Brief History of Matrices, As a Tool of Consolidated Financial Statements(*)

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Abstract

Consolidation of accounts has always generated problems for accountants and auditors because of its computational difficulties, especially when the subsidiaries have mutual stockholdings, circular stockholdings and indirect stockholdings. The application of mathematics to accounting is not always simple and clear, yet, referring to the fact that accounting has a structure, not obvious *a priori*, but real vector space in two dimensions, and we can apply methods based on linear algebra, classic technique of matrices and less traditional methods such as Markov chains. These methods are irreplaceable to understand and describe the logical groups and consolidations. We can go further in the analysis of relationships between companies in a group and we can have the ambition to rationalize and then to optimize these relationships using, generally, the optimal forms of mathematics, mathematics of symmetrical shapes, and the properties of the Euler characteristic and the Pythagorean regular polyhedrons, especially. Firstly we recall briefly the historical principles of application of matrix methods to the accounting, and the classical and non-classical methods to solve the general problem of consolidated financial statements.

Keywords: Consolidation - Consolidated Statements - Space Vector - Groups of companies - Holding - Matrices - regular polyhedron.

Jel Classification: M41, M49

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Introduction

Accounting has always been considered as part of mathematics, particularly in Antiquity and the Middle Ages, but mathematics has evolved a lot over the time. The former professional mathematicians, the *mathematicus* included accounting in their teaching job. The inclusion of accounting in the field of mathematics is right, because accounting, particularly basic accounting, uses the four arithmetic operations, addition, subtraction, multiplication and division. We might conclude immediately that accounting is the arithmetic operations used to explain the legal and economic. But this approach is not quite correct, because the accounting is not only the law and the economy using arithmetic. However, if you remove the mathematics of accounting, there is only the law and the economy, and if we ignore the law and the economy, it remains only arithmetic and in both cases the underlying nature of accounting disappears. If we refer to the epistemology of accounting, it can not exist without its two components, mathematical firstly, economic and legal secondly, like a complex number which can exist only in its two parts, real and complex. Do not forget that the ultimate goal is the description of reality, and at any time it must ensure that its formalism, logics or mathematics, is not in contradiction with reality. Accounting is a language, that is to say, a shell that can only exist if there is a content which gives it its real existence. Theoretical and applied mathematics, which have a special ability to serve content or reference are tolerated only recently in accounting, but still a limited number of accountants and auditors are aware of the relevance of mathematics in general, and linear algebra and matrices in particular, to solve complex problems, unsolvable with conventional tools. There is still much to be done to generalize the application of mathematical science in accounting, but the double-entry bookkeeping have, intrinsically, a vector space structure and can be described by the relationship between law and vectors and matrices, it is still possible to take advantage of certain privileged areas in the consolidated balance sheets and corporate groups for example.

1. Accounting History Represented by Linear Algebra Structures

Accounting users have some problems to use other tools than the four basic arithmetic operations, with a preference for addition or subtraction, sometimes division or multiplication to calculate costs and ratios, and they will sometimes use exponential or logarithmic computing to discount, but never beyond. They are not aware that accounting has a vector space structure. Demonstrations and the most comprehensive developments in this regard have been made by R. Sterling (1967) and J. Bouinot (1971). Having demonstrated that accounting was a vector space structure, it was possible to apply the techniques to accounting matrix. Indeed, in the history of mathematics, matrices have been developed when the theory of vector spaces has been established.

According to R. Mattessich, technology applied to matrix accounting is almost as old as humanity (Mattessich, 1989). He notes that in the early ages of civilization, prehistoric man was already using the principles of Input-Output to describe the physical reality of his immediate environment, using different kinds of tokens. From these findings, Mattessich (1957, 1964) using the work of Leontief showed that all economic transactions can be presented in journal form but also in the form of matrices, vectors, equations or algebraic flowcharts in a network. For him, the matrix structures were always implicitly present in accounting, including the protohistoric period. Magic squares and Latin squares, tables of specific numbers were also known for a very long time by the Chinese and Arabs.

1.1 Classical history of linear algebra and matrix theory

The classic history of linear algebra and matrix theory begins much later. (Athloen and McLaughlin, 1987; Vitulli, n.a). Everything started from the study of the coefficients of linear equations and determinants. Leibnitz began to use it in 1693 and Cramer presented its determinant formula to solve equations (Cramer's rule) in 1750. But already in 1700 Lagrange was an implicit user of matrices in its work on bilinear forms, calculating the maximum and minimum of functions with several variables (Lagrangian). A few years later, in 1800, Carl Friedrich Gauss developed the method known as Gaussian elimination and used it to solve some problems of geodesy. Some authors consider the Gaussian elimination was already known to the Chinese, because He Hui, in 263 AD, assigns Chang Ts'ang as the author of the method, to - 200 BC. More specifically, Gauss was mentioned in a manual about geodesy written by Wilhelm Jordan (1842-1899), a German researcher in geodesy, who studied the reliefs of Germany and Africa and founded the German Journal of Geodesy, combining the two names Gauss-Jordan. The Gauss-Jordan elimination, also called Gaussian elimination, and named in honor of Gauss and Wilhelm Jordan, is an algorithm in linear algebra for finding the solutions of a system of linear equations to identify the rank of a matrix or to calculate the inverse of a (square) invertible matrix. The technique of Gauss-Jordan elimination seems to have been erroneously attributed to Camille Jordan when it is Wilhelm Jordan who published it in 1888 in the third edition of his textbook on geodesy. The story of this algorithm has been studied by Althoen and McLaughlin (1987). Wilhelm Jordan is often confused with the French mathematician Camille Jordan, or the German physicist Pascual Jordan. The English man James Joseph Sylvester used first the term "matrix" to describe an array of numbers in 1850. James Joseph Sylvester (1814 - 1897) worked with Arthur Cayley on algebraic forms, especially on quadratic forms and their invariants and the theory of determinants. He has written hundreds of articles published in particular in the Cambridge and Dublin Mathematical Journal. He taught mathematics at the Royal Military Academy at Woolwich and at the American University Johns Hopkins. Then, Arthur Cayley worked on linear transformations and defined matrix multiplication, and the inverse matrix. Arthur Cayley (1821 - 1895) Professor at the University of Cambridge is one of the fathers of the modern British school of mathematics. He was the first to introduce the multiplication of matrices and the Cayley-Hamilton theorem that any square matrix is a solution of its characteristic polynomial. He outlined, in 1854, a first approach to the modern notion of group. The names of William Rowan Hamilton and Cayley are associated with the Cayley-Hamilton theorem stating that a square matrix is a root of its characteristic polynomial. In fact, the Cayley-Hamilton theorem was proved before them, by Ferdinand Georg Frobenius in 1878. Cayley theorem has been used extensively in this work, and Hamilton proved in the case of two dimensions. Cayley also highlighted the link between the determinants and matrices. William Rowan Hamilton (1805 - 1865) discovered the quaternions, and he showed that the linear operators in the space of quaternions are a special case of the theorem of Cayley-Hamilton. He also found the general method to solve equations of the fifth degree, and important elements about the development of quantum mechanics. Matrices experiencing new fields of application, especially in physics in 1925, when Werner Heisenberg gives the first matrix formulation of quantum mechanics, paving the way for all physicists using matrices and tensors. Modern mathematicians, John von Neumann and Alan Turing were also interested in matrices. John von Neumann, who made important discoveries in quantum mechanics, set theory and economics matrices used to develop the Minimax theorem which provides a rational approach to decision making in clashes between two competitors or adversaries. For its part, Turing, genius of computing, introduced the LU decomposition of a matrix in 1948. LU decomposition is a decomposition of a matrix as a product of a lower triangular matrix L (Low) and an upper triangular matrix U (Up). This decomposition is one of the methods used to solve systems of linear equations. Finally, we can mention Roger Penrose who developed the theory of generalized inverse matrices. When a matrix has two rows or two identical columns, its determinant is zero and it is not invertible. But sometimes, in the accounting field, we can find two sets of identical operations that nullify the determinant. In this case, we can use the generalized inverse matrices to find the solution, which is real, but impossible to find by classical Gauss-Jordan inversion.

For the application of accounting matrices, we must not neglect the work of F. Quesnay, without whom the matrix methods would be less developed (Quesnay, 1760). The Tableau économique of Quesnay can easily be transcribed as an economic system, or better Leontief inter-industry matrix. Describing the relations between social classes (producer's class, landowner's class, sterile class) with product streams and cash flows, he is an essential precursor of accounting matrix. Its *tableau économique* was a breakthrough. But the real founder of the matrix accounting was Augustus de Morgan. In the fifth edition of his book *Elements of arithmetic* (De Morgan, 1846, p. 180-189, 1st edition 1830), Appendix 7, a brief overview on how the accounts are kept, "when they are properly kept". De Morgan described in a few paragraphs how to keep accounts according to the conventional system, the correct way to hold the instruments to be used, the types of records used.

Table 1 – Accounting Matrix of Augustus de Morgan

	A debtor	B debtor	C debtor	D debtor	E debtor
A, creditor		23	19	32	4
B, creditor	17		6	11	25
C, creditor	9	41		10	2
D, creditor	14	28	16		3
E, creditor	15	4	60	1	

Source: De Morgan, 1846, p. 184

Quoting J. Jackson (1956), De Morgan is the most important author of 19th century in terms of innovative accounting records. The way opened by De Morgan remained unexplored until the work of WW Leontief who began studying the input-output matrix in 1947 and has perfected them until 1966 (Leontief, 1947, 1954, 1966, 1986). The great merit of Leontief was clear that the input-output analysis is an application of the neoclassical theory of general equilibrium. He extended his argument by saying that it is possible to go from big economic systems to small economic systems and from small systems to business groups and individual companies. In each case, the equilibrium is realized by a set of linear equations grouped together to form an input-output matrix for the determination of technical coefficients. The Leontief's works are an essential reference on matrix accounting and they are still relevant today. His works inspired Mattessich (1957) who drew the bases of a matrix formulation of accounting systems and Ijiri (1967), in his book called *Foundations of Accounting Measurement*.

1.2 The integration of the matrix computation in double-entry bookkeeping

With the history of accounting, we used four types of writing: pictograms, signs, words, and syllabic alphabets. We also used a limited number of types of accounts, at all ages and in all countries, alongside superimposed positions accounts (Sumerian accounts, accounts Egyptian), there are accounts in separate columns where one devotes a page register or documents the positive changes (inputs or revenue or jobs or flows), such as the left page and the facing page, the right for negative changes (outputs or expenditures or resources or credits). The third type of presentation of the route is married to columns, as in our current bank statements on the same sheet where the increases and decreases occupy two neighboring columns. Matrix accounting where an accounting record $C_y = (S\,E_y , S\,R_y)$, with $C_y \times (R^+ \times R^+)$ is at the coordinates of a row and a column is the fourth possibility, the last have been discovered. Let us briefly recall the calculating procedure for matrix applied to classical accounts. The initial situation is represented by a column vector (Si) where the accounts used are classified in the order of chart of account, receivable accounts with the sign (+) and the payable accounts with the sign (-). Current operations are then stored in a matrix of operations [Mo] with debited accounts in columns and credited accounts in rows, or, conversely, if desired, as shown in Exhibit 1. For the new situation after the movements described in the matrix [MB] must transpose it to get ^{t}[Mo] and then multiply the difference resulting from [^{t}Mo] - [Mo] by the column vector unit (S1).

We obtain (Si) = (t[Mi] - [Mi] x (S1)

The final vector of account after operations (Sf) is obtained by adding the vector to the initial position and the movement vector (Sf) = (Si) + (Sm). The vector (Sf) corresponds to pre-closing trial balance. If you want to establish adjusted trial balance, after performing year-end stocktaking operations, determine the final inventory, save the inventory records, save the adjustments of accruals. For this we must introduce the matrix of inventory operations that can be called [Mi]. As before, we must then calculate the transposed ^{t}[Mi] and multiply the difference by a column vector unit (S1).

We obtain (Si) = (t[Mi] - [Mi] x (S1)

The final situation after inventory adjustments is the sum of two vectors (Sf) and (If) or (S_F) = (Sf) + (Si). The vector (S_F) corresponds to the accounts of post-closing trial balance after the adjustments of accounts for expenses and revenues. Then, it is possible to establish the company's balance sheet with not balanced accounts. (Shank, 1972; Degos and Leclère, 1990). In addition to these basic operations, accounting matrix is useful for more complex accounting tasks, such as modeling systems or consolidation of budget balances and accounts (Degos and Leclère, 1990).

2. Emergence of Consolidated Financial Statements and Gauss-Jordan's Algorithm

Consolidation of accounts has been for a long time a specialty of English-speaking professionals (Finney, 1922; Nicholson, 1924; Newlove, 1926 Paton, 1932; Macbeath and Platt, 1951). Few years ago, Walker recalled this story (Walker, 1978). In France, the consolidation has been slower to emerge, particularly through the work of Veyrenc and Richard (1954) two former presidents of the Institute of Chartered Accountants. Consolidated accounts only became compulsory since 1985, but it existed since 1968, when the National Council of Accounting[1] has published the first work (Bensadon, 2010). On a theoretical level, J. Biabaut (1969) showed that the problem of consolidation was a special case of graph theory where it is necessary not only to enhance the edges, as in the general case, but also enhance the vertices (Biabaut, 1969, pp. . 43 et seq.) He extended the work of C. Flament on graph theory applied to the group structure (Flament, 1963). The consolidation allows the set-up of financial statements of a group of subsidiaries controlled by a holding company, for they are released, but also to be used for internal group.

1) The study of the National Council of Accounting included some imperfections, especially when it studied the reciprocal shareholdings and circular reciprocal shareholding. In addition, the method described to eliminate reciprocal shareholdings was wrong because it prescribed to eliminate the *apparent* reciprocal shareholding (first level) and not *real* reciprocal shareholding (the limit of the loop tends to infinity).

It allows the aggregation, that is to say adding the accounts of each company in a single unit. This aggregation, in classical form or matrix form presents no real problem, what does is the requirement to eliminate inter-company transactions and to make adjustments to make possible additions on the one hand, and the obligation to determine the nature and the actual percentage of control and participation of the parent company in its subsidiaries, especially when the control is not direct or when the control is reciprocal of the other hand[2]. In all these cases, accounting matrix can provide rational answers.

For a group of subsidiaries that are located within a scope of consolidation, there are three types of participation: associate societies undergo exclusive and majority interest of the holding company which owns a majority stake in their capital, companies linked where the parent company has a significant interest of at least 20% with minority interest, and in rare cases the interests of Community companies jointly controlled by two different holdings. The dependency link between the holding and its subsidiaries is realized by the percentage of control expressing the voting rights held and by the percentage of interest simply expressing the part of capital that the holding company owns in its subsidiaries, directly or indirectly. We are primarily interested in equity computations and we will send readers interested in French regulation CRC 99-02, approved by the Decree of 22 June 1999, which sets out the rules and procedures relating to the consolidated European Regulation 2005 IAS 7 Council in June 2002 of the European Parliament which constraints listed companies to present their consolidated accounts according to IFRS standards (IAS 27 and IAS 28) and International Regulations (new IAS 27 on financial statements, IAS 28 new on investments in associates and partners, supplemented by IFRS 10

2) We believe this is Newlove (1926), professor at Johns Hopkins University, in chapters 11 and 12 of his book, which dealt with the first fairly complete problems and indirect interests of reciprocal links. In chapter 12, p. 201-211, he studied shareholdings without reciprocal control and with reciprocal control.

on state financial statements, IFRS 11 on partnerships and IFRS 12 on the information of interests in other entities).

2.1 Shareholding, and reciprocal shareholding of a Group without matrix

Consider a simplified group consisting of a holding company and one or two subsidiaries. In this case there is no need to use the matrix computation and you can use simple techniques of classical algebra:

• Reciprocal shareholding in a mutual parent company and one subsidiary: Suppose a parent company SM that owns x % of the capital of the subsidiary FU that owns itself y % stake in the holding company. If the subsidiary has y % of the parent company, the other shareholders of the parent company have (100 - y) % of their company. They therefore hold [(100 - y)%] ′ x % of the subsidiary, but the stakes are reciprocal, were in the presence of an infinite loop, which is associated with a sum of terms in a geometric progression whose sum is:

$[(100 - y) \times x] \times [(100 - x^n y^n) / (100 - x, y)]$

and which tends to $[(100 - y) \times x] / (100 - x, y)$.

• Circular reciprocal shareholding between parent company and two subsidiaries: suppose a parent company SM that owns u % of its subsidiary FA, the subsidiary FA has v % of the other subsidiary FB and the subsidiary FB has w % of the parent company. The parent company has therefore a certain percentage to determine its subsidiary FB, but also the actual percentage of participation in FA is not apparent u % percentage. If we use the same reasoning as in the case of a single subsidiary, involving the shareholders of the parent company who have (100 - w) % of the company, we can determine the participation in SM as (100 - w) / (100 - uvw), participation in the subsidiary FA: $[(100 - w) \times u] / (100 - uvw)$ and participation in the subsidiary FB: $[(100 - w) \times uv] / (100 - uvw)$. With more than three companies, it is impossible to do simple operations from fractions and percentages and we have to use matrix techniques.

2.2 Shareholding, and reciprocal shareholding using matrix methods

Consider the example given table 2.

Table 2 – Shareholdings between Holding company (HC) and subsidiaries of Castor & Pollux Group

Company of Group	Company XY control (x%) of another company	Company XY is controled at (y%) by another
HC	SA (80%), SB (30%), SC (30%), SD (80%)	No control
SA	SB (10%)	HC (80%), SB (10%)
SB	SA (10%), SC (40%)	SA (10%), SC (20%)
SC	SB (20%), SD (20%)	SB (40%), SD (10%)
SD	SC (10%), SD (20%)	HC (80 %), SC (20%)

It is not easy to calculate, *a priori*, the interests of the holding company in each of its subsidiaries as a result of reciprocal links in the form of loops, the real interests being different than apparent interests. If (Y) is the vector of total shares of the holding company that is sought, (P) is the vector of visible minority and [A] matrix entries where each component $a_{i,j}$ is the participation of the society I in company J, and $a_{i,i} = 0$;

$$(Y) = \begin{bmatrix} y_H \\ y_A \\ y_B \\ y_C \\ y_D \end{bmatrix} ; \quad (P) = \begin{bmatrix} 1 \\ 0{,}10 \\ 0{,}40 \\ 0{,}20 \\ 0 \end{bmatrix} ; \quad [A] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0{,}80 & 0 & 0{,}10 & 0 & 0 \\ 0{,}30 & 0{,}10 & 0 & 0{,}20 & 0 \\ 0{,}30 & 0 & 0{,}40 & 0 & 0{,}10 \\ 0{,}80 & 0 & 0 & 0{,}20 & 0 \end{bmatrix}$$

By iteration shareholding in second order is [A] x (P), shareholding in 3rd order is $[A]^{2}$ x (P), and shareholding in n order is $[A]^{n-1}$ x (P), then (Y_n) = (P) + [A] x (P) + $[A]^{2}$ x (P) + ... + $[A]^{n-1}$ x (P),
But (Y) = lim (Y_n) when (Y_n) ® ¥ . If we multiply by [A] and if we subtract [A] ´ (Y_n) from (Y_n) after eliminating similar terms, as lim $[A]^{n} = 0$, we get:
(Y) – [A] x (Y) = (P).
Let [I] the identity matrix, we get:
(Y) – [A] x (Y) = [I – A] ´ (Y) = (P)
If we have [B] = [I - A] we can write: [B] x (Y) = (P) Þ (Y) = $[B]^{-1}$ x (P).

If we take the example of the Castor & Pollux Group of table 1, the matrix [B] is equal to:

$$[B] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -0{,}80 & 1 & -0{,}10 & 0 & 0 \\ -0{,}30 & -0{,}10 & 1 & -0{,}20 & 0 \\ -0{,}30 & 0 & -0{,}40 & 1 & -0{,}10 \\ -0{,}80 & 0 & 0 & -0{,}20 & 1 \end{bmatrix}$$

Its inverse $[B]^{-1}$ is equal to:

$$[B]^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0{,}851 & 1{,}011 & 0{,}110 & 0{,}022 & 0{,}002 \\ 0{,}504 & 0{,}110 & 1{,}100 & 0{,}224 & 0{,}022 \\ 0{,}594 & 0{,}045 & 0{,}449 & 1{,}112 & 0{,}111 \\ 0{,}919 & 0{,}009 & 0{,}090 & 0{,}222 & 1{,}022 \end{bmatrix}$$

And the vector of shares sought is equal to:

$$(Y) = [B]^{-1} \times (P) = \begin{bmatrix} 0 \\ 0{,}149 \\ 0{,}496 \\ 0{,}406 \\ 0{,}081 \end{bmatrix}$$

This corresponds to the following interests:

Table 3 – Real interest in Castor & Pollux Group

Company of Group	Holding interest	Visible interest	Minority interest	Consolidation method
HC	100 %		0 %	Parent company
SA	85,1 %	80 %	14,9 %	Full consolidation
SB	50,4 %	30 %	49,6 %	Full consolidation
SC	59,4 %	30 %	40,6 %	Full consolidation
SD	91,9 %	80 %	8,1 %	Full consolidation

The results, above, have been made, including the computing of inverse matrices, with the classical Gauss-Jordan method, from the percentages stored as fractions. There are other methods of computation, but one of the most attractive and the most magical is the method of Markov chains.

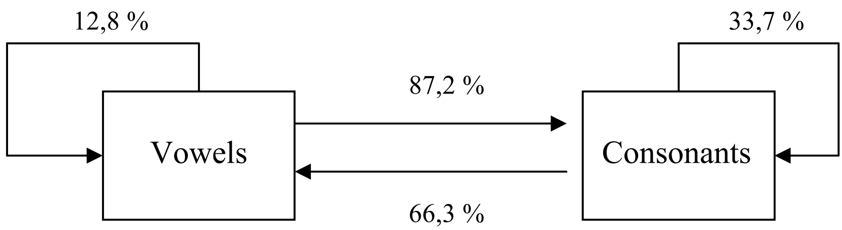
3. Consolidation, Markov chains, and Platonic solids

Andrei Andreyevich Markov (Андрей Андреевич Марков) was a Russian mathematician born June 2, 1856 and died July 20, 1922. He was a professor at the State University of St. Petersburg, Pafnouti Chebyshev (Пафнутий Львович Чебышёв) of to the Russian School of Mathematics and Statistics, founded by Daniel Bernoulli and Leonhard Euler who himself died in St. Petersburg. Member of the Academy of Sciences in St. Petersburg, Markov became interested in lexicography and the succession of letters of the Russian alphabet used in Pushkin's novel Eugene Onegin (Евгений Онегин). The succession of each letter of the novel, appearing not at random, but according to the previous letter with a certain probability, was the first work as a prelude to the discovery of Markov chains. By studying 20,000 characters of Eugene Onegin, he set up the transition matrix [P] as follows:

$$\text{Matrix } [P] = \begin{bmatrix} 12,8\,\% & 87,2\,\% \\ 66,3\,\% & 33,7\,\% \end{bmatrix}$$

Such a matrix is called stochastic, that is to say the sum of each row is equal to 1. This property is present in each transition matrix because the elements for row i are the likelihood of all possible transition from one state to another, without omitting any. If there are zeros in the transition matrix, this means that the transition is impossible. Fig. 1 shows the transition matrix of the above cases:

Figure 1 – Likelihood of succession of vowels and consonants (Markov, 1913)



In Russian, in Eugene Onegin are the likelihood that a vowel followed by a consonant is 87.2 % and the likelihood that a consonant followed by a vowel is 66.3%. The law limits associated with likelihood of transition is p = (43.2%, 56.2%), it is the overall probability of occurrence of vowels and consonants. These results were established in 1913, but Markov had already released its first findings in 1906, assuming a finite state space (Markov, 1906) and thirty years later, Kolmogorov (Андрей Николаевич Колмогоров) generalized the theory of Markov for the infinite state spaces.

The real and general problem studied by the Markov chains is (Kemeny et al, 1964): the process is in the initial state i. What is the likelihood that it is found in state j after n iterations at time n. As we get several states (or more vertices in a graph) it is necessary to calculate all the likelihood for all the initial positions of i at time 0 and for all end positions j at time n. These probabilities can be represented by a stochastic matrix, for example after n iterations for a Markov process with three states:

$$[P]^{n} = \begin{bmatrix} p^{n}_{11} & p^{n}_{21} & p^{n}_{31} \\ p^{n}_{12} & p^{n}_{22} & p^{n}_{32} \\ p^{n}_{13} & p^{n}_{23} & p^{n}_{33} \end{bmatrix}$$

This matrix, called transition matrix, is square, stochastic because all lines are equal to 1 and regular, since all terms are non negative. More if you can go from one state to any other, the matrix is ergodic. Markov chain assumes an initial state defined by: $(P)^{0} = (P^{0}_{1}, P^{0}_{2}, P^{0}_{3})$.

The probability that the vector (P) in state aj after n iterations is given by the vector $(P)^{n} = (P^{n}_{1}, P^{n}_{2}\ P^{n}_{3})$. Multiplying the vector $(P)^{0}$ of initial probabilities by n^{th} power of the transition matrix [P], we get the vector which components give the probabilities of being in each state after n iterations. In some cases, the vector is a fixed point of the matrix [P] in the following equation: $w = w \times (P)^{n}$. At the invariant point, the elements of the matrix are real non-probabilistic values and not probabilistic expression, and one of the paradoxes of Markov chains is that a probabilistic system tends to a non-probabilistic limit. Applications of Markov chains are innumerable, the most recent evaluation notes include rating agencies, and the probability of passage of any note, for example A, a better rating (AA) or a worst rating (BBB). Such a matrix has 8 states, from the highest grade AAA to failure. One can also evaluate for financial market indexes going from bearish stage where the index is stable, to a bullish situation, where the index is rising sharply, or a recession where index decreases. For consolidations, we use the Markov model, regardless of likelihood, but also noting that the model tends to a stable situation where there is absorbing states and an invariant point.

3.1 Markov Chains to better describe consolidated groups.

In the classical Gauss-Jordan method, we have a reductive view of minority interests, outside the Group. Instead, the use of Markov chains allows to better address this problem (Guérin and Pouget, 1972). If we keep the example of the Castor & Pollux Group, it is necessary to slightly modify the table 2 with table 4:

Table 4 – Links of Castor & Pollux Group with minority interests (MI)

Company of Group	Company XY control (x%) of another company	Company XY is controled at (y%) by another
HC	SA (80%), SB (30%), SC (30%), SD (80%)	HC (100%)
SA	SB (10%)	HC (80%), SB (10%), MI (10%)
SB	SA (10%), SC (40%), MI (40%)	SA (10%), SC (20%), MI (40%)
SC	SB (20%), SD (20%)	SB (40%), SD (10%), MI (20%)
SD	SC (10%), SD (20%)	HC (80 %), SC (20%)
MI	SA (10%), SB (40%), SC (20%)	MI (100%)

The table above can be represented by the incidence matrix [X] as follows:

$$[X] = \begin{bmatrix} 1 & 0,80 & 0,30 & 0,30 & 0,80 & 0 \\ 0 & 0 & 0,10 & 0 & 0 & 0 \\ 0 & 0,10 & 0 & 0,40 & 0 & 0 \\ 0 & 0 & 0,20 & 0 & 0,20 & 0 \\ 0 & 0 & 0 & 0,10 & 0 & 0 \\ 0 & 0,10 & 0,40 & 0,20 & 0 & 1 \end{bmatrix}$$

The matrix [X] will be «separated» into two matrices: the matrix [B] of dimensions (6.6) with only two values 1 for elements $b_{1,1}$ corresponding to the participation in the holding company itself, and $b_{6,6}$ corresponding to the participation of minority interest in themselves, and the matrix [A] which will include all the elements of the matrix [X], except the elements $a_{1,1}$ and $a_{6,6}$ which have the value 0 . We can check that [X] = [A] + [B].

$$[B] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} ; \quad [A] = \begin{bmatrix} 0 & 0,80 & 0,30 & 0,30 & 0,80 & 0 \\ 0 & 0 & 0,10 & 0 & 0 & 0 \\ 0 & 0,10 & 0 & 0,40 & 0 & 0 \\ 0 & 0 & 0,20 & 0 & 0,20 & 0 \\ 0 & 0 & 0 & 0,10 & 0 & 0 \\ 0 & 0,10 & 0,40 & 0,20 & 0 & 0 \end{bmatrix}$$

By calculating the difference between the unit matrix [I] and the matrix [A], reversing [I - A] and multiplying the matrix [B] by the inverse, we get the matrix $[X_n]$ which is a limit. We get: $[Xn] = [B] \times [I - A]^{-1}$

$$[I - A] = \begin{bmatrix} 1 & -0,80 & -0,30 & -0,30 & -0,80 & 0 \\ 0 & 1 & -0,10 & 0 & 0 & 0 \\ 0 & -0,10 & 1 & -0,40 & 0 & 0 \\ 0 & 0 & -0,20 & 1 & -0,20 & 0 \\ 0 & 0 & 0 & -0,10 & 1 & 0 \\ 0 & -0,10 & -0,40 & -0,20 & 0 & 1 \end{bmatrix}$$

The inverse matrix [I - A] is:

$$[I - A]^{-1} = \begin{bmatrix} 1 & 0,851 & 0,504 & 0,594 & 0,919 & 0 \\ 0 & 1,011 & 0,110 & 0,045 & 0,009 & 0 \\ 0 & 0,110 & 1,100 & 0,449 & 0,090 & 0 \\ 0 & 0,022 & 0,224 & 1,112 & 0,222 & 0 \\ 0 & 0,002 & 0,022 & 0,111 & 1,022 & 0 \\ 0 & 0,149 & 0,496 & 0,406 & 0,081 & 1 \end{bmatrix}$$

And its limit $[X_n]$ is:

$$[X_n] = \begin{bmatrix} 1 & 0,851 & 0,504 & 0,594 & 0,919 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0,149 & 0,496 & 0,406 & 0,081 & 1 \end{bmatrix}$$

We can read the matrix $[X_n]$ as follows: column 1, nobody controls the holding company. Column 2, the holding company controls 85.1 % of its subsidiary FA and minority shareholders controls up to 14.0 % of group. The results are similar to results found with the Gauss-Jordan method due to the use of fractions and not decimals rounded (85.1 % = 3785/4451 x 100 %), 14.9 % = 666 / 4451 x 100 %), and in the absence of collinearity phenomena. When the complexity of groups increases, the reciprocal shareholding and circular shareholding are difficult to detect at single glance, and facing this increased complexity, managers and auditors must control opacity phenomena and dilution of assets in a lot of subsidiaries. Matrix techniques are able to bring out the hidden reality under more or less misleading appearances.

3.2 Markov chains, Platonic solids and graphs to optimize the group structure

When we have to do some audit of a group, with a legal position of statutory auditors, we are often struck by the “architectural disorder” of groups formed randomly, depending on the opportunities for purchase or control. Often, when the group is unbalanced, for any reason whatsoever, it is not easy to restore a harmonious structure and many professionals do not have a solution to offer in front of this situation.

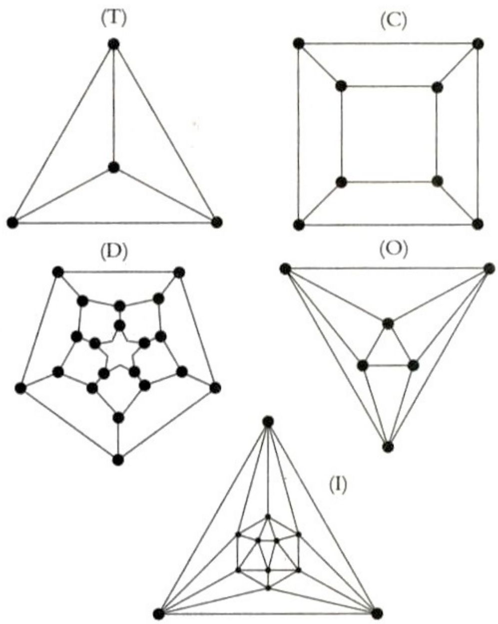
And yet, it seems that a solution exists, inspired by the architecture of perfect Pythagoreans solids, completely symmetrical, or even the architecture of imperfect solids described by Luca Pacioli and drawn by Leonardo da Vinci (Pacioli, 1509). We assume, probably to be checked, a coherent structure, such as symmetric, is much more resistant to external pressures and much easier to control, on the one hand, and a coherent structure allows better circulation information and products: tours can be shorter, more secure and rational. The richness of the regular polyhedron in terms of symmetry has enriched the mathematical theory of groups by the symmetry groups in space. First use was made by Johannes Kepler (1596) to represent the orbits of the five planets known at the time: Jupiter, Mars, Earth, Venus and Mercury. Luca Pacioli describes them not only in the *Divina proportione*, as we have seen above, but also in the *Summa de arithmetica* (Pacioli, 1494). A comprehensive study was made by L’Huillier (1812) and more recently by Joly (1979). Represented in three dimensions, the regular polyhedrons are not very easy to use, but can be represented by their graphs, from Descartes-Euler theorem, also called the Euler characteristic. According to this feature, the convex polyhedrons obey the relation $S - A + F = 2$, where S is the number of vertices, A is the number of edges and F is the number of faces. For five regular solids, we get:

Table 5 – Euler characteristics of regular polyhedrons

Polyhedron	Shape of faces	Number of sides	Number of vertices	Number of edges	Number of faces	Schläfli symbol	Vertex Config.
Tetrahedron	Triangle	3	4	6	4	{3,3}	3.3.3
Cube	Square	4	8	12	6	{4,3}	4.4.4
Octahedron	Triangle	3	6	12	8	{3,4}	3.3.3.3
Icosahedron	Triangle	3	12	30	20	{5,3}	5.5.5.
Dodecahedron	Pentagon	5	20	20	12	{3,5}	3.3.3.3.3

Each Platonic solid can therefore be denoted by a symbol {p, q} where p = the number of edges of each face (or the number of vertices of each face) and q = the number of faces meeting at each vertex (or the number of edges meeting at each vertex). The symbol {p, q}, called the Schläfli symbol, gives a combinatorial description of the polyhedron. The Schläfli symbols of the five Platonic solids are given in the table 5 right. Ignoring the solid faces, and studying only the edges and vertices we can represent some graphs (Source: Alsina, 2011), which brings us back to representations similar to those in Table 2 or Table 4, presented above.

Figure 2 – Graphs of Platonic solids



To build a group to consolidate with such graphs, we must first choose the position of the parent company, eg in the center of the graph and decide a majority stake in key subsidiaries. Then, each vertex must not receive more than 100 % of controls and that he does not carry more than 100 %. Let us take for example the case of the tetrahedron, the top left of the five graphs: we can try to see what would result in a structure where the parent company would control 90 % of its 3 subsidiaries directly and indirectly 5 % or even 50 % directly and 45 % indirectly, or finally if all companies would control up to another 10 %. Once we have chosen the structure, it is reduced to the problem of the previous paragraph: valuation of the Group's investments, excluding equity group valuation and determination of beneficial interests in the parent company in each of its subsidiaries. We could do the same with the other graphs. Matrices participations in different cases can be used to optimize the structure of the group step by step, and in addition, matrices participation can become a diagnostic tool:

- The number of columns equal to the number of vertices indicates the general nature of the structure: 4 columns and 4 companies for a tetrahedron, 6 columns and 6 companies for a cube, 8 columns and 8 companies to an octahedron, etc..

- The first row of the matrix shows the controls suffered by the parent company;

- In columns we see the percentage of control of the holding on each subsidiary;

- The first diagonal must be zero if a company has no own shares.

It is also possible to integrate a standard of participation: reciprocal shareholdings are regulated, in France, by Articles L. 223-29, L. 223-30 and L. 223-31 of the Commercial Code, but in other countries interests are not limited, for example in Italy. You can also try to determine the limits of self-control for each graph type. The last table gives some indications:

Table 6 – Limits of the self-control regarding the shape of graphs

Number of companies	Structure to choose	Theoritical maximum interest by company	Theoritical maximum interest for holding	Theoritical minimum interest by company	Theoritical minimum interest for holding
4	Tetrahedron	33 %	99 %	17 %	51 %
6	Cube	20 %	100 %	10 %	51 %
8	Octahedron	15 %	99 %	7 %	50 %
12	Icosahedron	8 %	99 %	4 %	50 %
20	Dodecahedron	5 %	100 %	3 %	50 %

With theoretical interests above - the group's control on itself would be absolute, and we must avoid it (Degos and Leclère, 1990, 1999, 2009) because if a group completely controls itself it prevents shareholders from voting.

Conclusion

There are still a lot of accounting problems related to corporate groups where the matrices and Markov chains have shown – historically – they were useful for the calculation of capital, reserves and results of the consolidated balance sheet of the holding company for the optimization of Group Treasury for planning investment projects of the group, using the real options method associated with Markov chains, to make simulations and predictions in the short, medium and long term. The matrix methods have historically provided to complex accounting not only a necessary scientific rigor but also a creativity that brings added value. "The accounting description should always make an effort to shape the reality and the application of the formalism alone is not enough" (Lassègue, 1962). But the mathematical formalism is a guarantee of rigor and control. And more, made in the form outlined above, accounting has a better ability to serve as a basis for economic calculation and its speculations, to support and validate the quantitative elements of strategy. Accounting, in itself, is not always intended to bind parameters other than by calculation; the economy may instead highlight the positive or contradictory aspects of a problem with all the nuances and all the necessary extensions. Mathematics, economics and accounting are often separated in research and teaching, but sometimes it is necessary to coordinate them and provide practical solutions to problems of performance management. Matrix methods are a relevant link between past accounting, Matrix methods are a relevant link between past accounting, and future.

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