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Normal paracontact metric space form on W_0-curvature tensor

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Abstract − In this article, normal paracontact metric space forms are investigated on W_0-curvature tensor. Characterizations of normal paracontact space forms are obtained on W_0-curvature tensor. Special curvature conditions established with the help of Riemann, Ricci, and concircular curvature tensors are discussed on W_0-curvature tensor. Through these curvature conditions, some important characterizations of normal paracontact metric space forms are obtained. Finally, the need for further research is discussed.

Keywords: *W_0-curvature tensors, semisymmetric manifold, normal paracontact space form*

Subject Classification (2020): 53C15, 53C25

1. Introduction

The study of paracontact geometry was initiated by Kenayuki and Williams [1]. Zamkovoy [2] studied paracontact metric manifolds and their subclasses. Recently, Welyczko [3-4] studied curvature and torsion of Frenet Legendre curves in 3-dimensional normal paracontact metric manifolds. In the recent years, contact metric manifolds and their curvature properties have been studied by many authors in [5-7].

In this article, normal paracontact metric space forms are investigated on W_0-curvature tensor. Characterizations of normal paracontact space forms are obtained on W_0-curvature tensor. Special curvature conditions established with the help of Riemann, Ricci, concircular curvature tensors are discussed on W_0-curvature tensor. Through these curvature conditions, some important characterizations of normal paracontact metric space forms are obtained.

2. Preliminaries

Take an n-dimensional differentiable M manifold. If it admits a tensor field ϕ of type $(1,1)$, a contravariant vector field ξ and a 1-form η satisfying the following conditions:

$$\phi^2X = X - \eta(X)\xi, \phi\xi = 0, \eta(\phi X) = 0, \eta(\xi) = 1 \tag{2.1}$$

and

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), g(X,\xi) = \eta(X) \tag{2.2}$$

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for all $X, Y, \xi \in \chi(M)$, (ϕ, ξ, η) is called almost paracontact structure and (M, ϕ, ξ, η) is called almost paracontact metric manifold. If the covariant derivative of ϕ satisfies

$$(\nabla_X \phi)Y = -g(X,Y)\xi - \eta(Y)X + 2\eta(X)\eta(Y)\xi \tag{2.3}$$

then, M is called a normal paracontact metric manifold, where ∇ is Levi-Civita connection. From (2.3), we can easily to see that

$$\phi X = \nabla_X \xi \tag{2.4}$$

for any $X \in \chi(M)$ [1].

Moreover, if such a manifold has constant sectional curvature equal to c, then it is the Riemannian curvature tensor is R given by

$$\begin{aligned} R(X,Y)Z = {} & \frac{c+3}{4}[g(Y,Z)X - g(X,Z)Y] + \frac{c-1}{4}[\eta(X)\eta(Z)Y \\ & -\eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi + g(\phi Y,Z)\phi X \\ & -g(\phi X,Z)\phi Y - 2g(\phi X,Y)\phi Z] \end{aligned} \tag{2.5}$$

for any vector fields $X, Y, Z \in \chi(M)$ [5].

In a normal paracontact metric space form by direct calculations, we can easily to see that

$$S(X,Y) = \frac{c(n-5)+3n+1}{4} g(X,Y) + \frac{(c-1)(5-n)}{4} \eta(X)\eta(Y) \tag{2.6}$$

which implies that

$$QX = \frac{c(n-5)+4n+1}{4} X + \frac{(c-1)(5-n)}{4} \eta(X)\xi \tag{2.7}$$

for any $X, Y \in \chi(M)$, where Q is the Ricci operator and S is the Ricci tensor of M.

Lemma 2.1. Let M be an n-dimensional normal paracontact metric manifold. In this case, the following equations hold.

$$R(\xi, X)Y = g(X,Y)\xi - \eta(Y)X \tag{2.8}$$

$$R(X, \xi)Y = -g(X,Y)\xi + \eta(Y)X \tag{2.9}$$

$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y \tag{2.10}$$

$$\eta(R(X,Y)Z) = g(\eta(X)Y - \eta(Y)X, Z) \tag{2.11}$$

$$S(X,\xi) = (n-1)\eta(X) \tag{2.12}$$

$$Q\xi = (n-1)\xi \tag{2.13}$$

where R, S, and Q are Riemann curvature tensor, Ricci curvature tensor, and Ricci operator, respectively.

Tripathi and Gunam [8] described a τ-curvature tensors of the $(1,3)$ type in an n-dimensional (M, g) semi-Riemann manifold. One of these tensors is defined as follows:

Definition 2.1. Let M be an n-dimensional semi-Riemannian manifold. The curvature tensor defined as

$$W_0(X,Y)Z = R(X,Y)Z - \frac{1}{n-1}[S(Y,Z)X - g(X,Z)QY] \tag{2.14}$$

is called the W_0-curvature tensor.

For the n-dimensional normal paracontact metric space form, if we choose $X = \xi$, $Y = \xi$, and $Z = \xi$, respectively in (2.14), then we get

$$W_0(X,Y)Z = R(X,Y)Z - \frac{1}{n-1}[S(Y,Z)X - g(X,Z)QY] \tag{2.15}$$

$$W_0(X,\xi)Z = 0 \tag{2.16}$$

$$W_0(X,Y)\xi = \frac{(n-5)(c-1)}{4(n-1)}[\eta(X)Y - \eta(X)\eta(Y)\xi] \tag{2.17}$$

Definition 2.2. Let M be a paracontact manifold. If its Ricci tensor S of type $(0,2)$ is of the form

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y) \tag{2.18}$$

then M is called η-Einstein manifold, where a, b are smooth functions on M. Moreover, if $b = 0$, then the manifold is called Einstein.

Definition 2.3. Let (M, g) be a semi-Riemannian manifold and the two-dimensional subspace Π of the tangent space $T_p(M)$. If $K(X_p, Y_p)$ is constant for each $p \in M$ and $X_p, Y_p \in T_p(M)$, then M is called a real space form, where $K(X_p, Y_p)$ is the section curvature of the Π plane.

3. Normal Paracontact Metric Space Forms on W_0-Curvature Tensor

In this section, the characterization of normal paracontact metric space form under special curvature conditions created by W_0-curvature tensor with Riemann, Ricci, concircular curvature tensors will be given. State and prove the following theorems.

Theorem 3.1. Let M be a n-dimensional normal paracontact metric space form. If M is W_0-flat, then M is an Einstein manifold.

Proof.

Assume that manifold M is W_0-flat. From (2.14), we can write

$$W_0(X,Y)Z = 0$$

for each $X, Y, Z \in \chi(M)$. Then, from (2.14), we obtain

$$R(X,Y)Z = \frac{1}{n-1}[S(Y,Z)X - g(X,Z)QY] \tag{3.1}$$

for each $X, Y, Z \in \chi(M)$. If we choose $Z = \xi$ in (3.1) and using (2.10) and (2.12), we obtain

$$\eta(X)QY = (n-1)\eta(X)Y \tag{3.2}$$

If we choose $X = \xi$ in (3.2) and take inner product both sides of the last equation by $Z \in \chi(M)$, then we get

$$S(Y,Z) = (n-1)g(Y,Z)$$

It is clear from the last equation that M is Einstein manifold. □

Theorem 3.2. Let M be the n-dimensional normal paracontact metric space form. If M is W_0-semisymmetric, then M is an Einstein manifold.

Proof.

Assume that M is W_0-semisymmetric. This means

$$(R(X,Y) \cdot W_0)(U,V,Z) = 0$$

for every $X, Y, Z, U, V \in \chi(M)$. Therefore, we can write

$$R(X,Y)W_0(U,V)Z - W_0(R(X,Y)U,V)Z - W_0(U,R(X,Y)V)Z - W_0(U,V)R(X,Y)Z = 0 \tag{3.3}$$

If we choose $X = \xi$ in (3.3) and make use of (2.8), we get

$$\begin{aligned} &g(Y,W_0(U,V)Z)\xi - \eta(W_0(U,V)Z)Y - g(Y,U)W_0(\xi,V)Z \\ &+\eta(U)W_0(Y,V)Z - g(Y,V)W_0(U,\xi)Z + \eta(V)W_0(U,Y)Z \\ &-g(Y,Z)W_0(U,V)\xi + \eta(Z)W_0(U,V)Y = 0 \end{aligned} \tag{3.4}$$

If we use (2.15)-(2.17) in (3.4), we obtain

$$\begin{aligned} &g(Y,W_0(U,V)Z)\xi - \eta(W_0(U,V)Z)Y + Ag(Y,U)g(V,Z)\xi \\ &-Ag(Y,U)\eta(Z)V + \eta(U)W_0(Y,V)Z + \eta(V)W_0(U,Y)Z \\ &-Ag(Y,Z)\eta(U)V + Ag(Y,Z)\eta(U)\eta(V)\xi + \eta(Z)W_0(U,V)Y = 0, \end{aligned} \tag{3.5}$$

where $A = \frac{(n-5)(c-1)}{4(n-1)}$. If we choose $U = \xi$ in (3.5) and use (2.15), we get

$$W_0(Y,V)Z + Ag(V,Z)Y - Ag(Y,Z)V = 0 \tag{3.6}$$

Putting (2.14) in (3.6), we have

$$R(Y,V)Z - \frac{1}{n-1}S(V,Z)Y + \frac{1}{n-1}g(Y,Z)QV + Ag(V,Z)Y - Ag(Y,Z)V = 0 \tag{3.7}$$

If we choose $Z = \xi$ in (3.5) and use (2.10) and (2.12), we get

$$\frac{1}{n-1}\eta(Y)QV + A\eta(V)Y - A\eta(Y)V = 0 \tag{3.8}$$

In (3.8), if we choose $Y = \xi$, and take inner product both sides of the equation by $Z \in \chi(M)$, we then have

$$S(V,Z) = \frac{(n-5)(c-1) + 4(n-1)}{4}g(V,Z) - \frac{(n-5)(c-1)}{4}\eta(V)\eta(Z)$$

□

Theorem 3.3. Let M be the n-dimensional normal paracontact metric space form. If M satisfies the curvature condition $W_0 \cdot R = 0$, then M is a real space form with constant scalar curvature.

Proof.

Assume that

$$(W_0(X,Y) \cdot R)(U,V,Z) = 0$$

for every $X,Y,Z,U,V \in \chi(M)$. Therefore, we can write

$$\begin{aligned} &W_0(X,Y)R(U,V)Z - R(W_0(X,Y)U,V)Z \\ &-R(U,W_0(X,Y)V)Z - R(U,V)W_0(X,Y)Z = 0 \end{aligned} \tag{3.9}$$

If we choose $X = \xi$ in (3.9) and make use of (2.15), we get

$$\begin{aligned} &-Ag(Y,R(U,V)Z)\xi + A\eta(R(U,V)Z)Y + Ag(Y,U)R(\xi,V)Z \\ &-A\eta(U)R(Y,V)Z + Ag(Y,V)R(U,\xi)Z - A\eta(V)R(U,Y)Z \\ &+Ag(Y,Z)R(U,V)\xi - A\eta(Z)R(U,V)Y = 0 \end{aligned} \tag{3.10}$$

If we use (2.8)-(2.10) in (3.10), we obtain

$$\begin{aligned} &-Ag(Y,R(U,V)Z)\xi + A\eta(R(U,V)Z)Y + Ag(Y,U)g(V,Z)\xi \\ &-Ag(Y,U)\eta(Z)V - A\eta(U)R(Y,V)Z - Ag(Y,V)g(U,Z)\xi \\ &+Ag(Y,V)\eta(Z)U - A\eta(V)R(U,Y)Z - A\eta(Z)R(U,V)Y \\ &+Ag(Y,Z)\eta(V)U - Ag(Y,Z)\eta(U)V = 0 \end{aligned} \tag{3.11}$$

If we choose $U = \xi$ in (3.11) and use (2.8), we get

$$-A[R(Y,V)Z - g(V,Z)Y + g(Y,Z)V] = 0 \tag{3.12}$$

□

Theorem 3.4. Let M be the n-dimensional normal paracontact metric space form. If M satisfies the curvature condition $W_0 \cdot W_0 = 0$, then M is an η-Einstein manifold.

Proof.

Assume that

$$(W_0(X,Y) \cdot W_0)(U,V,Z) = 0$$

for every $X,Y,Z,U,V \in \chi(M)$. Therefore, we can write

$$W_0(X,Y)W_0(U,V)Z - W_0(W_0(X,Y)U,V)Z - W_0(U,W_0(X,Y)V)Z - W_0(U,V)W_0(X,Y)Z = 0 \tag{3.13}$$

If we choose $X = \xi$ in (3.13) and make use of (2.15), we get

$$\begin{aligned}&-Ag(Y,W_0(U,V)Z)\xi + A\eta(W_0(U,V)Z)Y + Ag(Y,U)W_0(\xi,V)Z\\&-A\eta(U)W_0(Y,V)Z + Ag(Y,V)W_0(U,\xi)Z - A\eta(V)W_0(U,Y)Z\\&+Ag(Y,Z)W_0(U,V)\xi - A\eta(Z)W_0(U,V)Y = 0\end{aligned} \tag{3.14}$$

If we use (2.15)-(2.17) in (3.14), we obtain

$$\begin{aligned}&-Ag(Y,W_0(U,V)Z)\xi + A\eta(W_0(U,V)Z)Y - A^2g(Y,U)g(V,Z)\xi\\&+A^2g(Y,U)\eta(Z)V - A\eta(U)W_0(Y,V)Z - A\eta(V)W_0(U,Y)Z\\&+A^2g(Y,Z)\eta(U)V - A^2g(Y,Z)\eta(U)\eta(V)\xi - A\eta(Z)W_0(U,V)Y = 0\end{aligned} \tag{3.15}$$

If we choose $U = \xi$ in (3.15) and make the necessary adjustments using (2.15), we get

$$-A\{W_0(Y,V)Z + A[g(V,Z)Y - g(Y,Z)V]\} = 0 \tag{3.16}$$

Putting (2.14) in (3.16) and if we choose $Z = \xi$, we obtain

$$-A\left[A\eta(V)Y - (A+1)\eta(Y)V + \frac{1}{n-1}\eta(Y)QV\right] = 0 \tag{3.17}$$

If we choose $Y = \xi$ in (3.17), then we take inner product both sides of the equation by $Z \in \chi(M)$, we have

$$S(V,Z) = \frac{(n-5)(c-1) + 4(n-1)}{4}g(V,Z) - \frac{(n-5)(c-1)}{4}\eta(V)\eta(Z)$$

□

Corollary 3.1. Let M be the n-dimensional normal paracontact metric space form. If M satisfies the curvature condition $W_0 \cdot W_0 = 0$, then M is an Einstein manifold if and only if M is a real space form with constant scalar curvature $c = 1$.

Definition 3.1. Let M be an n-dimensional Riemannian manifold. The curvature tensor defined as

$$\tilde{Z}(X,Y)Z = R(X,Y)Z - \frac{r}{n(n-1)}[g(Y,Z)X - g(X,Z)Y] \tag{3.18}$$

is called the concircular curvature tensor.

For the n-dimensional normal paracontact metric space form, if we choose $X = \xi$, $Y = \xi$, and $Z = \xi$ in (3.18), respectively, then we get

$$\tilde{Z}(\xi,Y)Z = \left[1 - \frac{r}{n(n-1)}\right][g(Y,Z)\xi - \eta(Z)Y] \tag{3.19}$$

$$\tilde{Z}(X,\xi)Z = \left[1 - \frac{r}{n(n-1)}\right][-g(X,Z)\xi + \eta(Z)Y] \tag{3.20}$$

$$\tilde{Z}(X,Y)\xi = \left[1 - \frac{r}{n(n-1)}\right][\eta(Y)X - \eta(X)Y] \tag{3.21}$$

Theorem 3.5. Let M be the n-dimensional normal paracontact metric space form. If M satisfies the curvature condition $W_0 \cdot \tilde{Z} = 0$, then M is a real space form with constant scalar curvature.

Proof.

Assume that

$$\left(W_0(X,Y) \cdot \tilde{Z}\right)(U,V,Z) = 0$$

for every $X, Y, Z, U, V \in \chi(M)$. Therefore, we can write

$$W_0(X,Y)\tilde{Z}(U,V)Z - \tilde{Z}(W_0(X,Y)U,V)Z - \tilde{Z}(U,W_0(X,Y)V)Z - \tilde{Z}(U,V)W_0(X,Y)Z = 0 \tag{3.22}$$

If we choose $X = \xi$ in (3.22) and make use of (2.15), we get

$$\begin{aligned} &-Ag\left(Y,\tilde{Z}(U,V)Z\right)\xi + A\eta\left(\tilde{Z}(U,V)Z\right)Y + Ag(Y,U)\tilde{Z}(\xi,V)Z \\ &-A\eta(U)\tilde{Z}(Y,V)Z + Ag(Y,V)\tilde{Z}(U,\xi)Z - A\eta(V)\tilde{Z}(U,Y)Z \\ &+Ag(Y,Z)\tilde{Z}(U,V)\xi - A\eta(Z)\tilde{Z}(U,V)Y = 0 \end{aligned} \tag{3.23}$$

If we use (3.19)-(3.21) in (3.23), we obtain

$$\begin{aligned} &-Ag\left(Y,\tilde{Z}(U,V)Z\right)\xi + A\eta\left(\tilde{Z}(U,V)Z\right)Y + ABg(Y,U)\eta g(V,Z)\xi \\ &-ABg(Y,U)\eta(Z)V - A\eta(U)\tilde{Z}(Y,V)Z - ABg(Y,V)g(U,Z)\xi \\ &+ABg(Y,V)\eta(Z)U - A\eta(V)\tilde{Z}(U,Y)Z + ABg(Y,Z)\eta(V)U \\ &-ABg(Y,Z)\eta(U)V - A\eta(Z)\tilde{Z}(U,V)Y = 0 \end{aligned} \tag{3.24}$$

where $B = \left[1 - \frac{r}{n(n-1)}\right]$. If we choose $U = \xi$ in (3.24) and make the necessary adjustments using (3.19), we get

$$-A\{\tilde{Z}(Y,V)Z + B[g(Y,Z)V - g(V,Z)Y]\} = 0 \tag{3.25}$$

If we substitute the (3.18) in (3.25) and we make the necessary arrangements, we obtain

$$-A[R(Y,V)Z - g(V,Z)Y + g(Y,Z)V] = 0$$

□

Theorem 3.6. Let M be the n-dimensional normal paracontact metric space form. If M satisfies the curvature condition $W_0 \cdot S = 0$, then M is an Einstein manifold.

Proof.

Assume that

$$(W_0(X,Y) \cdot S)(U,V) = 0$$

for every $X, Y, U, V \in \chi(M)$. Therefore, we can write

$$S(W_0(X,Y)U,V) + S(U,W_0(X,Y)V) = 0 \tag{3.26}$$

If we choose $X = \xi$ in (3.26) and make use of (2.15), we get

$$\begin{aligned} &-A(n-1)g(Y,U)\eta(V) + A\eta(U)S(Y,V) \\ &-A(n-1)g(Y,V)\eta(U) + A\eta(V)S(U,Y) = 0 \end{aligned} \tag{3.27}$$

If we choose $U = \xi$ in (3.27), we have

$$\frac{(n-5)(c-1)}{4(n-1)}[S(Y,V) - (n-1)g(Y,V)] = 0$$

□

4. Conclusion

In this article, normal paracontact metric space forms are investigated on W_0-curvature tensor. Characterizations of normal paracontact space forms are obtained on W_0-curvature tensor. Special curvature conditions established with the help of Riemann, Ricci, concircular curvature tensors are discussed on W_0-curvature tensor. Through these curvature conditions, important characterizations of normal paracontact metric space forms are obtained.

Author Contributions

All the authors equally contributed to this work. They all read and approved the final version of the paper.

Conflict of Interest

All the authors declare no conflict of interest.

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