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Pseudoparallel invariant submanifolds of Kenmotsu manifolds

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Abstract — In this paper, we consider pseudoparallel invariant submanifolds, a particular class of invariant submanifolds of Kenmotsu manifolds, on W_8 curvature tensor and investigate some of their basic characterizations, such as W_8 pseudoparallel, W_8-2 pseudoparallel, W_8-Ricci generalized pseudoparallel, and W_8-2 Ricci generalized pseudoparallel. Moreover, we present some relations between these pseudoparallel invariant submanifolds and semi-parallel invariant submanifolds. We finally discuss the need for further research.

Keywords: *Kenmotsu manifold, pseudoparallel invariant submanifold,* 2*-pseudoparallel invariant submanifolds*

Subject Classification (2020): 53C15, 53D10

1. Introduction

In 1972, Kenmotsu [1] studied a class of contact Riemannian manifolds and called them Kenmotsu manifolds. He proved that if a Kenmotsu manifold satisfies the condition $R(a_1, a_2)\cdot R = 0$, then the manifold has negative curvature -1, where R is the Riemannian curvature tensor of (1,3)-type and $R(a_1, a_2)$ denotes the derivation of the tensor algebra at each point of the tangent space. Then, the properties of Kenmotsu manifolds have been studied by several authors, such as Haseeb [2], Wang and Liu [3], Wang and Wang [4], Özgür and De [5], Tripathi and Gupta [6], Singh et al. [7], Parakasha and Hadimani [8], De and De [9], and De and Pathak [10]. Afterward, the geometry of submanifolds has been examined on different manifolds, and many essential properties have been obtained [11–13].

The curvature tensor is one of the most important concepts used to learn the characterization of a manifold. The properties of manifolds, important for mathematics, physics, and engineering, are handled with the help of curvature tensors. One of the essential geometrical manifolds is the Kenmotsu manifolds. Kenmotsu manifolds are one-dimensional versions of complex manifolds. If an almost contact metric manifold satisfies the condition

$$(\nabla_{a_1}\varphi)a_2 = g(\varphi a_1, a_2)\xi - \eta(a_2)\varphi a_1$$

Then, the manifold is called a Kenmotsu manifold.

In this article, pseudoparallel invariant submanifolds for Kenmotsu manifolds are investigated. The Kenmotsu manifold is considered on the W_8-curvature tensor. Submanifolds of these manifolds with properties, such as W_8-pseudoparallel, W_8-2 pseudoparallel, W_8-Ricci generalized pseudoparallel, and W_8-2 Ricci generalized pseudoparallel has been investigated.

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2. Preliminaries

This section provides some basic properties to be required in the following sections. Let M be a $(2n+1)$-dimensional differentiable manifold. If it admits a tensor field φ of type (1,1), a vector field ξ, and a 1-form η satisfying the following conditions:

$$\varphi^2 a_1 = -a_1 + \eta(a_1)\xi \text{ and } \eta(\xi) = 1 \tag{2.1}$$

for all $a_1, a_2, \xi \in \chi(M)$, then (φ, ξ, η) is called almost contact structure and (M, φ, ξ, η) is called almost contact manifold. If there is a g metric on the $(2n+1)$-dimensional, almost contact manifold satisfying

$$g(a_1, \xi) = \eta(a_1) \tag{2.2}$$

and

$$g(\varphi a_1, \varphi a_2) = g(a_1, a_2) - \eta(a_1)\eta(a_2) \tag{2.3}$$

for all $a_1, a_2 \in \chi(M)$, then (φ, ξ, η, g) is called an almost contact metric structure and $(M, \varphi, \xi, \eta, g)$ is called an almost contact metric manifold.

On a $(2n+1)$-dimensional manifold M, we have

$$g(\varphi a_1, a_2) = -g(a_1, \varphi a_2)$$

for all $a_1, a_2 \in \chi(M)$, that is, φ is an anti-symmetric tensor field according to the g metric. The transformation Φ defined as

$$\Phi(a_1, a_2) = g(a_1, \phi a_2)$$

for all $a_1, a_2 \in \chi(M)$, is called the fundamental 2-form of the (φ, ξ, η, g) almost contact metric structure where

$$\eta \wedge \Phi^n \neq 0$$

Let M be a $(2n+1)$-dimensional almost contact metric manifold given the structure (φ, ξ, η, g). If $d\eta = 0$ and $d\Phi = 2\eta \wedge \Phi$ on an almost contact metric manifold M, then M is called an almost Kenmotsu manifold.

Let M be $(2n+1)$-dimensional almost contact metric manifold providing the structure (φ, ξ, η, g). If it satisfies the following conditions:

$$\begin{cases} \varphi^2 a_1 = -a_1 + \eta(a_1)\xi, \varphi\xi = 0 \\ \eta(\varphi a_1) = 0, \eta(a_1) = g(a_1, \xi)\eta(\xi) = 1 \\ (\nabla_{a_1}\varphi)a_2 = g(\varphi a_1, a_2)\xi - \eta(a_2)\varphi a_1 \end{cases} \tag{2.4}$$

for all $a_1, a_2, \xi \in \chi(M)$, then M is called the Kenmotsu manifold.

Lemma 2.1. [1] Let M be a $(2n+1)$-dimensional Kenmotsu manifold. In this case, the following equations are obtained.

$$\nabla_{a_1}\xi = -a_1 + \eta(a_1)\xi \tag{2.5}$$

$$(\nabla_{a_1}\eta)a_2 = g(a_1, a_2) - \eta(a_1)\eta(a_2) \tag{2.6}$$

$$R(a_1, a_2)\xi = \eta(a_1)a_2 - \eta(a_2)a_1 \tag{2.7}$$

$$R(\xi, a_1)a_2 = -g(a_1, a_2)\xi + \eta(a_2)a_1 \tag{2.8}$$

$$R(a_1, \xi)a_2 = g(a_1, a_2)\xi - \eta(a_2)a_1 \tag{2.9}$$

$$S(a_1, \xi) = -2n\eta(a_1) \tag{2.10}$$

and

$$Q\xi = -2n\xi \tag{2.11}$$

where ∇, R, S, and Q are Levi-Civita connections on M, Riemann tensor, Ricci tensor, and Ricci operator, respectively.

Definition 2.2. [6] Let M be a $(2n+1)$-dimensional semi-Riemannian manifold. Then, the W_8-curvature tensor is defined as

$$W_8(a_1, a_2)a_3 = R(a_1, a_2)a_3 - \frac{1}{2n}[S(a_2, a_3)a_1 - S(a_1, a_3)a_2] \tag{2.12}$$

for all $a_1, a_2, a_3 \in \chi(M)$.

If we choose $a_1 = \xi$, $a_2 = \xi$, and $a_3 = \xi$ in (2.12) for $(2n+1)$-dimensional Kenmotsu manifold respectively, we get

$$W_8(\xi, a_2)a_3 = -g(a_2, a_3)\xi + \eta(a_3)a_2 - \eta(a_2)a_3 - \frac{1}{2n}S(a_2, a_3)\xi \tag{2.13}$$

$$W_8(a_1, \xi)a_3 = g(a_1, a_3)\xi - \eta(a_1)a_3 \tag{2.14}$$

and

$$W_8(a_1, a_2)\xi = \eta(a_1)a_2 + \frac{1}{2n}S(a_1, a_2)\xi \tag{2.15}$$

Let $\widetilde{M}$ be immersed submanifold of a Kenmotsu manifold $M(\phi, \xi, \eta, g)$. Moreover, let the tangent and normal subspaces of $\widetilde{M}$ in $M(\phi, \xi, \eta, g)$ be $\Gamma(T\widetilde{M})$ and $\Gamma(T^{\perp}\widetilde{M})$, respectively. Then, Gauss and Weingarten formulas for $\Gamma(T\widetilde{M})$ and $\Gamma(T^{\perp}\widetilde{M})$ are

$$\nabla_{a_1} a_2 = \widetilde{\nabla}_{a_1} a_2 + \sigma(a_1, a_2) \tag{2.16}$$

and

$$\nabla_{a_1} a_5 = -A_{a_5}a_1 + \widetilde{\nabla}^{\perp}_{a_1} a_5 \tag{2.17}$$

respectively, for all $a_1, a_2 \in \Gamma(T\widetilde{M})$ and $a_5 \in \Gamma(T^{\perp}\widetilde{M})$, where $\widetilde{\nabla}$ and $\widetilde{\nabla}^{\perp}$ are the connections on $\widetilde{M}$ and $\Gamma(T^{\perp}\widetilde{M})$, respectively, σ and A are the second fundamental form and the shape operator of $\widetilde{M}$. There is a relation

$$g(A_{a_5}a_1, a_2) = g(\sigma(a_1, a_2), a_5)$$

between the second basic form and shape operator defined as above. The covariant derivative of the second fundamental form σ is defined as

$$(\nabla_{a_1}\sigma)(a_2, a_3) = \widetilde{\nabla}^{\perp}_{a_1} \sigma(a_2, a_3) - \sigma(\widetilde{\nabla}_{a_1} a_2, a_3) - \sigma(a_2, \widetilde{\nabla}_{a_1} a_3) \tag{2.18}$$

Specifically, if $\nabla\sigma = 0$, $\widetilde{M}$ is said to be a parallel second fundamental form.

Let $\tilde{R}$ be Riemann curvature tensor of $\widetilde{M}$. In this case, the Gauss equation can be expressed as

$$R(a_1,a_2)a_3 = \tilde{R}(a_1,a_2)a_3 + A_{\sigma(a_1,a_3)}a_2 - A_{\sigma(a_2,a_3)}a_1 + \left(\nabla_{a_1}\sigma\right)(a_2,a_3) - \left(\nabla_{a_2}\sigma\right)(a_1,a_3)$$

for all $a_1, a_2, a_3 \in \Gamma\left(T\widetilde{M}\right)$, where

$$\left(\widetilde{\nabla}_{a_1}\sigma\right)(a_2,a_3) - \left(\widetilde{\nabla}_{a_2}\sigma\right)(a_1,a_3) = 0$$

then it is called a curvature-invariant submanifold. Let $\widetilde{M}$ be a Riemannian manifold, T be $(0,k)$-type tensor field and A be $(0,2)$-type tensor field. In this case, Tachibana tensor field $Q(A,T)$ is defined as

$$Q(A,T)(X_1,\cdots,X_k;a_1,a_2) = -T\left((a_1 \wedge_A a_2)X_1,\cdots,X_k\right) - \cdots - T(X_1,\cdots,X_{k-1},(a_1 \wedge_A a_2)X_k) \qquad (2.19)$$

where

$$(a_1 \wedge_A a_2)a_3 = A(a_2,a_3)a_1 - A(a_1,a_3)a_2,\ k \geq 1,\ X_1,X_2,\cdots,X_k,a_1,a_2 \in \Gamma\left(T\widetilde{M}\right) \qquad (2.20)$$

Definition 2.3. [5] A submanifold of a Riemannian manifold (M,g) is said to be pseudoparallel, 2-pseudoparallel, Ricci-generalized pseudoparallel, and 2-Ricci generalized pseudoparallel if

$R \cdot \sigma$ and $Q(g,\sigma)$

$R \cdot \nabla\sigma$ and $Q(g,\nabla\sigma)$

$R \cdot \sigma$ and $Q(S,\sigma)$

$R \cdot \nabla\sigma$ and $Q(S,\nabla\sigma)$

are linearly dependent, respectively.

3. Pseudoparallel Invariant Submanifolds of Kenmotsu Manifold

Let $\widetilde{M}$ be the immersed submanifold of a $(2n+1)$-dimensional Kenmotsu manifold $M(\phi,\xi,\eta,g)$. If $\phi\left(T_{a_1}M\right) \subset T_{a_1}M$ in every a_1 point, the $\widetilde{M}$ manifold is called an invariant submanifold. We note that all of the properties of an invariant submanifold inherit the ambient manifold. Throughout this paper, let $\widetilde{M}$ be an invariant submanifold of the Kenmotsu manifold $M(\phi,\xi,\eta,g)$. Therefore, for all $a_1, a_2 \in \Gamma\left(T\widetilde{M}\right)$,

$$\sigma(\phi a_1,a_2) = \sigma(a_1,\phi a_2) = \phi\sigma(a_1,a_2) \qquad (3.1)$$

and

$$\sigma(a_1,\xi) = 0 \qquad (3.2)$$

Lemma 3.1. Let $\widetilde{M}$ be the invariant submanifold of the $(2n+1)$-dimensional Kenmotsu manifold $M(\phi,\xi,\eta,g)$. The second fundamental form σ of $\widetilde{M}$ is parallel if and only if $\widetilde{M}$ is the total geodesic submanifold.

Consider the invariant submanifolds of the Kenmotsu manifold on the W_8-curvature tensor.

Definition 3.2. Let $\widetilde{M}$ be the invariant submanifold of the $(2n+1)$-dimensional Kenmotsu manifold $M(\phi,\xi,\eta,g)$. If $W_8 \cdot \sigma$ and $Q(g,\sigma)$ are linearly dependent, $\widetilde{M}$ is called W_8-pseudoparallel submanifold.

In this mean, it can be said that there is a function k_1 on the set $M_1 = \left\{x \in \widetilde{M} \mid \sigma(x) \neq g(x)\right\}$ such that

$$W_8 \cdot \sigma = k_1 Q(g,\sigma)$$

If $k_1 = 0$ specifically, $\widetilde{M}$ is called a W_8-semi-parallel submanifold.

Theorem 3.3. Let $\widetilde{M}$ be an invariant submanifold of a $(2n+1)$-dimensional Kenmotsu manifold $M(\phi,\xi,\eta,g)$. If $\widetilde{M}$ is W_8-pseudoparallel submanifold, then $\widetilde{M}$ is either a total geodesic or $k_1 = -1$.

Proof.

Assume that $\widetilde{M}$ is a W_8-pseudoparallel submanifold. Thus, for all $a_1, a_2, a_4, a_5 \in \Gamma(T\widetilde{M})$,

$$(W_8(a_1,a_2)\cdot\sigma)(a_4,a_5) = k_1 Q(g,\sigma)(a_4,a_5;a_1,a_2) \tag{3.3}$$

From (3.3), it is clear that

$$R^{\perp}(a_1,a_2)\sigma(a_4,a_5) - \sigma(W_8(a_1,a_2)a_4,a_5) - \sigma(a_4,W_8(a_1,a_2)a_5) = -k_1\left\{\sigma\left((a_1\wedge_g a_2)a_4,a_5\right) + \sigma\left(a_4,(a_1\wedge_g a_2)a_5\right)\right\}$$

That is, we can write

$$\begin{gathered}R^{\perp}(a_1,a_2)\sigma(a_4,a_5) - \sigma(W_8(a_1,a_2)a_4,a_5) - \sigma(a_4,W_8(a_1,a_2)a_5) = \\ -k_1\{g(a_2,a_4)\sigma(a_1,a_5) - g(a_1,a_4)\sigma(a_2,a_5) + g(a_2,a_5)\sigma(a_4,a_1) - g(a_1,a_5)\sigma(a_4,a_2)\}\end{gathered} \tag{3.4}$$

If we choose $a_5 = \xi$ in (3.4) and make use of (2.15) and (3.2), then

$$\eta(a_1)\sigma(a_4,a_2) = k_1[\eta(a_2)\sigma(a_4,a_1) - \eta(a_1)\sigma(a_4,a_2)] \tag{3.5}$$

If we choose $a_1 = \xi$ in (3.5), we obtain

$$(k_1+1)\sigma(a_4,a_2) = 0$$

□

Corollary 3.4. Let $\widetilde{M}$ be an pseudoparallel invariant submanifold of a $(2n+1)$-dimensional Kenmotsu manifold $M(\phi,\xi,\eta,g)$. Then, $\widetilde{M}$ *is* W_8-semi-parallel if and only if $\widetilde{M}$ is totally geodesic.

Definition 3.5. Let $\widetilde{M}$ be the invariant submanifold of a $(2n+1)$-dimensional Kenmotsu manifold $M(\phi,\xi,\eta,g)$. If $W_8\cdot\nabla\sigma$ and $Q(g,\nabla\sigma)$ are linearly dependent, then $\widetilde{M}$ is called W_8-2 pseudoparallel submanifold.

In this case, there is a function k_2 on the set $M_2 = \{x \in \widetilde{M} \mid \nabla\sigma(x) \neq g(x)\}$ such that

$$W_8\cdot\nabla\sigma = k_2 Q(g,\nabla\sigma)$$

If $k_2 = 0$ specifically, $\widetilde{M}$ is called a W_8-2 semiparallel submanifold.

Theorem 3.6. Let $\widetilde{M}$ be an invariant submanifold of a $(2n+1)$-dimensional Kenmotsu manifold $M(\phi,\xi,\eta,g)$. If $\widetilde{M}$ is W_8-2 pseudoparallel submanifold, then $\widetilde{M}$ is either a total geodesic submanifold or $k_2 = -1$.

Proof.

Assume that $\widetilde{M}$ is a W_8-2 pseudoparallel submanifold. Therefore,

$$(W_8(a_1,a_2)\cdot\nabla\sigma)(a_4,a_5,a_3) = k_2 Q(g,\nabla\sigma)(a_4,a_5,a_3;a_1,a_2) \tag{3.6}$$

for all $a_1,a_2,a_4,a_5,a_3 \in \Gamma(T\widetilde{M})$. If we choose $a_1 = a_3 = \xi$ in (3.6), then

$$R^{\perp}(\xi,a_2)(\nabla_{a_4}\sigma)(a_5,\xi)-(\nabla_{W_8(\xi,a_2)a_4}\sigma)(a_5,\xi)-(\nabla_{a_4}\sigma)(W_8(\xi,a_2)a_5,\xi)-(\nabla_{a_4}\sigma)(a_5,W_8(\xi,a_2)\xi)= -k_2\left\{\left(\nabla_{(\xi\wedge_g a_2)a_4}\sigma\right)(a_5,\xi)+(\nabla_{a_4}\sigma)\left((\xi\wedge_g a_2)a_5,\xi\right)+(\nabla_{a_4}\sigma)\left(a_5,(\xi\wedge_g a_2)\xi\right)\right\} \tag{3.7}$$

Calculate all the expressions in (3.7). Thus, by using (2.18) and taking into account Lemma 3.1, we can write

$$\begin{aligned} R^{\perp}(\xi,a_2)(\nabla_{a_4}\sigma)(a_5,\xi) &= R^{\perp}(\xi,a_2)\{\widetilde{\nabla}^{\perp}_{a_4}\sigma(a_5,\xi)-\sigma(\widetilde{\nabla}_{a_4}a_5,\xi)-\sigma(a_5,\widetilde{\nabla}_{a_4}\xi)\} \\ &= -R^{\perp}(\xi,a_2)\sigma(a_5,-a_4+\eta(a_4)\xi) \\ &= R^{\perp}(\xi,a_2)\sigma(a_5,a_4) \end{aligned} \tag{3.8}$$

secondly,

$$\begin{aligned} (\nabla_{W_8(\xi,a_2)a_4}\sigma)(a_5,\xi) &= \widetilde{\nabla}^{\perp}_{W_8(\xi,a_2)a_4}\sigma(a_5,\xi)-\sigma(\widetilde{\nabla}_{W_8(\xi,a_2)a_4}a_5,\xi)-\sigma(a_5,\widetilde{\nabla}_{W_8(\xi,a_2)a_4}\xi) \\ &= -\sigma(a_5,-W_8(\xi,a_2)a_4+\eta(W_8(\xi,a_2)a_4)\xi) \\ &= \eta(a_4)\sigma(a_5,a_2)-\eta(a_2)\sigma(a_5,a_4) \end{aligned} \tag{3.9}$$

Moreover,

$$\begin{aligned} (\nabla_{a_4}\sigma)(W_8(\xi,a_2)a_5,\xi) &= \widetilde{\nabla}^{\perp}_{a_4}\sigma(W_8(\xi,a_2)a_5,\xi)-\sigma(\widetilde{\nabla}_{a_4}W_8(\xi,a_2)a_5,\xi)-\sigma(W_8(\xi,a_2)a_5,\widetilde{\nabla}_{a_4}\xi) \\ &= -\sigma\left(-g(a_2,a_5)\xi+\eta(a_5)a_2-\eta(a_2)a_5-\frac{1}{2n}S(a_2,a_5)\xi,-a_4+\eta(a_4)\xi\right) \\ &= \eta(a_5)\sigma(a_2,a_4)-\eta(a_2)\sigma(a_5,a_4) \end{aligned} \tag{3.10}$$

$$\begin{aligned} (\nabla_{a_4}\sigma)(a_5,W_8(\xi,a_2)\xi) &= (\nabla_{a_4}\sigma)(a_5,a_2-\eta(a_2)\xi) \\ &= (\nabla_{a_4}\sigma)(a_5,a_2)-(\nabla_{a_4}\sigma)(a_5,\eta(a_2)\xi) \\ &= (\nabla_{a_4}\sigma)(a_5,a_2)-\widetilde{\nabla}^{\perp}_{a_4}\sigma(a_5,\eta(a_2)\xi)+\sigma(\widetilde{\nabla}_{a_4}a_5,\eta(a_2)\xi)+\sigma(a_5,\widetilde{\nabla}_{a_4}\eta(a_2)\xi) \\ &= (\nabla_{a_4}\sigma)(a_5,a_2)-\eta(a_2)\sigma(a_5,a_4) \end{aligned} \tag{3.11}$$

$$\begin{aligned} \left(\nabla_{(\xi\wedge_g a_2)a_4}\sigma\right)(a_5,\xi) &= \widetilde{\nabla}^{\perp}_{(\xi\wedge_g a_2)a_4}\sigma(a_5,\xi)-\sigma\left(\widetilde{\nabla}_{(\xi\wedge_g a_2)a_4}a_5,\xi\right)-\sigma\left(a_5,\widetilde{\nabla}_{(\xi\wedge_g a_2)a_4}\xi\right) \\ &= -\eta(a_4)\sigma(a_5,a_2) \end{aligned} \tag{3.12}$$

$$\begin{aligned} (\nabla_{a_4}\sigma)\left((\xi\wedge_g a_2)a_5,\xi\right) &= \widetilde{\nabla}^{\perp}_{a_4}\sigma\left((\xi\wedge_g a_2)a_5,\xi\right)-\sigma(\widetilde{\nabla}_{a_4}(\xi\wedge_g a_2)a_5,\xi)-\sigma\left((\xi\wedge_g a_2)a_5,\widetilde{\nabla}_{a_4}\xi\right) \\ &= -\sigma(g(a_2,a_5)\xi-g(\xi,a_5)a_2,-a_4+\eta(a_4)\xi) \\ &= -\eta(a_5)\sigma(a_2,a_4) \end{aligned} \tag{3.13}$$

and

$$\begin{aligned} (\nabla_{a_4}\sigma)(a_5,(\xi\wedge_g a_2)\xi) &= (\nabla_{a_4}\sigma)(a_5,\eta(a_2)\xi-a_2) \\ &= (\nabla_{a_4}\sigma)(a_5,\eta(a_2)\xi)-(\nabla_{a_4}\sigma)(a_5,a_2) \\ &= \eta(a_2)\sigma(a_5,a_4)-(\nabla_{a_4}\sigma)(a_5,a_2) \end{aligned} \tag{3.14}$$

If we substitute (3.8)-(3.14) in (3.7), we have

$$R^{\perp}(\xi,a_2)\sigma(a_5,a_4)-\eta(a_4)\sigma(a_5,a_2)-\eta(a_5)\sigma(a_2,a_4)+3\eta(a_2)\sigma(a_5,a_4)-(\nabla_{a_4}\sigma)(a_5,a_2)= \\ -k_2\{-\eta(a_5)\sigma(a_4,a_2)-\eta(a_4)\sigma(a_2,a_5)+\eta(a_2)\sigma(a_5,a_4)-(\nabla_{a_4}\sigma)(a_5,a_2)\} \quad (3.15)$$

If we choose $a_5=\xi$ in (3.5) and use,

$$(\nabla_{a_4}\sigma)(\xi,a_2)=\sigma(a_4,a_2)$$

we get

$$[k_2+1]\sigma(a_2,a_4)=0$$

which proves the assertions.

□

Corollary 3.7. Let $\widetilde{M}$ be an invariant pseudoparallel submanifold of a $(2n+1)$-dimensional Kenmotsu manifold $M(\phi,\xi,\eta,g)$. Then, $\widetilde{M}$ *is* W_8-2 semi-parallel if and only if $\widetilde{M}$ is totally geodesic.

Definition 3.8. Let $\widetilde{M}$ be the invariant submanifold of a $(2n+1)$-dimensional Kenmotsu manifold $M(\phi,\xi,\eta,g)$. If $W_8\cdot\sigma$ and $Q(S,\sigma)$ are linearly dependent, $\widetilde{M}$ is called W_8-Ricci generalized pseudoparallel submanifold.

In this case, there is a function k_3 on the set $M_3=\{x\in\widetilde{M} \mid \sigma(x)\neq S(x)\}$ such that

$$W_8\cdot\sigma=k_3Q(S,\sigma)$$

If $k_3=0$ specifically, $\widetilde{M}$ is called a W_8-Ricci generalized semiparallel submanifold.

Theorem 3.9. Let $\widetilde{M}$ be the invariant submanifold of a $(2n+1)$-dimensional Kenmotsu manifold $M(\phi,\xi,\eta,g)$. If $\widetilde{M}$ is W_8-Ricci generalized pseudoparallel submanifold, then $\widetilde{M}$ is either a total geodesic or $k_3=\frac{1}{2n}$.

Proof.

Assume that $\widetilde{M}$ is a W_8-Ricci generalized pseudoparallel submanifold. Therefore, we have

$$(W_8(a_1,a_2)\cdot\sigma)(a_4,a_5)=k_3Q(S,\sigma)(a_4,a_5;a_1,a_2)$$

that is

$$R^{\perp}(a_1,a_2)\sigma(a_4,a_5)-\sigma(W_8(a_1,a_2)a_4,a_5)-\sigma(a_4,W_8(a_1,a_2)a_5)= \\ -k_3\{\sigma((a_1\wedge_S a_2)a_4,a_5)+\sigma(a_4,(a_1\wedge_S a_2)a_5)\} \quad (3.16)$$

for all $a_1,a_2,a_4,a_5\in\Gamma(T\widetilde{M})$. If we choose $a_1=a_5=\xi$ in (3.6), we get

$$-\sigma(a_4,W_8(\xi,a_2)\xi)=k_3S(\xi,\xi)\sigma(a_4,a_2) \quad (3.17)$$

If we use (2.10) and (2.15) in (3.7), we have

$$(1-2nk_3)\sigma(a_4,a_2)=0$$

□

Corollary 3.10. Let $\widetilde{M}$ be an invariant pseudoparallel submanifold of a $(2n+1)$-dimensional Kenmotsu manifold $M(\phi,\xi,\eta,g)$. Then, $\widetilde{M}$ is W_8-Ricci generalized semi-parallel if and only if $\widetilde{M}$ is totally geodesic.

Definition 3.11. *Let* $\widetilde{M}$ be an invariant pseudoparallel submanifold of a $(2n+1)$-dimensional Kenmotsu manifold $M(\phi,\xi,\eta,g)$. If $W_8 \cdot \widetilde{\nabla}\sigma$ and $Q(S,\widetilde{\nabla}\sigma)$ are linearly dependent, $\widetilde{M}$ is called W_8-2 Ricci generalized pseudoparallel submanifold.

Then, there is a function k_4 on the set $M_4 = \{x \in \widetilde{M} \mid \nabla\sigma(x) \neq S(x)\}$ such that

$$W_8 \cdot \nabla\sigma = k_4 Q(S,\nabla\sigma)$$

If specifically, $k_4 = 0$, M is called a W_8-2 Ricci generalized semiparallel submanifold.

Theorem 3.12. Let $\widetilde{M}$ be an invariant pseudoparallel submanifold of a $(2n+1)$-dimensional Kenmotsu manifold $M(\phi,\xi,\eta,g)$. If $\widetilde{M}$ is W_8-2 Ricci generalized pseudoparallel submanifold, then $\widetilde{M}$ is either a total geodesic or $k_4 = \frac{1}{2n}$.

Proof.

Assume that $\widetilde{M}$ is a W_8-2 Ricci generalized pseudoparallel submanifold. Thus, we can write

$$(W_8(a_1,a_2)\cdot\nabla\sigma)(a_4,a_5,a_3) = k_4 Q(S,\nabla\sigma)(a_4,a_5,a_3;a_1,a_2) \tag{3.18}$$

for all $a_1,a_2,a_4,a_5,a_3 \in \Gamma(T\widetilde{M})$. If we choose $a_1 = a_5 = \xi$ in (3.18), we can write

$$\begin{aligned} &R^{\perp}(\xi,a_2)(\nabla_{a_4}\sigma)(\xi,a_3) - (\nabla_{W_8(\xi,a_2)a_4}\sigma)(\xi,a_3) - (\nabla_{a_4}\sigma)(W_8(\xi,a_2)\xi,a_3) - (\nabla_{a_4}h)(\xi,W_8(\xi,a_2)a_3) \\ &= -k_4\{(\nabla_{(\xi\wedge_S a_2)a_4}\sigma)(\xi,a_3) + (\nabla_{a_4}\sigma)((\xi\wedge_S a_2)\xi,a_3) + (\widetilde{\nabla}_{a_4}\sigma)(\xi,(\xi\wedge_S a_2)a_3)\} \end{aligned} \tag{3.19}$$

Calculate all the expressions in (3.9). Firstly, making use of (2.18), (3.1), and Lemma 3.1, we can write

$$\begin{aligned} R^{\perp}(\xi,a_2)(\nabla_{a_4}\sigma)(\xi,a_3) &= R^{\perp}(\xi,a_2)\{\widetilde{\nabla}^{\perp}_{a_4}\sigma(\xi,a_3) - \sigma(\widetilde{\nabla}_{a_4}a_3,\xi) - \sigma(a_3,\widetilde{\nabla}_{a_4}\xi)\} \\ &= R^{\perp}(\xi,a_2)\sigma(a_3,a_4) \end{aligned} \tag{3.20}$$

For the same reason, we can write

$$\begin{aligned} (\nabla_{W_8(\xi,a_2)a_4}\sigma)(\xi,a_3) &= \widetilde{\nabla}^{\perp}_{W_8(\xi,a_2)a_4}\sigma(\xi,a_3) - \sigma(\widetilde{\nabla}_{W_8(\xi,a_2)a_4}\xi,a_3) - \sigma(\xi,\widetilde{\nabla}_{W_8(\xi,a_2)a_4}a_3) \\ &= \eta(a_4)\sigma(a_2,a_3) - \eta(a_2)\sigma(a_3,a_4) \end{aligned} \tag{3.21}$$

$$\begin{aligned} (\nabla_{a_4}\sigma)(W_8(\xi,a_2)\xi,a_3) &= (\nabla_{a_4}\sigma)(a_2 + \eta(a_2)\xi,a_3) \\ &= (\nabla_{a_4}\sigma)(a_2,a_3) - \eta(a_2)\sigma(a_4,a_3) \end{aligned} \tag{3.22}$$

$$\begin{aligned} (\nabla_{a_4}\sigma)(\xi,W_8(\xi,a_2)a_3) &= \widetilde{\nabla}^{\perp}_{a_4}\sigma(\xi,W_8(\xi,a_2)a_3) - \sigma(\widetilde{\nabla}_{a_4}\xi,W_8(\xi,a_2)a_3) - \sigma(\xi,\widetilde{\nabla}_{a_4}W_8(\xi,a_2)a_3) \\ &= \eta(a_3)\sigma(a_4,a_2) - \eta(a_2)\sigma(a_3,a_4) \end{aligned} \tag{3.23}$$

$$\begin{aligned} (\nabla_{(\xi\wedge_S a_2)a_4}\sigma)(\xi,a_3) &= \widetilde{\nabla}^{\perp}_{(\xi\wedge_S a_2)a_4}\sigma(\xi,a_3) - \sigma(\widetilde{\nabla}_{(\xi\wedge_S a_2)a_4}\xi,a_3) - \sigma(\xi,\widetilde{\nabla}_{(\xi\wedge_S a_2)a_4}a_3) \\ &= 2n\eta(a_4)\sigma(a_2,a_3) \end{aligned} \tag{3.24}$$

$$\begin{aligned} (\nabla_{a_4}\sigma)((\xi\wedge_S a_2)\xi,a_3) &= (\nabla_{a_4}h)(S(a_2,\xi)\xi - S(\xi,\xi)a_2,a_3) \\ &= (\nabla_{a_4}h)(-2n\eta(a_2)\xi + 2na_2,a_3) \\ &= -2n\{\widetilde{\nabla}^{\perp}_{a_4}\sigma(\eta(a_2)\xi,a_3) - \sigma(\widetilde{\nabla}_{a_4}\eta(a_2)\xi,a_3) - \sigma(\eta(a_2)\xi,\widetilde{\nabla}_{a_4}a_3) + 2n(\nabla_{a_4}\sigma)(a_2,a_3)\} \\ &= 2n(\nabla_{a_4}\sigma)(a_2,a_3) - 2n\eta(a_2)\sigma(a_4,a_3) \end{aligned} \tag{3.25}$$

In the same way, we have

$$\begin{aligned}(\nabla_{a_4}\sigma)(\xi,(\xi \wedge_S a_2)a_3) &= (\nabla_{a_4}\sigma)(\xi, S(a_2,a_3)\xi - S(\xi,a_3)a_2) \\ &= (\nabla_{a_4}\sigma)(\xi, S(a_2,a_3)\xi) + (\tilde{\nabla}_{a_4}\,\sigma)(\xi, 2n\eta(a_3)a_2) \\ &= 2n\eta(a_3)\sigma(a_4,a_2)\end{aligned} \tag{3.26}$$

Consequently, we substitute (3.20)-(3.26) in (3.19), we obtain

$$\begin{aligned}R^{\perp}(\xi,a_2)\sigma(a_3,a_4) - \eta(a_4)\sigma(a_2,a_3) + 3\eta(a_2)\sigma(a_4,a_3) - \eta(a_3)\sigma(a_4,a_2) - (\nabla_{a_4}\sigma)(a_2,a_3) = \\ -k_4\{2n\eta(a_4)\sigma(a_2,a_3) - 2n\eta(a_2)\sigma(a_4,a_3) + 2n\eta(a_3)\sigma(a_4,a_2) + 2n(\nabla_{a_4}\sigma)(a_2,a_3)\}\end{aligned} \tag{3.27}$$

If we choose $a_3 = \xi$ in (3.27) and use

$$(\nabla_{a_4}\sigma)(a_2,\xi) = \sigma(a_4,a_2)$$

we get

$$[2nk_4 - 1]\sigma(a_4,a_2) = 0$$

□

Corollary 3.13. Let $\tilde{M}$ be a pseudoparallel invariant submanifold of a $(2n+1)$-dimensional Kenmotsu manifold $M(\phi,\xi,\eta,g)$. Then, $\tilde{M}$ is W_8-2 Ricci generalized semi-parallel if and only if $\tilde{M}$ is totally geodesic.

4. Conclusion

This paper considered pseudoparallel invariant submanifolds of Kenmotsu manifolds on W_8 curvature tensor and researched some basic characterizations, such as W_8 pseudoparallel, W_8-2 pseudoparallel, W_8-Ricci generalized pseudoparallel, and W_8-2 Ricci generalized pseudoparallel. In addition, the paper provided some relations between these pseudoparallel invariant submanifolds and semi-parallel invariant submanifolds. In future studies, pseudoparallel invariant submanifolds of Kenmotsu manifolds can also be characterized using other curvature tensors. Besides, this topic is worth studying on the other manifolds.

Author Contributions

All the authors equally contributed to this work. This paper is derived from the first author’s master’s thesis supervised by the second author. They all read and approved the final version of the paper.

Conflict of Interest

All the authors declare no conflict of interest.

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