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A Smarandache Curve According to the Darboux Frame and the Kinematics of this Curve in Minkowski Space

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Abstract. In this study, the curvatures and frames of the Smarandache curve formed by a unt speed timelike curve taken on a time like surface according to Darboux frame in Semi-Euclidean space $\mathbb{E}_1^3$ were created. Then, the image curve on the screw surface was obtained by applying translational and rotational motion using the quaternion multiplication under the same frame of this curve. Finally, the curvature and torsion of the obtained image curve were calculated.

Keywords: Smarandache curve · Quaternon · Screw surface · Image curve.

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1 Introduction

In differential geometry, curves have many important places and properties. In the theory of curves especially helice, geodesic, circle and many other types of curves are studied. Researchers closely follow the work on these curves. Researchers have worked with various curves besides existing studies. Smarandache curves are one of them. This curve is defined as a regular curve produced by the Frenet frame of a regular unit speed curve.

Smarandache curves have been studied by some authors. A. T. Ali has introduced some special Smarandache curves in the Euclidean space [1]. He introduced some special Smarandache curves in three-dimensional Euclidean space. In his master's thesis, Karaman F. created all Smarandache curves that can occur under Frenet frames and examined. The curvatures and torsions of these curves are calculated individually [2].

These curves have been studied on various frames such as Frenet, Darboux and Bishop. . Bekta and S. Yce studied these curves on the Darboux frame [3]. They obtained these curves by using the relationship formulas between Frenet and Darboux frame in their studies. They also calculated the curvature and torsion of these curves.

The theory of quaternions was discovered by Hamilton. Thus, kinematic geometry has become one of the most important fields of science. Curves and surfaces could be studied in more detail in this geometry. Kinematic surface, screw surface, quaternionic curve are at the beginning of these studies.

Whether the curve is timelike or spacelike and the surface on which the curve is located is timelike or spacelike has been the subject of practice for different authors. Therefore, the properties of these curves studied have taken their place in different applications [4,5].

In this study, a new image curve is defined, which is formed by rotating the curve whose position vector is Smarandache curve by an angle θ around the vector $\vartheta = (\vartheta_x, \vartheta_y, \vartheta_z)$ and its translation by $\lambda \in \mathbb{R}$.

The curvatures and frames of the Smarandache curve formed by a spacelike curve with unit velocity taken on a timelike surface according to the Darboux frame in Semi-Euclidean space were created. Then, the image curve on the screw surface was obtained by applying translational and rotational motion (screw motion) using the quaternion multiplication under the same frame of this curve. Finally, the curvature and torsion of the obtained image curve were calculated.

2 Preliminiaries

Any two vectors $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$ in $\mathbb{R}^3_1$ the product of scalar and vector is defined as,

$$\langle , \rangle_L : \mathbb{R}^3_1 \times \mathbb{R}^3_1 \to \mathbb{R}$$

$$\langle u, v \rangle_L = u_1 v_1 + u_2 v_2 - u_3 v_3$$

$$\times_L : \mathbb{R}^3_1 \times \mathbb{R}^3_1 \to \mathbb{R}^3_1$$

$$u \times_L v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{-k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

functions are called Lorentzian scalar and vector product, defined according to the sign $(+, +, -)$ in $\mathbb{R}^3_1$ space. Together with this, scalar product $\mathbb{R}^3_1$ space is also called Lorentz space. It is denoted by $\mathbb{R}^3_1$ or $\mathbb{E}^3_1$. If $\langle u, v\rangle_L = 0$ these vectors are said to be pseudo orthogonal in the Lorentz sense.

For any $u \in \mathbb{R}^3_1$ in this space,

i. If $\langle u, u\rangle_L = 0$ then u lightlike or null vector
ii. If $\langle u, u\rangle_L < 0$ then u timelike vector
iii. If $\langle u, u\rangle_L > 0$ then u spacelike vector

It is also defined $\|u\|_L = \sqrt{|\langle u, u\rangle_L|}$ using the absolute value as the norm of u. If $\langle u, u\rangle_L = \pm 1$ the vector u is called the unit vector. The set of split quaternions is defined as follows,

$$\hat{\mathbb{H}} = \{q_1 + q_2\mathbf{i} + q_3\mathbf{j} + q_4\mathbf{k} : q_i \in \mathbb{R}, \mathbf{i}^2 = -1, \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = 1\}$$

Any $q = q_1 + q_2\mathbf{i} + q_3\mathbf{j} + q_4\mathbf{k}$ split quaternion can be written as $q = s_q + v_q$. The number $s_q = q_1$ is called the scalar part and the part of $v_q = q_2\mathbf{i} + q_3\mathbf{j} + q_4\mathbf{k}$ is called the vector part of the q split quaternion. If $s_q = 0$ then, q split quaternion is called a pure split quaternion. The conjugate any $q = q_1 + q_2 i + q_3 j + q_4 k$ split quaternion is defined as,

$$\overline{q} = q_1 - q_2\mathbf{i} - q_3\mathbf{j} - q_4\mathbf{k}$$

The sum and product of two split quaternions are defined as

$$\begin{aligned} q + p &= (s_q + v_q) + (s_p + v_p) \\ &= (q_1 + p_1) + (q_2 + p_2)\mathbf{i} + (q_3 + p_3)\mathbf{j} + (q_4 + p_4)\mathbf{k} \\ qp &= (s_q + v_q)(s_p + v_p) = s_q s_p - \langle v_q, v_p\rangle_L + s_q v_p + s_p v_q + v_q \times_L v_p \end{aligned}$$

3 Screw Motion in 3-Dimensional Semi-Euclid Space $\mathbb{E}^3_1$

Each unit real quaternion corresponds to a spherical rotation in Euclidean space. A similar situation exists for split quaternions. We can say that the split quaternions match with the three-dimensional Minkowski space, whose vector part is denoted by $\mathbb{E}^3_1$. Our unit spheres in this space are one sheet hyperboloid and two sheet hyperboloid.

In that case, the spheres in the Lorentz meaning are the movements on these hyperboloids expressed above it is possible to examine the movements on these hyperboloid spheres with the help of split quaternions. We know that each unit real quaternion corresponds to a large circular arc on the unit sphere. Similarly, it is possible to define large circles on spheres that is hyperboloids in the Lorentz meaning.

Let's take two surfaces $M, M^v \in \mathbb{E}_1^3$ where $p \in M$. The rotation of θ angle around the timelike vector N is expressed as follows

$$q = (\cosh\frac{\theta}{2} + \sinh\frac{\theta}{2}N)$$

If there is a function f defined as follows, then the M^v surface is called the timelike kinematic screw surface of the M surface where $\lambda \in \mathbb{R}$

$$f : M \to M^v$$

$$p \to f(p) = qp\overline{q} + \lambda N$$

Accordingly, we can define the kinematic screw motion function as follows

$$f(p) = (\cosh\frac{\theta}{2} + \sinh\frac{\theta}{2}N)p(\cosh\frac{\theta}{2} - \sinh\frac{\theta}{2}N) + \lambda N$$

4 Curves in $\mathbb{E}_1^3$

Let $M = M(u, v)$ be a differentiable spacelike surface in $\mathbb{E}_1^3$. Let spacelike curve ξ on this surface. This spacelike curve has a frame at each point which we call the Darboux frame, we denote T, V, N. Here unit vector T is expressed as the unit tangent vector of the curve, the unit vector N is the unit normal vector of the surface at any point s and the unit vector V is expressed as the unit geodesic vector defined as $V = -N \times T$. Because of the unit vector N is timelike, that vectors V and T will be spacelike vectors. Thus, the following relationships exist between these vectors.

$$N = T \times V \quad T = -V \times N \quad V = -N \times T$$

Among these vectors, vector T is the common vector of both Darboux and Frenet frame so all other vectors are in the same plane. If the vectors in the Frenet frame are rotated about the tangent vector which is the spacelike vector in the positive direction by the hyperbolic angle $\varphi = \varphi(s)$ the new frame obtained will be Darboux frame. The angle here is the hyperbolic angle between V and N. Thus, the relation between Frenet and Darboux frame for this case is as follows,

$$\begin{bmatrix} T \\ V \\ N \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cosh\varphi & \sinh\varphi \\ 0 & \sinh\varphi & \cosh\varphi \end{bmatrix} \begin{bmatrix} \mathbb{T} \\ \mathbb{N} \\ \mathbb{B} \end{bmatrix}$$

Derivative formulas according to s semi-parametric arc of Darboux frame are expressed as follows.

$$\frac{d}{ds}\begin{bmatrix} T \\ V \\ N \end{bmatrix} = \begin{bmatrix} 0 & \kappa_g & \kappa_n \\ -\kappa_g & 0 & \tau_g \\ \kappa_n & \tau_g & 0 \end{bmatrix} \begin{bmatrix} \mathbb{T} \\ V \\ N \end{bmatrix}$$

Where κ_n, τ_g, τ_g are named as geodesic curvature, normal curvature and geodesic torsion respectively. We can easily write the following equations by using the relations we have expressed and the trigonometric relations of the hyperbolic angles,

$$\kappa^2 = \kappa_g^2 - \kappa_n^2$$
$$\kappa_g = \kappa \cosh \varphi$$
$$\kappa_n = \kappa \sinh \varphi$$
$$\tau_g = \tau \frac{d\varphi}{ds}$$

Lemma 1. *In the differential geometry, a curve $\alpha(s)$ lying on surface M that the followings are well known*

i. If $\alpha(s)$ is a geodesic curve $\Longleftrightarrow \kappa_g = 0$
ii. If $\alpha(s)$ is an asymptotic line $\Longleftrightarrow \kappa_n = 0$
iii. If $\alpha(s)$ principal line $\Longleftrightarrow \tau_g = 0$

5 TV-Smarandache Curve Created by a Timelike Curve on a Timelike Surface and its Image Curve in $\mathbb{E}_1^3$

Let M be a timelike surface in $\mathbb{E}_1^3$ and a spacelike curve $\alpha = \alpha(s)$ on M. This curve is located on a timelike surface M. Let's take the Frenet frame $\{\mathbb{T}(s), \mathbb{B}(s), \mathbb{N}(s)\}$ and the Darboux frame at point s of α as $\{T(s), V(s), N(s)\}$. Here, vector $N(s)$ is the unit normal vector of surface M at point s and $V(s) = T(s)xN(s)$. Since M is a timelike surface, $N(s)$ is a spacelike vector and $\alpha(s)$ is a spacelike curve, so the $T(s) = \alpha'(s)$ tangent vector is spacelike. Then the relationship between frame vectors of curve can be given as

$$\langle T(s), T(s)\rangle = -1 \quad \langle T(s), N(s)\rangle = 0 \quad \langle \mathbb{T}(s), \mathbb{T}(s)\rangle = -1$$
$$\langle N(s), N(s)\rangle = 1 \quad \langle T(s), V(s)\rangle = 0 \quad \langle \mathbb{N}(s), \mathbb{N}(s)\rangle = 1$$
$$\langle V(s), V(s)\rangle = 1 \quad \langle V(s), N(s)\rangle = 0 \quad \langle \mathbb{B}(s), \mathbb{B}(s)\rangle = 1$$

$\alpha(s)$ curve consisting of T ,V Darboux frame vectors curve of α is called the TV-Smarandache curve. With Darbux frame $\{T, V, N\}$. The TV-Smarandache curve is defined as follows

$$\beta(s^*) = \frac{1}{\sqrt{2}}(T(s) + V(s))$$

Frenet elements of this curve are as follows

$$
\begin{aligned}
\mathbb{T}_\beta &= \frac{1}{\sqrt{\left|2\kappa_g^2 - (\tau_g + \kappa_n)^2\right|}} \Big(\kappa_g T(s) + \kappa_g V(s) + (\kappa_n - \tau_g) N(s) \Big) \\
\mathbb{N}_\beta &= \frac{\bar{q}_1 T(s) + \bar{q}_2 V(s) + \bar{q}_3 N(s)}{\sqrt{|-q_1^2 + q_2^2 - q_3^2|}} \\
\mathbb{B}_\beta - \mathbb{T}_\beta \times \mathbb{N}_\beta &= \frac{\bar{q}_1 T(s) + \bar{q}_2 V(s) + \bar{q}_3 N(s)}{(\kappa_n - \tau_g)\sqrt{|-q_1^2 + q_2^2 - q_3^2|}}
\end{aligned}
$$

where

$$
\begin{aligned}
q_1 &= \kappa_g(\tau_g' - \kappa_n') + (\tau_g - \kappa_n)(\kappa_g' + \kappa_g^2 - \tau_g \kappa_n + \kappa_n^2) \\
q_2 &= \kappa_g(\tau_g' - \kappa_n') - (\tau_g - \kappa_n)(\kappa_g' + \kappa_g^2 + \tau_g \kappa_n - \tau_g^2) \\
q_3 &= \kappa_g(\tau_g - \kappa_n)^2
\end{aligned}
$$

$$
\begin{aligned}
\bar{q}_1 &= (\tau_g - \kappa_n) q_1 - \kappa_g q_3 \\
\bar{q}_2 &= -(\tau_g - \kappa_n) q_1 - \kappa_g q_3 \\
\bar{q}_3 &= \kappa_g(q_1 - q_2)
\end{aligned}
$$

$$
\kappa_\beta = \left\| \frac{d\mathbb{T}_\beta}{ds^*} \right\| = \frac{\sqrt{2}\sqrt{|-q_1^2 + q_2^2 - q_3^2|}}{(\kappa_n - \tau_g)^3}
$$

$$
\tau_\beta(s^*) = \frac{\frac{(\kappa_n - \tau_g)}{(2\sqrt{2}\kappa^2)} \left[\begin{array}{r} -3\kappa_g(\tau_g'^2 + \kappa_n'^2) + \tau_g'\left(6\kappa_n'\kappa_g + \Big(3\kappa_g' + 5\kappa_g^2 + \tau_g(\kappa_n - \tau_g)\Big)(\kappa_n - \tau_g)\right) \\ - \kappa_n'(\kappa_n - \tau_g)(3\kappa_g' + 5\kappa_g^2 + \tau_g(\kappa_n - \tau_g)) + (\kappa_n - \tau_g) \end{array} \right]}{\kappa_g \Big(2(\kappa_n - \tau_g)^2(\kappa_n + \tau_g) - \tau_g'' + \kappa_n'' \Big) + (\kappa_n - \tau_g)(5\kappa_g \kappa_g' + 2\kappa_g^3 + \kappa_g'')}
$$

The unit normal vector of the surface M and the unit normal vector V are obtained as follows

$$
\mathbb{T}_\beta = \mathbb{T}_\beta
$$

$$
V_\beta = \cosh\varphi \mathbb{T}_\beta + \sinh\varphi \mathbb{B}_\beta
$$

$$V_\beta = \frac{\left[\begin{array}{r}\left(q_1\cosh\varphi + \frac{\overline{q}_1\sinh\varphi}{\tau_g-\kappa_n}\right)T(s) + \left(q_2\cosh\varphi + \frac{\overline{q}_2\sinh\varphi}{\tau_g-\kappa_n}\right)V(s)\\ + \left(q_3\cosh\varphi + \frac{\overline{q}_3\sinh\varphi}{\tau_g-\kappa_n}\right)N(s)\end{array}\right]}{\sqrt{|-q_1^2+q_2^2-q_3^2|}}$$

$$N_\beta = -\sinh\varphi\mathbb{N}_\beta + \cosh\varphi\mathbb{B}_\beta$$

$$V_\beta = \frac{\left[\begin{array}{r}\left(q_1\sinh\varphi + \frac{\overline{q}_1\cosh\varphi}{\tau_g-\kappa_n}\right)T(s) + \left(q_2\sinh\varphi + \frac{\overline{q}_2\cosh\varphi}{\tau_g-\kappa_n}\right)V(s)\\ + \left(q_3\sinh\varphi + \frac{\overline{q}_3\cosh\varphi}{\tau_g-\kappa_n}\right)N(s)\end{array}\right]}{\sqrt{|-q_1^2+q_2^2-q_3^2|}}$$

The geodesic curvature, normal curvature and geodosic torsion of the TV-Smarandache curve are as follows,

$$(\kappa_g)_\beta = \kappa_\beta\cosh\varphi = \frac{1}{\nu^2}\langle\beta'', V_\beta\rangle$$

$$(\kappa_g)_\beta = \frac{\left[\begin{array}{l}-(\kappa_g' + \kappa_g^2 - \tau_g\kappa_n - \kappa_n^2)(q_1\cosh\varphi + \frac{\overline{q}_1\sinh\varphi}{\tau_g-\kappa_n})\\ \quad + (\kappa_g' + \kappa_n^2 + \tau_g\kappa_n - \tau_g^2)(q_2\cosh\varphi + \frac{\overline{q}_2\sinh\varphi}{\tau_g-\kappa_n})\\ \qquad + (-\tau_g' + \kappa_n' - \tau_g\kappa_g + \kappa_g\kappa_n)(q_3\cosh\varphi + \frac{\overline{q}_3\sinh\varphi}{\tau_g-\kappa_n})\end{array}\right]}{\nu^2\sqrt{2}\sqrt{|-q_1^2+q_2^2-q_3^2|}}$$

$$(\kappa_n)_\beta = \kappa_\beta\sinh\varphi = \frac{1}{\nu^2}\langle\beta'', N_\beta\rangle$$

$$(\kappa_g)_\beta = \frac{\left[\begin{array}{l} -(\kappa_g' + \kappa_g^2 - \tau_g\kappa_n - \kappa_n^2)(-q_1 \sinh\varphi + \frac{\overline{q}_1 \cosh\varphi}{\tau_g - \kappa_n}) \\ \quad + (\kappa_g' + \kappa_n^2 + \tau_g\kappa_n - \tau_g^2)(-q_2 \sinh\varphi + \frac{\overline{q}_2 \cosh\varphi}{\tau_g - \kappa_n}) \\ \qquad + (-\tau_g' + \kappa_n' - \tau_g\kappa_g + \kappa_g\kappa_n)(-q_3 \sinh\varphi + \frac{\overline{q}_3 \cosh\varphi}{\tau_g - \kappa_n}) \end{array}\right]}{\nu^2\sqrt{2}\sqrt{|-q_1^2 + q_2^2 - q_3^2|}}$$

$$(\tau_g)_\beta = \tau_g + \frac{d\varphi}{ds} = \frac{1}{\nu}\langle T_\beta', N_\beta\rangle$$

$$(\tau_g)_\beta = \sqrt{2}\frac{\left[\begin{array}{r} -q_1\left(-q_1 \sinh\varphi + \frac{\overline{q}_1 \cosh\varphi}{\tau_g - \kappa_n}\right) + q_2\left(-q_2 \sinh\varphi + \frac{\overline{q}_2 \cosh\varphi}{\tau_g - \kappa_n}\right) \\ + q_3\left(-q_3 \sinh\varphi + \frac{\overline{q}_3 \cosh\varphi}{\tau_g - \kappa_n}\right) \end{array}\right]}{\nu(\kappa_n - \tau_g)^3\sqrt{|-q_1^2 + q_2^2 - q_3^2|}}$$

where $\nu = \|\beta'(s)\|$

Now let's create the image curve of the curve we created. Let $\alpha = \alpha(s)$ timelike curve be given on a timelike surface $M = M(u, \vartheta)$ in $\mathbb{E}_1^3$ Semi-Euclidean space.

$$f(\beta(s^*)) = \gamma(s^*) = \Big(\cosh\frac{\theta}{2} + \sinh\frac{\theta}{2}N(s)\Big)\frac{1}{\sqrt{2}}\Big(T(s) + V(s)\Big)\Big(\cosh\frac{\theta}{2} - \sinh\frac{\theta}{2}N(s)\Big) + \lambda N(s)$$

By choosing $\lambda = \frac{1}{\sqrt{2}}$ specifically and using the quaternion product our curve formed as a result of screw motion can be written as follows

$$\gamma(s^*) = \frac{1}{\sqrt{2}}\left((1 - \sinh\theta)T(s) + (1 + \sinh\theta)V(s) + N(s)\right)$$

Considering that this curve is a curve in $\mathbb{E}_1^3$ semi-Euclidean space and not a unit speed curve. Let's calculate the curvature and torsion of this curve and obtain the Frenet frame. Then let's create the Darboux frame using this frame with the help of the translation matrix and find the curvatures of this curve. For this let's first take the derivative of the image curve with respect to s.

$$\gamma'(s^*) = \frac{1}{\sqrt{2}}(A_1T(s) + A_2V(s) + A_3N(s))$$

Where

$$\begin{aligned}
A_1 &= -\cosh\theta + \kappa_g(1+\sinh\theta) + \kappa_n \\
A_2 &= \cosh\theta + \kappa_g(1-\sinh\theta) + \tau_g \\
A_3 &= (\kappa_n - \tau_g) - (\tau_g + \kappa_n)\sinh\theta
\end{aligned}$$

Thus, we can express the tangent vector field as follows

$$\mathbb{T}_\gamma = \frac{\gamma'}{\|\gamma'\|} = \frac{A_1T(s)+A_2V(s)+A_3N(s)}{\sqrt{|A_1^2+A_2^2-A_3^2|}}$$

where

$$\rho = \|\gamma'\| = \frac{1}{\sqrt{2}}\sqrt{|A_1^2+A_2^2-A_3^2|}$$

Now let's calculate the second and third derivatives of the $\gamma(s^*)$ image curve

$$\gamma'' = \frac{1}{\sqrt{2}}(B_1T(s)+B_2V(s)+B_3N(s))$$

$$\gamma''' = \frac{1}{\sqrt{2}}(C_1T(s)+C_2V(s)+C_3N(s))$$

where

$$\begin{aligned}
B_1 &= 2\kappa_g\cosh\theta - \sinh\theta + \kappa^2\cosh^2\theta(1-\sinh\theta) + (\kappa_g' - \kappa_n\tau_g)(1+\sinh\theta) + \kappa_n' + \kappa_g\tau_g \\
B_2 &= -2\kappa_g\cosh\theta + \sinh\theta + (\kappa_g^2 - \tau_g^2)(1+\sinh\theta) + (\kappa_g' + \kappa_n\tau_g)(1-\sinh\theta) + \kappa_g' + \kappa_g\kappa_n \\
B_3 &= -2(\tau_g+\kappa_n)\cosh\theta + (\kappa_g\kappa_n - \tau_g'')(1+\sinh\theta) + (\kappa_n' - \kappa_g\tau_g)(1-\sinh\theta) - \tau_g^2 + \kappa_g^2 + \kappa_n
\end{aligned}$$

$$\begin{aligned}
C_1 =& \cosh\theta\left(3\kappa_g' - 3\kappa_n\tau_g - 3\kappa^2\cosh^2\theta - 1\right) + 3\kappa_g\sinh\theta \\
&+ (1-\sinh\theta)\left(2\kappa\kappa'\cosh 2\theta + 2\kappa^2\sinh 2\theta + \kappa_g\kappa_g' + \kappa_n\kappa_n'\right) + \kappa_n^2(1+\kappa_n) + \tau_g(\kappa_g' - \kappa_n\tau_g) \\
&+ (1+\sinh\theta)\left(\kappa_g' - 2\kappa_n\tau_g' - \kappa_n'\tau_g + \kappa_g(\kappa^2\cosh 2\theta - \tau_g^2) + \kappa_g\tau_g' + \kappa_n''\right) \\
C_2 =& \cosh\theta\left(3\kappa_g^2 - 3\kappa_g' - 3\tau_g^2 - 3\kappa_n\tau_g + 1\right) - 3\kappa_g\sinh\theta \\
&+ (1-\sinh\theta)\left(\kappa_g(\kappa^2\cosh 2\theta - \tau_g^2) + 2\kappa_n'\tau_g + \kappa_n\tau_g' + \kappa_g''\right) + \kappa^2\tau_g\cosh 2\theta + 2\kappa_g\kappa_n' \\
&+ (1+\sinh\theta)\left(3\kappa_g\kappa_g' - 3\tau_g\tau_g'\right) + \kappa_n(\kappa_g' + \tau_g) - \tau_g^3 + \kappa_g'' \\
C_3 =& \cosh\theta\left[3(\kappa_g\kappa_n - \tau_g' - \kappa_n') + \kappa_g\tau_g + 2\tau_g^2\right] - 3\sinh\theta(\kappa_n+\tau_g) - \tau_g(2\tau_g' + \kappa_g') \\
&+ (1-\sinh\theta)\left(\kappa_n(\kappa^2\cosh 2\theta - \tau_g^2) - 2\kappa_g'\tau_g - \kappa_g\tau_g' + \kappa_n'' - \kappa_n\tau_g^2\right) \\
&+ (1+\sinh\theta)\left(2\kappa_n\kappa_g' + \kappa_g\kappa_n' - \tau_g\kappa^2\cosh 2\theta - \tau_g^3 - \tau_g''\right) + 3\kappa_n\kappa_n'
\end{aligned}$$

Accordingly, the normal and binormal vector fields of the image curve are as follows,

$$
\begin{aligned}
\mathbb{B}_\gamma &= \frac{\gamma' \times \gamma''}{\|\gamma' \times \gamma''\|} \\
&= \frac{(A_2B_3 - B_2A_3)T(s) + (A_3B_1 - B_3A_1)V(s) + (A_1B_2 - B_1A_2)N(s)}{\sqrt{|-(A_2B_3 - B_2A_3)^2 + (A_3B_1 - B_3A_1)^2 - (A_1B_2 - B_1A_2)^2|}}.
\end{aligned}
$$

$$
\mathbb{N}_\gamma = -\,\mathbb{T}_\gamma \times \mathbb{B}_\gamma
$$

$$
= \frac{\begin{array}{c} -[A_1(A_3B_1 - B_1A_3) - A_2(A_2B_3 - B_2A_3)]T(s) \\ -[A_3(A_2B_3 - B_2A_3) - A_1(A_1B_2 - B_1A_2)]V(s) \\ -[A_2(A_2B_3 - B_2A_3) - A_3(A_3B_1 - B_3A_1)]N(s) \end{array}}{\sqrt{|-A_1^2 + A_2^2 - A_3^2|}\sqrt{|-(A_2B_3 - B_2A_3)^2 + (A_3B_1 - B_3A_1)^2 - (A_1B_2 - B_1A_2)^2|}}
$$

When these results are evaluated together the curvature and torsion of the image curve are written as follows

$$
\begin{aligned}
\kappa_\gamma &= \frac{\|\gamma' \times \gamma''\|}{\|\gamma'\|^3} \\
&= \frac{\sqrt{|-(A_2B_3 - B_2A_3)^2 + (A_3B_1 - B_3A_1)^2 - (A_1B_2 - B_1A_2)^2|}}{2\rho^3}.
\end{aligned}
$$

$$
\tau_\gamma = \frac{\sqrt{2}\,[(A_3B_2 - B_3A_2)C_1 + (A_3B_1 - B_3A_1)C_2 - (A_2B_1 - B_2A_1)C_3]}{-(A_2B_3 - B_2A_3)^2 + (A_3B_1 - B_3A_1)^2 - (A_1B_2 - B_1A_2)^2}
$$

Thus, the Darboux frame of the image curve is found as follows

$$
\mathbb{T}_\gamma = \frac{\gamma'}{\|\gamma'\|} = \frac{(A_1T(s) + A_2V(s) + A_3N(s))}{\sqrt{|-A_1^2 + A_2^2 - A_3^2|}}
$$

$$
V_\gamma = \cosh\varphi \mathbb{N}_\gamma + \sinh\varphi \mathbb{B}_\gamma
$$

$$
= \frac{\begin{array}{c} -\Big[(A_1(A_3B_1 - B_3A_1) - A_2(A_2B_3 - B_2A_3))\cosh\varphi \\ +\sqrt{|-A_1^2 + A_2^2 - A_3^2|}(A_2B_3 - B_2A_3)\sinh\varphi\Big]T(s) \\ -\Big[A_3(A_2B_3 - B_2A_3) - A_1(A_2B_1 - B_2A_1)\cosh\varphi \\ +\sqrt{|-A_1^2 + A_2^2 - A_3^2|}(A_3B_1 - B_3A_1)\sinh\varphi\Big]V(s) \\ -\Big[A_3(A_2B_3 - B_2A_3) - A_1(A_2B_1 - B_2A_1)\cosh\varphi \\ +\sqrt{|-A_1^2 + A_2^2 - A_3^2|}(A_1B_2 - B_1A_2)\sinh\varphi\Big]N(s) \end{array}}{\sqrt{|-(A_2B_3 - B_2A_3)^2 + (A_3B_1 - B_3A_1)^2 - (A_1B_2 - B_1A_2)^2|}\sqrt{|-A_1^2 + A_2^2 - A_3^2|}}
$$

$$N_\gamma = \frac{\begin{array}{l} \sinh\varphi\mathbb{N}_\gamma + \cosh\varphi\mathbb{B}_\gamma \\ +\Big[(A_1(A_3B_1 - B_3A_1) - A_2(A_2B_3 - B_2A_3))\sinh\varphi \\ \quad +\sqrt{|-A_1^2 + A_2^2 - A_3^2|}(A_2B_3 - B_2A_3)\cosh\varphi\Big]T(s) \\ \quad +\Big[A_3(A_2B_3 - B_2A_3) - A_1(A_2B_1 - B_2A_1)\sinh\varphi \\ \quad\quad +\sqrt{|-A_1^2 + A_2^2 - A_3^2|}(A_3B_1 - B_3A_1)\cosh\varphi\Big]V(s) \\ \quad\quad +\Big[A_3(A_2B_3 - B_2A_3) - A_1(A_2B_1 - B_2A_1)\sinh\varphi \\ \quad\quad\quad +\sqrt{|-A_1^2 + A_2^2 - A_3^2|}(A_1B_2 - B_1A_2)\cosh\varphi\Big]N(s) \end{array}}{\sqrt{|-(A_2B_3 - B_2A_3)^2 + (A_3B_1 - B_3A_1)^2 - (A_1B_2 - B_1A_2)^2|}\sqrt{|-A_1^2 + A_2^2 - A_3^2|}}$$

Finally, we can find the curvature of our image curve as follows,

$$(\kappa_g)_\gamma = \kappa_\gamma \cosh\varphi = \frac{\langle \gamma'', V_\gamma \rangle}{\rho^2}$$

$$(\kappa_g)_\gamma = \frac{\begin{array}{l} -B_1\Big[(A_1(A_3B_1 - B_3A_1) - A_2(A_2B_3 - B_2A_3))\cosh\varphi \\ \quad +\sqrt{|-A_1^2 + A_2^2 - A_3^2|}(A_2B_3 - B_2A_3)\sinh\varphi\Big] \\ \quad +B_2\Big[A_3(A_2B_3 - B_2A_3) - A_1(A_1B_2 - B_1A_2)\cosh\varphi \\ \quad\quad +\sqrt{|-A_1^2 + A_2^2 - A_3^2|}(A_3B_1 - B_3A_1)\sinh\varphi\Big] \\ \quad\quad -B_3\Big[A_2(A_2B_3 - B_2A_3) - A_3(A_3B_1 - B_1A_3)\cosh\varphi \\ \quad\quad\quad +\sqrt{|-A_1^2 + A_2^2 - A_3^2|}(A_1B_2 - B_1A_2)\sinh\varphi\Big] \end{array}}{\sqrt{2}\rho^2\sqrt{|-(A_2B_3 - B_2A_3)^2 + (A_3B_1 - B_3A_1)^2 - (A_1B_2 - B_1A_2)^2|}\sqrt{|-A_1^2 + A_2^2 - A_3^2|}}$$

$$(\kappa_n)_\gamma = \kappa_\gamma \sinh\varphi = \frac{\langle \gamma'', V_\gamma \rangle}{2\rho^2}$$

$$(\kappa_n)_\gamma = \frac{\begin{aligned}&-B_1\Big[(A_1(A_3B_1-B_3A_1)-A_2(A_2B_3-B_2A_3))\sinh\varphi\\&\quad+\sqrt{|-A_1^2+A_2^2-A_3^2|}(A_2B_3-B_2A_3)\cosh\varphi\Big]\\&\quad\quad+B_2\Big[A_3(A_2B_3-B_2A_3)-A_1(A_1B_2-B_1A_2)\sinh\varphi\\&\quad\quad\quad+\sqrt{|-A_1^2+A_2^2-A_3^2|}(A_3B_1-B_3A_1)\cosh\varphi\Big]\\&\quad\quad\quad\quad-B_3\Big[A_2(A_2B_3-B_2A_3)-A_3(A_3B_1-B_3A_1)\sinh\varphi\\&\quad\quad\quad\quad\quad+\sqrt{|-A_1^2+A_2^2-A_3^2|}(A_1B_2-B_1A_2)\cosh\varphi\Big]\end{aligned}}{\sqrt{2}\rho^2\sqrt{|-(A_2B_3-B_2A_3)^2+(A_3B_1-B_3A_1)^2-(A_1B_2-B_1A_2)^2|}\sqrt{|-A_1^2+A_2^2-A_3^2|}}$$

$$(\tau_g)_\gamma = \frac{\langle T'_\gamma, N_\gamma\rangle}{\rho}$$

$$\mathbb{T}'_\gamma = \frac{\begin{aligned}&\left[-A_1^2A_1'+A_1A_2A_2'-A_1A_3A_3'+A_1'+A_2\kappa_g+A_3'\kappa_n\right]T(s)\\&\quad+\left[A_2^2A_2'-A_1A_2A_1'-A_2A_3A_3'+A_2'+A_2\kappa_g+A_3\tau_g\right]V(s)\\&\quad\quad+\left[-A_3^2A_3'+A_2A_3A_2'-A_1A_3A_1'+A_3'-A_1\kappa_n-A_2\tau_g\right]N(s)\end{aligned}}{\sqrt{|-A_1^2+A_2^2-A_3^2|}}$$

$$(\kappa_n)_\gamma = \frac{\begin{aligned}&[(A_1(A_3B_1-B_3A_1)-A_2(A_2B_3-B_2A_3))\sinh\varphi\\&\quad+\sqrt{|-A_1^2+A_2^2-A_3^2|}(A_2B_3-B_2A_3)\cosh\varphi]\\&\quad\quad[-A_1^2A_1'+A_1A_2A_2'-A_1A_3A_3'+A_1'+A_2\kappa_g+A_3'\kappa_n]\\&\quad\quad\quad-[A_3(A_2B_3-B_2A_3)-A_1(A_1B_2-B_1A_2)\sinh\varphi\\&\quad\quad\quad\quad+\sqrt{|-A_1^2+A_2^2-A_3^2|}(A_3B_1-B_3A_1)\cosh\varphi]\\&\quad\quad\quad\quad\quad[A_2^2A_2'-A_1A_2A_1'-A_2A_3A_3'+A_2'+A_2\kappa_g+A_3\tau_g]\\&\quad\quad\quad\quad\quad\quad-[A_2(A_2B_3-B_2A_3)-A_3(A_3B_1-B_3A_1)\sinh\varphi\\&\quad\quad\quad\quad\quad\quad\quad+\sqrt{|-A_1^2+A_2^2-A_3^2|}(A_1B_2-B_1A_2)\cosh\varphi]\\&\quad\quad\quad\quad\quad\quad\quad\quad[-A_3^2A_3'+A_2A_3A_2'-A_1A_3A_1'+A_3'-A_1\kappa_n-A_2\tau_g]\end{aligned}}{\rho(-A_1^2+A_2^2-A_3^2)\sqrt{-(A_2B_3-B_2A_3)^2+(A_3B_1-B_3A_1)^2-(A_1B_2-B_1A_2)^2}}$$

6 Discussion and Conclusion

In this study, we created the image curve of a Smarandache curve using the quaternion product method. We have obtained the Darboux frame of this curve and the corresponding Frenet frame. We calculated the curvature and torsion of the image curve. We also calculated the principal normal curvature, geodesic curvature and geodesic torsion of the image curve separately. This work can also be extended for other researchers of the image curve. Calculations for other curves can be made using the same method. Thus, the properties of the new curve formed by the kinematic motion of the curve can be discovered.

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