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Research Article

Non-linear mixed Jordan triple 1-$*$-product on von Neumann algebras

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Abstract

It is shown that if M and N are two von Neumann algebras, one of which has no central abelian projection with $\psi : M \to N$ satisfying mixed Jordan triple 1-$*$-product, i.e.,

$$\psi(A \circ B \bullet C) = \psi(A) \circ \psi(B) \bullet \psi(C)$$

for all $A, B, C \in M$, then there exists a bijective map $\Psi : M \to N$ such that $\Psi(A) = \psi(I)\psi(A)$ with $\psi(I)^2 = I$, whenever $\psi(I)$ is central, and there exist a central projection $\mathfrak{P} \in M$ such that the restriction of ψ to $M\mathfrak{P}$ is a linear $*$-isomorphism, and to $M(I - \mathfrak{P})$ is a conjugate linear $*$-isomorphism.

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1. Notations and introduction

Let M be a von Neumann algebra and $A, B \in M$. We express $A \bullet_\lambda B = AB + \lambda BA^*$, the Jordan λ-$*$-product. For $\lambda = \pm 1$, we say Jordan 1-$*$-product and Jordan (-1)-$*$-product, respectively. Traditionally, numerous algebraists were already committed to analyse those mappings that aren't necessarily additive preserved Jordan $*$-products on various algebras. The study of non-linear preserving problems is one of the premier areas in matrix theory as well as operator theory. A variety of research objectives on certain algebras such as von Neumann algebras, operator algebras, prime $*$-algebras, etc were discussed in depth [2, 3, 7–11, 14–16] and references therein. The first implementation of this theory was presented by Šemrl [17]. In addition, with the relation to quadratic functionals, the Jordan (-1)-$*$-product was introduced and studied by him. In [1], Bai and Du revealed that the sum of linear and conjugate linear $*$-isomorphisms would be any bijective map on von Neumann algebras without central abelian projections, which preserved the Jordan (-1)-$*$-product. Quite few generalizations throughout the last result can be found [4, 6, 7, 11] done by plenty of authors.

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Throughout this line of questioning, recently Huo *et al.* [5] extended the above-mentioned interpellation for Jordan triple $\eta-$product. Specifically, he stated that: "Assume that ψ is a bijection between two von Neumann algebras that is not necessarily linear, of which one has abelian projections which is not central with $\psi(I)=I$ and having the Jordan triple η-$*$-product. If η is not real, then ψ is a linear $*$-isomorphism, and if η is real, then ψ is the sum of a linear $*$-isomorphism and a conjugate linear $*$-isomorphism". Additionally, they also addressed a conjecture that whether this result is relevant without $\psi(I)=I$. In 2017, Li and Lu [8] provided the affirmative response to this problem and developed the consequence on von Neumann algebras for Jordan's triple 1-$*$-product, of which one has abelian projections which is not central. In this article, we also provide a constructive response to the above problem but not only dismantle the presumption of $\psi(I)=I$, we demonstrate the result in a somewhat broader sense by considering mixed Jordan 1-$*$-product which is defined as for any $A,B,C\in M$,

$$A\circ B\bullet C=(AB+BA)\bullet C=ABC+BAC+CB^*A^*+CA^*B^*.$$

Within this manuscript, we are primarily interested in exploring how non-linear maps are formed on von Neumann algebras satisfying mixed Jordan triple 1-$*$-product i.e., $\psi(A\circ B\bullet C)=\psi(A)\circ\psi(B)\bullet\psi(C)$ for all $A,B,C\in M$. Over few years some significant work drawn an attention of researchers has been consecrated to the evaluation of mixed Lie and Jordan triple products and derivations ([12, 18–20]). Such studies reported above encourage us to prove the following:

Theorem 1.1. *Let M and N be two von Neumann algebras, one of which has no central abelian projection. Define a map $\psi: M\to N$ such that*

$$\psi(A\circ B\bullet C)=\psi(A)\circ\psi(B)\bullet\psi(C)$$

for all $A,B,C\in M$. If $\psi(I)$ is central, then there exists a bijective map $\Psi: M\to N$ such that $\Psi(A)=\psi(I)\psi(A)$ with $\psi(I)^2=I$ and there exsits a central projection $\mathfrak{P}\in M$ such that the restriction of ψ to $M\mathfrak{P}$ is a linear $*$-isomorphism and the restriction of ψ to $M(I-\mathfrak{P})$ is a conjugate linear $*$-isomorphism.

We systematize the proof of aforementioned result in two sections. Section 2 presents some preliminary notions and useful lemmas that are essential to show ψ is additive. In Section 3, we shall provide numerous constructive remarks and lemmas to elaborate the essertion of Theorem 1.1.

2. Additivity of ψ

Theorem 2.1. *Let M and N be two von Neumann algebras and define a bijective map $\psi: M\to N$ such that*

$$\psi(A\circ B\bullet C)=\psi(A)\circ\psi(B)\bullet\psi(C)$$

for all $A,B,C\in M$. Then ψ is additive.

Proof. Take into account that $\mathfrak{P}_1\in M$ and $\mathfrak{P}_2=I-\mathfrak{P}_1$ are projections, whereas I is an unit element of M. We write $M_{jk}=\mathfrak{P}_jM\mathfrak{P}_k$ for $j,k=1,2$. Then by Peire's decomposition of M, we have $M=M_{11}\oplus M_{12}\oplus M_{21}\oplus M_{22}$. It should be noted that any operator $A\in M$ can be written as $A=A_{11}+A_{12}+A_{21}+A_{22}$.

In view of the approximately facts, the verification of the theorem is given within the presentation of the following lemmas:

Lemma 2.2. $\psi(0)=0$.

Proof. Due to ψ being surjective, there is $A\in M$ such that $\psi(A)=0$. Thus

$$\psi(0)=\psi(0\circ 0\bullet A)=\psi(0)\circ\psi(0)\bullet\psi(A)=0.$$

□

Lemma 2.3. *Let $A_{12} \in M_{12}$ and $A_{21} \in M_{21}$. Then $\psi(A_{12}+A_{21}) = \psi(A_{12}) + \psi(A_{21})$.*

Proof. Let $\Phi = (A_{12}+A_{21}) - \psi^{-1}(\psi(A_{12}) + \psi(A_{21}))$. Then, we have

$$\begin{aligned}\psi(A_{12}+A_{21}) \circ \psi(\mathfrak{P}_2) \bullet \psi(\mathfrak{P}_1) &= \psi((A_{12}+A_{21}) \circ \mathfrak{P}_2 \bullet \mathfrak{P}_1)\\ &= \psi(A_{12} \circ \mathfrak{P}_2 \bullet \mathfrak{P}_1) + \psi(A_{21} \circ \mathfrak{P}_2 \bullet \mathfrak{P}_1)\\ &= \psi(A_{12}) \circ \psi(\mathfrak{P}_2) \bullet \psi(\mathfrak{P}_1) + \psi(A_{21}) \circ \psi(\mathfrak{P}_2) \bullet \psi(\mathfrak{P}_1)\\ &= (\psi(A_{12}) + \psi(A_{21})) \circ \psi(\mathfrak{P}_2) \bullet \psi(\mathfrak{P}_1).\end{aligned}$$

Apply ψ^{-1} on both sides of above expression. This gives $\Phi \circ \mathfrak{P}_2 \bullet \mathfrak{P}_1 = 0$, which yields $\Phi_{21} = 0$. Similarly, we can show that $\Phi_{12} = 0$ by replacing $\mathfrak{P}_2$ by $\mathfrak{P}_1$ and $\mathfrak{P}_1$ by $\mathfrak{P}_2$, respectively. Next, we have

$$\begin{aligned}\psi(I) \circ \psi(\mathfrak{P}_1 - \mathfrak{P}_2) \bullet \psi(A_{12}+A_{21}) &= \psi(I \circ (\mathfrak{P}_1 - \mathfrak{P}_2) \bullet (A_{12}+A_{21}))\\ &= \psi(I \circ (\mathfrak{P}_1 - \mathfrak{P}_2) \bullet A_{12}) + \psi(I \circ (\mathfrak{P}_1 - \mathfrak{P}_2) \bullet A_{21})\\ &= \psi(I) \circ \psi(\mathfrak{P}_1 - \mathfrak{P}_2) \bullet \psi(A_{12})\\ &\quad + \psi(I) \circ \psi(\mathfrak{P}_1 - \mathfrak{P}_2) \bullet \psi(A_{21})\\ &= \psi(I) \circ \psi(\mathfrak{P}_1 - \mathfrak{P}_2) \bullet (\psi(A_{12}) + \psi(A_{21})).\end{aligned}$$

Again, impose ψ^{-1} in last relation, we get $I \circ (\mathfrak{P}_1 - \mathfrak{P}_2) \bullet \Phi = 0$. This further implies $\Phi_{11} = \Phi_{22} = 0$. Thus $\Phi = 0$ i.e.,

$$\psi(A_{12}+A_{21}) = \psi(A_{12}) + \psi(A_{21}).$$

□

Lemma 2.4. *For any $A_{11} \in M_{11}, A_{12} \in M_{12}$ and $A_{21} \in M_{21}$,*

(i) $\psi(A_{11}+A_{12}+A_{21}) = \psi(A_{11}) + \psi(A_{12}) + \psi(A_{21})$;
(ii) $\psi(A_{12}+A_{21}+A_{22}) = \psi(A_{12}) + \psi(A_{21}) + \psi(A_{22})$.

Proof. Let $\Theta = (A_{11}+A_{12}+A_{21}) - \psi^{-1}(\psi(A_{11}) + \psi(A_{12}) + \psi(A_{21}))$. Then by Lemma 2.3, we have

$$\begin{aligned}\psi(A_{11}+A_{12}+A_{21}) \circ \psi(\mathfrak{P}_1) \bullet \psi(\mathfrak{P}_2) &= \psi((A_{11}+A_{12}+A_{21}) \circ \mathfrak{P}_1 \bullet \mathfrak{P}_2)\\ &= \psi(A_{11} \circ \mathfrak{P}_1 \bullet \mathfrak{P}_2) + \psi(A_{12} \circ \mathfrak{P}_1 \bullet \mathfrak{P}_2)\\ &\quad + \psi(A_{21} \circ \mathfrak{P}_1 \bullet \mathfrak{P}_2)\\ &= \psi(A_{11}) \circ \psi(\mathfrak{P}_1) \bullet \psi(\mathfrak{P}_2) + \psi(A_{12}) \circ \psi(\mathfrak{P}_1)\\ &\quad \bullet \psi(\mathfrak{P}_2) + \psi(A_{21}) \circ \psi(\mathfrak{P}_1) \bullet \psi(\mathfrak{P}_2)\\ &= (\psi(A_{11}) + \psi(A_{12}) + \psi(A_{21})) \circ \psi(\mathfrak{P}_1) \bullet \psi(\mathfrak{P}_2).\end{aligned}$$

The last expression yields $\Theta \circ \mathfrak{P}_1 \bullet \mathfrak{P}_2 = 0$, and hence $\Theta_{12} = 0$. Similarly, we can get $\Theta_{21} = 0$. Now, we only need to show $\Theta_{11} = \Theta_{22} = 0$. It follows from the hypothesis and

Lemma 2.3 that

$$\begin{aligned}
\psi\left(\frac{I}{2}\right) &\circ \psi(\mathfrak{P}_1 - \mathfrak{P}_2) \bullet \psi(A_{11} + A_{12} + A_{21}) \\
&= \psi\left(\frac{I}{2} \circ (\mathfrak{P}_1 - \mathfrak{P}_2) \bullet (A_{11} + A_{12} + A_{21})\right) \\
&= \psi\left(\frac{I}{2} \circ (\mathfrak{P}_1 - \mathfrak{P}_2) \bullet A_{11}\right) + \psi\left(\frac{I}{2} \circ (\mathfrak{P}_1 - \mathfrak{P}_2) \bullet A_{12}\right) \\
&\quad + \psi\left(\frac{I}{2} \circ (\mathfrak{P}_1 - \mathfrak{P}_2) \bullet A_{21}\right) \\
&= \psi\left(\frac{I}{2}\right) \circ \psi\left(\mathfrak{P}_1 - \mathfrak{P}_2\right) \bullet \psi\left(A_{11}\right) + \psi(\frac{I}{2}) \circ \psi(\mathfrak{P}_1 - \mathfrak{P}_2) \bullet \psi(A_{12}) \\
&\quad + \psi\left(\frac{I}{2}\right) \circ \psi(\mathfrak{P}_1 - \mathfrak{P}_2) \bullet \psi(A_{21}) \\
&= \psi\left(\frac{I}{2}\right) \circ \psi(\mathfrak{P}_1 - \mathfrak{P}_2) \bullet (\psi(A_{11}) + \psi(A_{12}) + \psi(A_{21})).
\end{aligned}$$

Reasoning as above, we obtain $\Theta_{11} = \Theta_{22} = 0$, and hence

$$\psi(A_{11} + A_{12} + A_{21}) = \psi(A_{11}) + \psi(A_{12}) + \psi(A_{21}).$$

Similarly, we can show

$$\psi(A_{12} + A_{21} + A_{22}) = \psi(A_{12}) + \psi(A_{21}) + \psi(A_{22}).$$

This completes the proof. □

Lemma 2.5. *For any $A_{ij} \in M_{ij}$, $1 \leq i, j \leq 2$, we have*

$$\psi(\sum_{i,j=1}^{2} A_{ij}) = \sum_{i,j=1}^{2} \psi(A_{ij}).$$

Proof. Assume that $\nabla = \sum_{i,j=1}^{2} A_{ij} - \psi^{-1}(\sum_{i,j=1}^{2} \psi(A_{ij}))$. In view of Lemma 2.4($i$) and $\mathfrak{P}_1 \circ I \bullet A_{22} = 0$, we have

$$\begin{aligned}
\psi(\mathfrak{P}_1) \circ \psi(I) \bullet \psi(\sum_{i,j=1}^{2} A_{ij}) &= \psi(\mathfrak{P}_1 \circ I \bullet \sum_{i,j=1}^{2} A_{ij}) \\
&= \psi(\mathfrak{P}_1 \circ I \bullet A_{11}) + \psi(\mathfrak{P}_1 \circ I \bullet A_{12}) \\
&\quad + \psi(\mathfrak{P}_1 \circ I \bullet A_{21}) + \psi(\mathfrak{P}_1 \circ I \bullet A_{22}) \\
&= \psi(\mathfrak{P}_1) \circ \psi(I) \bullet \psi(A_{11}) + \psi(\mathfrak{P}_1) \circ \psi(I) \bullet \psi(A_{12}) \\
&\quad + \psi(\mathfrak{P}_1) \circ \psi(I) \bullet \psi(A_{21}) + \psi(\mathfrak{P}_1) \circ \psi(I) \bullet \psi(A_{22}) \\
&= \psi(\mathfrak{P}_1) \circ \psi(I) \bullet \sum_{i,j=1}^{2} \psi(A_{ij}).
\end{aligned}$$

Apply ψ^{-1} on both sides of above expression which yields $\mathfrak{P}_1 \circ I \bullet \nabla = 0$, and hence $\nabla_{11} = \nabla_{12} = \nabla_{21} = 0$. We can show in similar manner that $\nabla_{22} = 0$. Thus $\nabla = 0$ i.e.,

$$\psi(\sum_{i,j=1}^{2} A_{ij}) = \sum_{i,j=1}^{2} \psi(A_{ij}).$$

□

Lemma 2.6. *For any $A_{ij}, B_{ij} \in M_{ij}$ with $i \neq j$, $\psi(A_{ij} + B_{ij}) = \psi(A_{ij}) + \psi(B_{ij})$.*

Proof. Since $\frac{I}{2} \circ (\mathfrak{P}_i + A_{ij}) \bullet (\mathfrak{P}_j + B_{ij}) = A_{ij} + B_{ij} + A_{ij}^* + B_{ij}A_{ij}^*$, so it follows from Lemma 2.4 that

$$\begin{aligned}
\psi(A_{ij} + B_{ij}) + \psi(A_{ij}^*) + \psi(B_{ij}A_{ij}^*) &= \psi\left(\frac{I}{2} \circ (\mathfrak{P}_i + A_{ij}) \bullet (\mathfrak{P}_j + B_{ij})\right) \\
&= \psi\left(\frac{I}{2}\right) \circ \psi(\mathfrak{P}_i + A_{ij}) \bullet \psi(\mathfrak{P}_j + B_{ij}) \\
&= \psi\left(\frac{I}{2}\right) \circ (\psi(\mathfrak{P}_i) + \psi(A_{ij})) \bullet (\psi(\mathfrak{P}_j) + \psi(B_{ij})) \\
&= \psi\left(\frac{I}{2}\right) \circ \psi(\mathfrak{P}_i) \bullet \psi(\mathfrak{P}_j) + \psi(\frac{I}{2}) \circ \psi(\mathfrak{P}_i) \bullet \psi(B_{ij}) \\
&\quad + \psi\left(\frac{I}{2}\right) \circ \psi(A_{ij}) \bullet \psi(\mathfrak{P}_j) \\
&\quad + \psi(\frac{I}{2}) \circ \psi(A_{ij}) \bullet \psi(B_{ij}) \\
&= \psi\left(\frac{I}{2} \circ \mathfrak{P}_i \bullet \mathfrak{P}_j\right) + \psi\left(\frac{I}{2}) \circ \mathfrak{P}_i \bullet B_{ij}\right) \\
&\quad + \psi\left(\frac{I}{2} \circ A_{ij} \bullet \mathfrak{P}_j\right) + \psi\left(\frac{I}{2} \circ A_{ij} \bullet B_{ij}\right) \\
&= \psi(B_{ij}) + \psi(A_{ij} + A_{ij}^*) + \psi(B_{ij}A_{ij}^*) \\
&= \psi(A_{ij}) + \psi(B_{ij}) + .\psi(A_{ij}^*) + \psi(B_{ij}A_{ij}^*).
\end{aligned}$$

Thus

$$\psi(A_{ij} + B_{ij}) = \psi(A_{ij}) + \psi(B_{ij}).$$

□

Lemma 2.7. *For any* $A_{ii}, B_{ii} \in M_{ii}$, $\psi(A_{ii} + B_{ii}) = \psi(A_{ii}) + \psi(B_{ii})$.

Proof. Suppose $\Pi = (A_{ii} + B_{ii}) - \psi^{-1}(\psi(A_{ii}) + \psi(B_{ii}))$. It is easy to find

$$\begin{aligned}
\psi(\mathfrak{P}_j) \circ \psi(I) \bullet \psi(A_{ii} + B_{ii}) &= \psi(\mathfrak{P}_j \circ I \bullet (A_{ii} + B_{ii})) \\
&= \psi(\mathfrak{P}_j \circ I \bullet A_{ii}) + \psi(\mathfrak{P}_j \circ I \bullet B_{ii}) \\
&= \psi(\mathfrak{P}_j) \circ \psi(I) \bullet \psi(A_{ii}) + \psi(\mathfrak{P}_j) \circ \psi(I) \bullet \psi(B_{ii}) \\
&= \psi(\mathfrak{P}_j) \circ (\psi(A_{ii}) + \psi(B_{ii})) \bullet \psi(\mathfrak{P}_j).
\end{aligned}$$

From above, we have $\mathfrak{P}_j \circ I \bullet \Pi = 0$. This yields $\Pi_{ij} = \Pi_{ji} = \Pi_{jj} = 0$. Next, according to Lemma 2.5 and Lemma 2.6, for any $C_{ij} \in M_{ij}$ with $i \neq j$, we have

$$\begin{aligned}
\psi(\mathfrak{P}_i \circ (A_{ii} + B_{ii}) \bullet C_{ij}) &= \psi(A_{ii}C_{ij} + A_{ii}C_{ij} + B_{ii}C_{ij} + B_{ii}C_{ij}) \\
&= \psi(A_{ii}C_{ij} + A_{ii}C_{ij}) + \psi(B_{ii}C_{ij} + B_{ii}C_{ij}) \\
&= \psi(\mathfrak{P}_i \circ A_{ii} \bullet C_{ij}) + \psi(\mathfrak{P}_i \circ B_{ii} \bullet C_{ij}) \\
&= \psi(\mathfrak{P}_i) \circ \psi(A_{ii}) \bullet \psi(C_{ij}) + \psi(\mathfrak{P}_i) \circ \psi(B_{ii}) \bullet \psi(C_{ij}) \\
&= \psi(\mathfrak{P}_i) \circ (\psi(A_{ii}) + \psi(B_{ii})) \bullet \psi(C_{ij})
\end{aligned}$$

On the other hand,

$$\psi(\mathfrak{P}_i \circ (A_{ii} + B_{ii}) \bullet C_{ij}) = \psi(\mathfrak{P}_i) \circ \psi(A_{ii} + B_{ii}) \bullet \psi(C_{ij}).$$

Hence $\mathfrak{P}_i \circ \Pi \bullet C_{ij} = 0$. This gives $\Pi_{ii} = 0$. Thus $\Pi = 0$ i.e.,

$$\psi(A_{ii} + B_{ii}) = \psi(A_{ii}) + \psi(B_{ii}).$$

□

Lemma 2.8. ψ *is an additive map.*

Proof. It follows from Lemmas 2.2-2.7 that ψ is additive. □

3. Proof of Theorem 1.1

While we proceed upon our scientific findings, we could present some information, including some few conceptual details. The unital von Neumann algebra M is indeed a weakly closed, self-adjoint operator algebra in the Hilbert space H. The collection $Z(M) = \{S \in M : ST = TS \text{ for all } T \in M\}$ is referred as the center of M. If $\mathfrak{P} \in Z(M)$ or $\mathfrak{P}M\mathfrak{P}$ abelian, then the projection $\mathfrak{P}$ is called the central abelian projection. Aware that perhaps the central carrier of A, denoted by $\overline{A}$, seems to be the smallest $\mathfrak{P}$ central projection that meets $\mathfrak{P}A = A$. The central carrier of A can be viewed as the projection onto the closed subspace spanned by $\{BA(x) : B \in M, x \in H\}$. The core of A, denoted by $\underline{A}$, is $\sup\{S \in Z(M) : S = S^*, S \leq A\}$, if A is self-adjoint. If $\mathfrak{P}$ is a projection, then it is obvious that $\underline{\mathfrak{P}}$ is the largest central Q projection that satisfies $Q \leq \mathfrak{P}$. If $\underline{\mathfrak{P}} = 0$, then the projection $\mathfrak{P}$ is said to be core-free. It is straightforward to see it now $\underline{\mathfrak{P}} = 0$ if and only if $\overline{(I - \mathfrak{P})} = I$. Following remarks are critical for the proof of our main result:

Remark 3.1. [13, Lemma 4] "If M is a von Neumann algebra with no central abelian projection $\mathfrak{P} \in M$, then there exists a projection $\mathfrak{P} \in M$ such that $\underline{\mathfrak{P}} = 0$ and $\overline{\mathfrak{P}} = I$."

Remark 3.2. [8, Lemma 2.2] "Let M be a von Neumann algebra on a Hilbert space H. Let A be an operator in M and $\mathfrak{P} \in M$ is a projection with $\overline{\mathfrak{P}} = I$. If $AB\mathfrak{P} = 0$ for all $B \in M$, then $A = 0$. Consequently, if $\mathfrak{Z} \in Z(M)$, then $\mathfrak{Z}\mathfrak{P} = 0$ implies $\mathfrak{Z} = 0$."

Remark 3.3. [8, Lemma 2.3] "Let M be a von Neumann algebra and $A \in M$. Then $AB + BA^* = 0$ for all $B \in M$ implies that $A = -A^* \in Z(M)$."

We can see from Theorem 2.1, ψ would be an additive map. Throughout the succeeding arguments, the unit elements of the algebra M and N weren't differentiated and we'll see within next proof that this does not impact our argument. We'll demonstrate that theorem progressively through implementing:

Lemma 3.4. $2I = \psi(I)^2 + (\psi(I)^*)^2$.

Proof. Let $A \in M$ such that $\psi(A) = I$. Then it follows from the additivity of ψ that

$$4I = 4\psi(A) = \psi(I \circ I \bullet A) = \psi(I) \circ \psi(I) \bullet I = 2(\psi(I)^2 + (\psi(I)^*)^2). \tag{3.1}$$

This implies $2I = \psi(I)^2 + (\psi(I)^*)^2$. ☐

Lemma 3.5. *Let* $\Theta_{\mathfrak{P}} = \frac{1}{2}(\psi(I)\psi(\mathfrak{P}) + \psi(I)^*\psi(\mathfrak{P})^*)$, *where* $\mathfrak{P} \in M$ *is a projection. Then* $\Theta_{\mathfrak{P}}$ *is a projection such that* $\psi(\mathfrak{P}) = \psi(I)\Theta_{\mathfrak{P}}$.

Proof. From the hypothesis, we have

$$\begin{aligned} 4\psi(\mathfrak{P}) &= \psi(I \circ \mathfrak{P} \bullet I) \\ &= \psi(I) \circ \psi(\mathfrak{P}) \bullet \psi(I) \\ &= 2\psi(I)\psi(\mathfrak{P}) \bullet \psi(I) \\ &= 2\psi(I)(\psi(I)\psi(\mathfrak{P}) + \psi(I)^*\psi(\mathfrak{P})^*) \\ &= 4\psi(I)\Theta_{\mathfrak{P}}. \end{aligned}$$

Also

$$\begin{aligned} 4\psi(\mathfrak{P}) &= \psi(I \circ \mathfrak{P} \bullet \mathfrak{P}) \\ &= \psi(I) \circ \psi(\mathfrak{P}) \bullet \psi(\mathfrak{P}) \\ &= 2\psi(I)\psi(\mathfrak{P}) \bullet \psi(\mathfrak{P}) \\ &= 2\psi(\mathfrak{P})^2\psi(I) + 2\psi(\mathfrak{P})\psi(\mathfrak{P})^*\psi(I)^* \\ &= 2\psi(\mathfrak{P})(\psi(I)\psi(\mathfrak{P}) + \psi(I)^*\psi(\mathfrak{P})^*) \\ &= 4\psi(\mathfrak{P})\Theta_{\mathfrak{P}} = 4\psi(I)\Theta_{\mathfrak{P}}^2. \end{aligned}$$

It follows from the last two relations that

$$\psi(\mathfrak{P}) = \psi(I)\Theta_{\mathfrak{P}}^2. \tag{3.2}$$

Since $\Theta_{\mathfrak{P}}$ is self-adjoint, so we have

$$\psi(\mathfrak{P})^* = \psi(I)^*\Theta_{\mathfrak{P}}^2. \tag{3.3}$$

Multiply (3.2) by $\psi(I)$ and (3.3) by $\psi(I)^*$, and add the so obtained relations, we get

$$\psi(I)\psi(\mathfrak{P}) + \psi(I)^*\psi(\mathfrak{P})^* = (\psi(I)^2 + (\psi(I)^*)^2)\Theta_{\mathfrak{P}}^2. \tag{3.4}$$

Observe from Lemma 3.4 that $2\Theta_{\mathfrak{P}} = 2\Theta_{\mathfrak{P}}^2$. Therefore $\Theta_{\mathfrak{P}}$ is a projection. □

Lemma 3.6. *For any $A_{12} \in M_{12}$ and a projection $\mathfrak{P} \in M$, $\psi(A_{12}) = \Theta_{\mathfrak{P}}\psi(A_{12}) + \psi(A_{12})\Theta_{\mathfrak{P}}$.*

Proof. It follows from Lemma 3.5 that

$$\begin{aligned} 2\psi(A_{12}) &= \psi(I \circ \mathfrak{P} \bullet A_{12}) \\ &= \psi(I) \circ \psi(\mathfrak{P}) \bullet \psi(A_{12}) \\ &= 2\psi(I)\psi(\mathfrak{P})\psi(A_{12}) + 2\psi(A_{12})\psi(I)^*\psi(\mathfrak{P})^* \\ &= 2\psi(I)^2\Theta_{\mathfrak{P}}\psi(A_{12}) + 2(\psi(I)^2)^*\psi(A_{12})^*\Theta_{\mathfrak{P}}. \end{aligned}$$

Multiply both sides of above equation by $\Theta_{\mathfrak{P}}$, we get $\Theta_{\mathfrak{P}}\psi(A_{12})\Theta_{\mathfrak{P}} = 0$. Similarly, if we multply above expression from left and right by $I - \Theta_{\mathfrak{P}}$, we obtain $(I - \Theta_{\mathfrak{P}})\psi(A_{12})(I - \Theta_{\mathfrak{P}}) = 0$. Therefore,

$$\psi(A_{12}) = \Theta_{\mathfrak{P}}\psi(A_{12}) + \psi(A_{12})\Theta_{\mathfrak{P}}.$$

Hence the proof. □

Lemma 3.7. $\psi(I)^2 = I$.

Proof. By the hypothesis, we can choose a projection $\mathfrak{P}' \in N$, where N has no central abelian projection, such that $\underline{\mathfrak{P}'} = 0$ and $\overline{\mathfrak{P}'} = I$. Let $\mathcal{N} \in N$ such that $\mathcal{N} = \mathfrak{P}'\mathcal{N}(I - \mathfrak{P}')$. Assume that $\mathfrak{P} = \frac{1}{2}(\psi^{-1}(I)\psi^{-1}(\mathfrak{P}') + \psi^{-1}(I)^*\psi^{-1}(\mathfrak{P}')^*)$. It is clear from Lemma 3.5 and 3.6 that $\mathfrak{P}$ is a projection and $\psi^{-1}(\mathcal{N}) = \mathfrak{P}\psi^{-1}(\mathcal{N}) + \psi^{-1}(\mathcal{N})\mathfrak{P}$. Now, it follows that

$$\begin{aligned} \psi(\mathfrak{P}) &= \frac{1}{2}\psi(\psi^{-1}(I)\psi^{-1}(\mathfrak{P}') + \psi^{-1}(I)^*\psi^{-1}(\mathfrak{P}')^*) \\ &= \frac{1}{4}\psi(\psi^{-1}(I) \circ \psi^{-1}(\mathfrak{P}') \bullet I) \\ &= \frac{1}{4}(I \circ \mathfrak{P}' \bullet \psi(I)) = \psi(I)\mathfrak{P}'. \end{aligned}$$

Also, we have

$$\begin{aligned} \mathcal{N} &= \psi(\mathfrak{P}\psi^{-1}(\mathcal{N}) + \psi^{-1}(\mathcal{N})\mathfrak{P}) \\ &= \frac{1}{2}\psi(I \circ \mathfrak{P} \bullet \psi^{-1}(\mathcal{N})) \\ &= \frac{1}{2}(\psi(I) \circ \psi(\mathfrak{P}) \bullet \mathcal{N}) \\ &= \psi(I)\psi(\mathfrak{P})\mathcal{N} + \psi(I)^*\mathcal{N}\psi(\mathfrak{P})^* \\ &= \psi(I)^2\mathfrak{P}'\mathcal{N} + (\psi(I)^*)^2\mathcal{N}\mathfrak{P}' \\ &= \psi(I)^2\mathcal{N}. \end{aligned}$$

This gives $(I - \psi(I)^2)\mathcal{N} = 0$. Thus, $(I - \psi(I)^2)\mathfrak{P}'\mathcal{N}(I - \mathfrak{P}') = 0$ and since $\overline{(I - \mathfrak{P}')} = I$, in view of Remark 3.1, we obtain $(I - \psi(I)^2)\mathfrak{P}' = 0$. Note that $(I - \psi(I)^2) \in Z(N)$ and $\overline{\mathfrak{P}'} = I$, then again by Remark 3.1, we have $I - \psi(I)^2 = 0$ i.e., $\psi(I)^2 = I$. This completes the proof. □

Now, for any $A \in M$, define a map $\Psi : M \to N$ by $\Psi(A) = \psi(I)\psi(A)$. Then Ψ has the following characteristics:

Lemma 3.8. (a) Ψ *is an additive bijective map satisfying*

$$\Psi(A \circ B \bullet C) = \Psi(A) \circ \Psi(B) \bullet \Psi(C) \quad \text{for all} \quad A, B, C \in M;$$

(b) $\Psi(I) = I$;
(c) $\Psi(A^*) = \Psi(A)^*$ *for all* $A \in M$;
(d) $\mathfrak{P}$ *is a projection in* M *iff* $\Psi(\mathfrak{P})$ *is a projection in* N.

Proof. For any $A, B \in M$, we have

$$\Psi(A + B) = \psi(I)\psi(A + B) = \psi(I)\psi(A) + \psi(I)\psi(B) = \Psi(A) + \Psi(B).$$

On the other hand, for any $A, B, C \in M$, we have

$$\begin{aligned}
\Psi(A \circ B \bullet C) &= \psi(I)\psi(A \circ B \bullet C) \\
&= \psi(I)(\psi(A) \circ \psi(B) \bullet \psi(C)) \\
&= \psi(I)((\psi(A)\psi(B) + \psi(B)\psi(A)) \bullet \psi(C)) \\
&= (\psi(I)\psi(A)\psi(B) + \psi(I)\psi(B)\psi(A))\psi(C)) \\
&+ (\psi(C)(\psi(I)\psi(B)^*\psi(A)^* + \psi(I)\psi(A)^*\psi(B)^*).
\end{aligned}$$

An application of Lemma 3.7 gives

$$\begin{aligned}
\Psi(A \circ B \bullet C) &= (\psi(I)\psi(A)\psi(I)\psi(B) + \psi(I)\psi(B)\psi(I)\psi(A))\psi(I)\psi(C)) \\
&+ (\psi(I)\psi(C)(\psi(I)^*\psi(B)^*\psi(I)^*\psi(A)^* + \psi(I)^*\psi(A)^*\psi(I)^*\psi(B)^*) \\
&= (\psi(I)\psi(A)\psi(I)\psi(B) + \psi(I)\psi(B)\psi(I)\psi(A)\psi(I)\psi(C)) \\
&+ (\psi(I)\psi(C)((\psi(I)\psi(B))^*(\psi(I)\psi(A))^* + (\psi(I)\psi(A))^*(\psi(I)\psi(B))^* \\
&= (\Psi(A)\Psi(B) + \Psi(B)\Psi(A))\Psi(C) + \Psi(C)(\Psi(B)^*\Psi(A)^* + \Psi(A)^*\Psi(B)^*) \\
&= \Psi(A) \circ \Psi(B) \bullet \Psi(C)
\end{aligned}$$

This completes the proof.

(b) It follows directly from hypothesis and Lemma 3.7.

(c) By the hypothesis, we have

$$2(\Psi(A) + \Psi(A^*)) = 2\Psi(A + A^*) = \Psi(I \circ A \bullet I) = I \circ \Psi(A) \bullet I = 2(\Psi(A) + \Psi(A)^*).$$

Above relation yields $\Psi(A^*) = \Psi(A)^*$ for all $A \in M$.

(d) Since $\Psi(A) = \psi(I)\psi(A)$ for all $A \in M$, so for $A = \mathfrak{P}$ and from Lemma 3.5, we have $\Psi(\mathfrak{P}) = \psi(I)\psi(\mathfrak{P}) = \psi(I)^2\Theta_{\mathfrak{P}} = \Theta_{\mathfrak{P}}$. As we know $\Theta_{\mathfrak{P}}$ is a projection. Thus $\Psi(\mathfrak{P})$ is also, as we asserted. □

As N has no central abelian projections, it follows from Lemma 3.4 that there exists a projection $Q_1 \in N$ such that $\underline{Q_1} = 0$ and $\overline{Q_1} = I$. Then by Lemma 3.8 (d), $\mathfrak{P}_1 = \Psi^{-1}(Q_1)$ is a projection in M. We denote $A_{ij} = \mathfrak{P}_i M \mathfrak{P}_j$ and $B_{ij} = \mathfrak{P}_i N \mathfrak{P}_j$, respectively. Keep it into mind, we now prove the following:

Lemma 3.9. *For any* $A_{ij} \in M_{ij}$ *and* $B_{ij} \in N_{ij}$, $1 \leq i, j \leq 2$, *we have* $\Psi(A_{ij}) = B_{ij}$.

Proof. First we prove for $i = 1, j = 2$. It follows from hypothesis that

$$\begin{aligned}
2\Psi(A_{12}) &= \Psi(I \circ \mathfrak{P}_1 \bullet A_{12}) \\
&= I \circ Q_1 \bullet \Psi(A_{12}) \\
&= 2Q_1\Psi(A_{12}) + 2\Psi(A_{12})Q_1.
\end{aligned}$$

Multiply above relation, first by Q_1 on both sides and then by $Q_2 = I - Q_1$, we obtain $\Psi(A_{12}) = B_{12} + B_{21}$ for some $B_{12} \in N_{12}$ and $B_{21} \in N_{21}$. Next, we prove that $B_{21} = 0$. Observe that

$$\begin{aligned} 0 &= \Psi(I \circ A_{12} \bullet \mathfrak{P}_1) \\ &= I \circ \Psi(A_{12}) \bullet Q_1 \\ &= 2(B_{21} + B_{21}^*). \end{aligned}$$

This gives $B_{21} = 0$, and hence $\Psi(B_{12}) \subseteq M_{12}$. Since Ψ is a bijection, so we can easily obtain $\Psi(B_{12}) = M_{12}$. Similarly, we can show $\Psi(B_{21}) = M_{21}$. □

Lemma 3.10. *For any $A_{ii} \in M_{ii}$ and $B_{ii} \in N_{ii}$, we have $\Psi(A_{ii}) \subseteq B_{ii}$.*

Proof. For $j \neq i$, we have

$$\begin{aligned} 0 &= \Psi(I \circ \mathfrak{P}_j \bullet A_{ii}) \\ &= I \circ Q_j \bullet \Psi(A_{ii}) \\ &= 2(Q_j \Psi(A_{ii}) + \Psi(A_{ii}) Q_j). \end{aligned}$$

This yields $Q_i \Psi(A_{ii}) Q_i = \Psi(A_{ii}) \subseteq B_{ii}$. □

Lemma 3.11. *For any $A_{ij}, B_{ij} \in M_{ij}$, $i \leq i, j \leq 2$, we have*
(a) $\Psi(A_{11}B_{12}) = \Psi(A_{11})\Psi(B_{12})$ *and* $\Psi(A_{22}B_{21}) = \Psi(A_{22})\Psi(B_{21})$;
(b) $\Psi(A_{12}B_{21}) = \Psi(A_{12})\Psi(B_{21})$ *and* $\Psi(A_{21}B_{12}) = \Psi(A_{21})\Psi(B_{12})$;
(c) $\Psi(A_{11}B_{11}) = \Psi(A_{11})\Psi(B_{11})$ *and* $\Psi(A_{22}B_{22}) = \Psi(A_{22})\Psi(B_{22})$;
(d) $\Psi(A_{12}B_{22}) = \Psi(A_{12})\Psi(B_{22})$ *and* $\Psi(A_{21}B_{11}) = \Psi(A_{21})\Psi(B_{11})$.

Proof. (a) It follows from Lemma 3.9 and 3.10 that $\Psi(B_{12}A_{11}^*) = \Psi(B_{12})\Psi(A_{11})^* = 0$. Thus

$$\begin{aligned} 2\Psi(A_{11}B_{12}) + 2\Psi(B_{12}A_{11}^*) &= \Psi(I \circ A_{11} \bullet B_{12}) \\ &= I \circ \Psi(A_{11}) \bullet \Psi(B_{12}) \\ &= 2\Psi(A_{11})\Psi(B_{12}) + 2\Psi(B_{12})\Psi(A_{11})^*. \end{aligned}$$

This implies $\Psi(A_{11}B_{12}) = \Psi(A_{11})\Psi(B_{12})$. Similarly, we can show $\Psi(A_{22}B_{21}) = \Psi(A_{22})\Psi(B_{21})$. Next, to show ($b$), see from Lemma 3.9 that $\Psi(B_{21})\Psi(A_{12})^* = 0$. Therefore,

$$2\Psi(A_{12}B_{21}) = \Psi(A_{12} \circ I \bullet B_{21}) = 2\Psi(A_{12})\Psi(B_{21}).$$

Hence, $\Psi(A_{12}B_{21}) = \Psi(A_{12})\Psi(B_{21})$. Equivalently, one can easily show $\Psi(A_{21}B_{12}) = \Psi(A_{21})\Psi(B_{12})$. Now, we establish ($c$). Let $X_{12} \in N_{12}$ such that $C_{12} = \Psi^{-1}(X_{12}) \in M_{12}$ from Lemma 3.9. It follows from (a) that

$$\Psi(A_{11}B_{11})X_{12} = \Psi(A_{11}B_{11}C_{12}) = \Psi(A_{11})\Psi(B_{11}C_{12}) = \Psi(A_{11})\Psi(B_{11})X_{12}$$

for all $X_{12} \in M_{12}$. Since $\overline{Q_2} = I$, it follows from Remark 3.1 and 3.2 that $\Psi(A_{11}B_{11}) = \Psi(A_{11})\Psi(B_{11})$. Similarly, we can show $\Psi(A_{22}B_{22}) = \Psi(A_{22})\Psi(B_{22})$. Finally, to prove ($d$), we see from Lemma 3.9 that $E_{21} = \Psi^{-1}(Y_{21}) \in M_{21}$ for any $Y_{21} \in N_{21}$. So

$$\Psi(A_{12}B_{22})Y_{21} = \Psi(A_{12}B_{22}E_{21}) = \Psi(A_{12})\Psi(B_{22}E_{21}) = \Psi(A_{12})\Psi(B_{22})Y_{21}.$$

Reasoning as above, we obtain $\Psi(A_{12}B_{22}) = \Psi(A_{12})\Psi(B_{22})$. Similarly, we can have $\Psi(A_{21}B_{11}) = \Psi(A_{21})\Psi(B_{11})$. □

Lemma 3.12. *Ψ is a $\mathbb{R}$-linear $*$-ismomorphism.*

Proof. Since we know from Lemma 3.8 that Ψ is additive, so it follows from Lemma 3.11 that

$$\begin{aligned}\Psi(AB) &= \Psi(A_{11}B_{11}+A_{11}B_{12}+A_{12}B_{12}+A_{12}B_{22}\\ &\quad +A_{21}B_{11}+A_{21}B_{12}+A_{22}B_{21}+A_{22}B_{22})\\ &= \Psi(A_{11}B_{11})+\Psi(A_{11}B_{12})+\Psi(A_{12}B_{12})+\Psi(A_{12}B_{22})\\ &\quad +\Psi(A_{21}B_{11})+\Psi(A_{21}B_{12})+\Psi(A_{22}B_{21})+\Psi(A_{22}B_{22})\\ &= \Psi(A_{11})\Psi(B_{11})+\Psi(A_{11})\Psi(B_{12})+\Psi(A_{12})\Psi(B_{12})+\Psi(A_{12})\Psi(B_{22})\\ &\quad +\Psi(A_{21})\Psi(B_{11})+\Psi(A_{21})\Psi(B_{12})+\Psi(A_{22})\Psi(B_{21})+\Psi(A_{22})\Psi(B_{22})\\ &= \Psi(A)\Psi(B)\end{aligned}$$

for all $A, B \in M$. Therefore, Ψ is an isomorphism, and hence $*$-ismomorphism by Lemma 3.8(c). Now we show Ψ is $\mathbb{R}-$linear. Thus, for every $\eta \in \mathbb{R}$, there exist two rational sequences $\{r_n\}, \{s_n\}$ such that $r_n \leq \eta \leq s_n$ and $\lim r_n = \lim s_n = \eta$ when $n \to \infty$. It is clear that Ψ preserves positive elements, then Φ preserves order. So, by the additivity of Ψ, we have

$$r_nI = \Psi(r_nI) \leq \Psi(\eta I) \leq \Psi(s_nI) = s_nI.$$

Hence,

$$\Psi(\eta I) = \eta I$$

for $\eta \in \mathbb{R}$. It means that Ψ is $\mathbb{R}$-linear. Thereby the proof is completed. □

Lemma 3.13. *The restriction of Ψ to $M\mathfrak{P}$ is linear and restriction to $M(I-\mathfrak{P})$ is conjugate linear.*

Proof. By Lemma 3.12, $\Psi(iI)^2 = \Psi((iI)^2) = -\Psi(I) = -I$. Also by Lemma 3.8(c), $\Psi(iI)^* = \Psi((iI)^*) = -\Psi(iI)$. Let $F = \frac{I-i\Psi(iI)}{2}$. Then it is easy to verify that F is a central projection in M. Let $\mathfrak{P} = \Psi^{-1}(F)$. Then by Lemma 3.8$(d)$, $\mathfrak{P}$ is a central projection in N. Moreover, for $A \in N$, there hold

$$\Psi(iA\mathfrak{P}) = \Psi(A)\Psi(\mathfrak{P})\Psi(iI) = i\Psi(A)\Psi(\mathfrak{P})(2F-I),$$

and

$$\Psi(iA(I-\mathfrak{P})) = \Psi(A)\Psi(I-\mathfrak{P})\Psi(iI) = -i\Psi(A)(I-F) = -i\Psi(A(I-\mathfrak{P})).$$

That is, the restriction of Ψ to $M\mathfrak{P}$ is linear and restriction to $M(I-\mathfrak{P})$ is conjugate linear. This together with Lemmas 3.8, 3.11 and 3.12 completes the proof of Theorem 1.1. □

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