lambda(pqx)}{x^2\lambda(x)}+\frac{\mu(pqy)}{y^2\mu(y)}+\frac{\sigma(pqz)}{z^2\sigma(z)}\right]+p\left[\frac{\lambda(q^2x)}{x^2\lambda(x)}+\frac{\mu(q^2y)}{y^2\mu(y)}+\frac{\sigma(q^2z)}{z^2\sigma(z)}\right]=0. \tag{3.5}$$

Using the method in [1], let

$$
\begin{aligned}
&\lambda(x)=\alpha^{\log_p qx}\\
&\lambda(px)=\alpha\alpha^{\log_p qx}=\alpha\lambda(x)\\
&\lambda(p^2x)=\alpha^2\lambda(x)\\
&\lambda(pqx)=\alpha^{\log_p pq}\lambda(x)\\
&\lambda(q^2x)=\alpha^{\log_p q^2}\alpha^{\log_p qx}=\alpha^{2\log_p q}\lambda(x)\\
&\mu(y)=\alpha^{\log_p qy}\\
&\mu(py)=\alpha\alpha^{\log_p qy}=\alpha\mu(y)\\
&\mu(p^2y)=\alpha^2\mu(y)\\
&\mu(pqy)=\alpha^{\log_p pq}\mu(y)\\
&\mu(q^2y)=\alpha^{\log_p q^2}\alpha^{\log_p qy}=\alpha^{2\log_p q}\mu(y)\\
&\sigma(z)=\alpha^{\log_p qz}\\
&\sigma(pz)=\alpha\alpha^{\log_p qz}=\alpha\sigma(z)\\
&\sigma(p^2z)=\alpha^2\sigma(z)\\
&\sigma(pqz)=\alpha^{\log_p pq}\sigma(z)\\
&\sigma(q^2z)=\alpha^{\log_p q^2}\alpha^{\log_p qz}=\alpha^{2\log_p q}\sigma(z).
\end{aligned}
$$

Substituting these equalities in (3.5), we obtain the characteristic equation

$$q\alpha^2-(p+q)\alpha\alpha^{\log_p q}+p\alpha^{2\log_p q}=0. \tag{3.6}$$

Note that the characteristic equation obtained is different from that obtained in q-calculus (compare with [1]). Roots of (3.6) are $\alpha_1=p$ and $\alpha_2=1$. In this case the solutions are

$$\beta(x,y,z)=c_1p^{\log_p qx}+c_2p^{\log_p qy}+c_3p^{\log_p qz}$$

$$\gamma(x,y,z)=c_1+c_2+c_3.$$

4. The Falling Field Problem

Imagine that the body with a mass m is falling from a height of h from ground height first velocity υ_0. The falling equation of the field in the ordinary sense is defined by

$$m\frac{d\upsilon}{dt}=-mg-mk\upsilon, \tag{4.1}$$

with a non-negative fixed number k, [8, 10]. The beginning conditions are as follows

$$\begin{cases}\upsilon(0)=\upsilon_0\\ x(0)=h\end{cases} \tag{4.2}$$

where $x(t)$ is the ground clearance at time t and its $\frac{dx(t)}{dt}=\upsilon(t)$ velocity at time t. We define the equation of motion (4.1) in the (p,q)-calculus sense as follows

$$\frac{d_{p,q}\upsilon}{d_{p,q}t}=-g-k\upsilon;\quad (p,q\in(0,1)). \tag{4.3}$$

We accept that the solution of equation (4.3) is in form of the series:

$$\upsilon(t)=\sum_{n=0}^{\infty}c_nt^n. \tag{4.4}$$

From (4.3) and (4.4), we have the following equation

$$
\begin{aligned}
\frac{d_{p,q}\upsilon}{d_{p,q}t} &= \sum_{n=1}^{\infty}[n]_{p,q}c_nt^{n-1}\\
&= \sum_{n=0}^{\infty}[n+1]_{p,q}c_{n+1}t^n,
\end{aligned}
$$

where $[0]_{p,q}=0$. If we substitute $\frac{d_{p,q}\upsilon}{d_{p,q}t}$ and $\upsilon(t)$ in (4.3), we get

$$\sum_{n=0}^{\infty}[n+1]_{p,q}c_{n+1}t^n=-g-k\sum_{n=0}^{\infty}c_nt^n$$

and

$$[1]_{p,q}c_1+\sum_{n=1}^{\infty}[n+1]_{p,q}c_{n+1}t^n=-g-kc_0-k\sum_{n=1}^{\infty}c_nt^n$$

which gives

$$\begin{aligned} c_1 &= \frac{-g-kc_0}{[1]_{p,q}} \\ c_{n+1} &= \frac{-kc_n}{[n+1]_{p,q}}, \quad n\geq 1. \end{aligned} \tag{4.5}$$

From (4.5), we have

$$\begin{aligned} c_2 &= \frac{-kc_1}{[2]_{p,q}}=\frac{(-1)^2kg+(-k)^2c_0}{[1]_{p,q}[2]_{p,q}} \\ c_3 &= \frac{-kc_2}{[3]_{p,q}}=\frac{(-1)^3k^2g+(-k)^3c_0}{[1]_{p,q}[2]_{p,q}[3]_{p,q}} \\ c_4 &= \frac{-kc_3}{[4]_{p,q}}=\frac{(-1)^4k^3g+(-k)^4c_0}{[1]_{p,q}[2]_{p,q}[3]_{p,q}[4]_{p,q}} \\ &\vdots \\ c_n &= \frac{-kc_{n-1}}{[n]_{p,q}}=\frac{(-1)^nk^{n-1}g+(-k)^nc_0}{[1]_{p,q}[2]_{p,q}[3]_{p,q}...[n]_{p,q}}, \quad n\geq 1. \end{aligned}$$

Taking advantage of the fact that we can express the n-term coefficient (p,q)-factorially, we get

$$c_n=\frac{(-1)^nk^{n-1}g+(-k)^nc_0}{[n]_{p,q}!}, \quad n\geq 1. \tag{4.6}$$

Using (4.4) and (4.6), we obtain the momentary velocity as

$$\begin{aligned} \upsilon(t) &= c_0+\sum_{n=1}^{\infty}c_nt^n \\ &= c_0+\sum_{n=1}^{\infty}\left[\frac{k^{n-1}g(-1)^n+(-k)^nc_0}{[n]_{p,q}!}\right]t^n \\ &= c_0+\sum_{n=1}^{\infty}\left[\frac{(g/k)(-kt)^n+(-kt)^nc_0}{[n]_{p,q}!}\right]. \end{aligned}$$

When the instantaneous velocity obtained is regulated, we have

$$\upsilon(t)=c_0+(\frac{g}{k}+c_0)\sum_{n=1}^{\infty}\frac{(-kt)^n}{[n]_{p,q}!}.$$

By rewriting the instantaneous velocity in terms of the $e_{p,q}(-kt)$, we get

$$\upsilon(t)=c_0+(\frac{g}{k}+c_0)[e_{p,q}(-kt)-1].$$

Using the first initial condition in (4.2), we get $\upsilon_0=c_0$ and hence $\upsilon(t)$ occurs

$$\upsilon(t)=c_0+(\frac{g}{k}+\upsilon_0)[e_{p,q}(-kt)-1]. \tag{4.7}$$

If we rearrange (4.7), we obtain instantaneous velocity as

$$\upsilon(t)=-\frac{g}{k}+(\frac{g}{k}+\upsilon_0)e_{p,q}(-kt). \tag{4.8}$$

Since the velocity of the body is the change of its vertical height $x(t)$ with respect to time, we can rewrite equation (4.8) as

$$D_{p,q}x(t)=-\frac{g}{k}+(\frac{g}{k}+\upsilon_0)e_{p,q}(-kt). \tag{4.9}$$

Integrating (4.9), we get

$$\int_0^t D_{p,q}x(s)d_{p,q}s=\int_0^t(-\frac{g}{k})d_{p,q}s+(\frac{g}{k}+\upsilon_0)\int_0^t e_{p,q}(-ks)d_{p,q}s$$

and therefore,

$$x(t)-x(0)=-\frac{g}{k}[s]_0^t+(\frac{g}{k}+\upsilon_0)\left[-\frac{e_{p,q}(-ks)}{k}\right]_0^t$$

or

$$x(t)=h-\frac{gt}{k}+(\frac{g}{k}+\upsilon_0)(-\frac{e_{p,q}(-kt)}{k}+\frac{1}{k})$$

that is;

$$x(t)=h-\frac{gt}{k}+\frac{1}{k}(\frac{g}{k}+\upsilon_0)(1-e_{p,q}(-kt)). \tag{4.10}$$

Finally, (4.8) and (4.10) are exact solutions of the falling field problem and these solutions are as in ordinary Newtonian mechanics when $q \longrightarrow 1 = p$.

5. The (p,q)-Bernoulli's Equation

We define the (p,q)-like of the Bernoulli's equation as follows :

$$D_{p,q}y(t)+\alpha(t)y(t)=\beta(t)y^k(t);(k\in\mathbb{R}). \tag{5.1}$$

Theorem 5.1. *The solution of the (p,q)-Bernoulli's difference equation is as follows*

$$y(t)=y((\frac{q}{p})^jt)+\sum_{i=1}^{j}\frac{q^{i-1}}{p^i}t(p-q)[-\alpha(\frac{q^{i-1}}{p^i}t)y(\frac{q^{i-1}}{p^i}t)+\beta(\frac{q^{i-1}}{p^i}t)y^k(\frac{q^{i-1}}{p^i}t)]$$

where for $0<q<p<1$ and $p\neq q$.

Proof. If we use the (2.1) to solve the equation (5.1), we get the following equality

$$\frac{y(pt)-y(qt)}{(p-q)t}=-\alpha(t)y(t)+\beta(t)y^k(t).$$

$$\begin{aligned}
y(pt)-y(qt) &= -t(p-q)\alpha(t)y(t)+t(p-q)\beta(t)y^k(t)\\
y(t)-y(\frac{qt}{p}) &= -\frac{t}{p}(p-q)\alpha(\frac{t}{p})y(\frac{t}{p})+\frac{t}{p}(p-q)\beta(\frac{t}{p})y^k(\frac{t}{p})\\
y(\frac{qt}{p})-y((\frac{q}{p})^2t) &= -\frac{qt}{p^2}(p-q)\alpha(\frac{qt}{p^2})y(\frac{qt}{p^2})+\frac{qt}{p^2}(p-q)\beta(\frac{qt}{p^2})y^k(\frac{qt}{p^2})\\
y((\frac{q}{p})^2t)-y((\frac{q}{p})^3t) &= -\frac{q^2t}{p^3}(p-q)\alpha(\frac{q^2t}{p^3})y(\frac{q^2t}{p^3})+\frac{q^2t}{p^3}(p-q)\beta(\frac{q^2t}{p^3})y^k(\frac{q^2t}{p^3})\\
y((\frac{q}{p})^3t)-y((\frac{q}{p})^4t) &= -\frac{q^3t}{p^4}(p-q)\alpha(\frac{q^3t}{p^4})y(\frac{q^3t}{p^4})+\frac{q^3t}{p^4}(p-q)\beta(\frac{q^3t}{p^4})y^k(\frac{q^3t}{p^4})\\
&\vdots\\
y((\frac{q}{p})^{j-1}t)-y((\frac{q}{p})^jt) &= -\frac{q^{j-1}t}{p^j}(p-q)\alpha(\frac{q^{j-1}t}{p^j})y(\frac{q^{j-1}t}{p^j})+\frac{q^{j-1}t}{p^j}(p-q)\beta(\frac{q^{j-1}t}{p^j})y^k(\frac{q^{j-1}t}{p^j}).
\end{aligned}$$

Finally, if these equalities are additived side by side,

$$y(t)-y((\frac{q}{p})^jt)=\sum_{i=1}^{j}\frac{q^{i-1}}{p^i}t(p-q)[-\alpha(\frac{q^{i-1}}{p^i}t)y(\frac{q^{i-1}}{p^i}t)+\beta(\frac{q^{i-1}}{p^i}t)y^k(\frac{q^{i-1}}{p^i}t)]$$

and therefore

$$y(t)=y((\frac{q}{p})^jt)+\sum_{i=1}^{j}\frac{q^{i-1}}{p^i}t(p-q)[-\alpha(\frac{q^{i-1}}{p^i}t)y(\frac{q^{i-1}}{p^i}t)+\beta(\frac{q^{i-1}}{p^i}t)y^k(\frac{q^{i-1}}{p^i}t)]$$

is obtained. □

For $p=1$, equation reduces to the q-analogue Bernoulli's equation (see[3]).

Remark 5.2. *If we take $j\to\infty$, then we obtain that:*

$$y(t)=y(0)+\sum_{i=1}^{\infty}\frac{q^{i-1}}{p^i}t(p-q)[-\alpha(\frac{q^{i-1}}{p^i}t)y(\frac{q^{i-1}}{p^i}t)+\beta(\frac{q^{i-1}}{p^i}t)y^k(\frac{q^{i-1}}{p^i}t)].$$

6. Conclusion

In this article, the (p,q)-calculus theory has been used to solve the fallen field problem. The explicit solutions for the (p,q)-momentary velocity and the (p,q)-vertical path have been deduced. The deduced explicit solutions are denoted in nominal of the (p,q)-exponential function. Also (p,q)-calculus is applied for the Laplace equation and the equation is solved. Finally, the solution of Bernoulli's equation has been given in the (p,q)-calculus.

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