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# MODELLING REAL VALUED FUNCTIONS VIA OPTICAL LENSES[1]

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**Abstract**

In this study, we modeled real valued functions using freeform lenses. In our model, the bottom surface of the lens is flat whereas its top surface is determined by a function, $f(x)$. We consider vertically coming light rays with x-coordinate $x$. Our aim is to find $f(x)$ such that $x$ is mapped to $F(x)$, the horizontal position where the light ray leaves the bottom surface. We have found the nonlinear differential equation for a generic lens to model a given function. Namely, given $F(x)$, the solution of the differential equation gives us the lens surface $f(x)$. Finally, we have calculated the lens surface for four functions numerically and have provided their plots respectively.

**Keywords:** Freeform lens design, ray optics, mathematical modelling

## 1. Introduction

In physics, there are two ways to model light. First is the wave model (as supported by interference experiments such as the Young's double-slit experiment) and the second is the particle model (which is supported by the corpuscle theory of Newton, or more recently by the photo-electric effect as explained by Einstein). In the past, scientists tried to determine which model was the right one and today we know that both are equally valid. This is known in quantum mechanics as *the wave-particle duality*. This is also valid for entities thought primarily as particles, e.g. electrons, which also show wave-like interference effect under right conditions. In our study, we restrict ourselves with what is known as *ray optics*, in which light propagates as a ray.

One may consider a ray of light as a sequence of photons (i.e. quanta of light) one following another on a straight line and change direction when a new medium is met. Moreover, each

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medium has a quantity known as *refractive index* which determines the speed of light in the medium and in general depends on the wavelength (or frequency, equivalently) of light. In this study we considered a monochromatic light ray, that is with single color. The deflection of light depends on refractive indices of media and the angle that is made with the surface normal. (See Figure 1.) The quantities just mentioned are related to through an equation known as the *Snell's Law*:

$$n_1 \sin(\theta_1) = n_2 \sin(\theta_2) \tag{1}$$

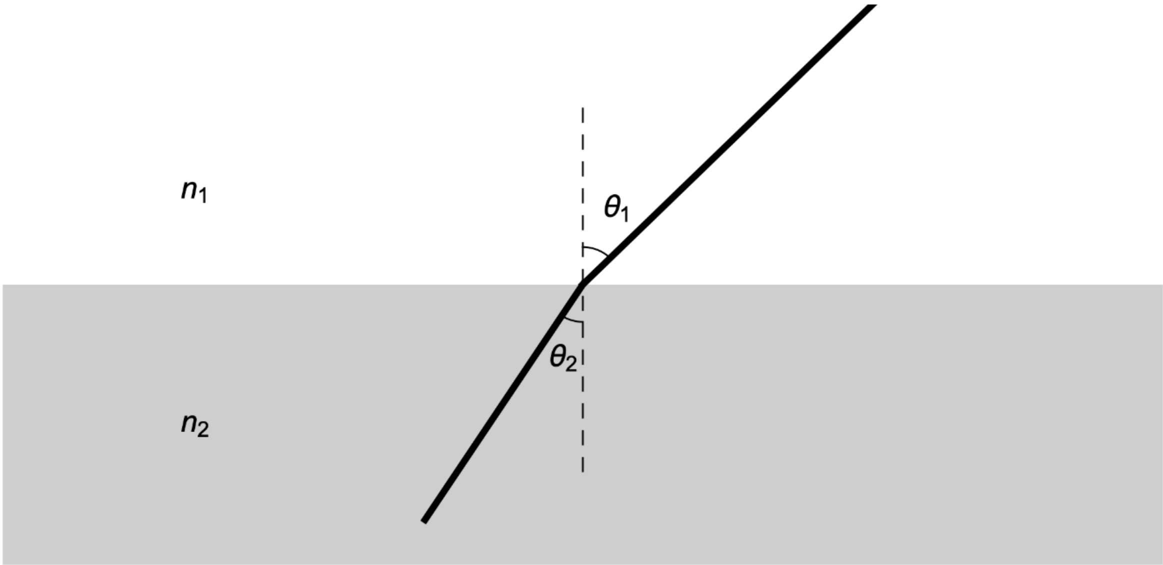


**Figure 1.** Dashed line is the surface normal. Refractive indices of media are denoted as $n_1, n_2$ and the angles the rays make with the surface normal as $\theta_1, \theta_2$. See Equation (1) for the relation between these four quantities: known as the Snell's Law.

The study of engineering a lens surface for an application where the surface is a degree of freedom is known as freeform lens design in the literature, see Refs. [2,4] for a review. This concept has a wide range of applications such as "green energy, aerospace, illumination and biomedical engineering" [2].

In this study, we are concerned with how to engineer an optical lens that *materializes* a function from $F: \mathbb{R} \to \mathbb{R}$. By *materializing* a function, we mean that the surface of a lens should be designed in a way so that the incoming ray with $x$ x-coordinate is mapped to $F(x)$. Our model has an immediate generalization to functions with cylindrical symmetry since they are in the same manner only a function of single variable: the distance from the origin.

In general, the study of freeform lens design is concerned with image forming [5] (may be useful for people needing custom-made optics in order to improve their sight, readers may also search the Internet for 'freeform lens' to see that there are companies that produce such elements) or non-image forming (e.g. illumination) [1,6,7,8] of some region of space.

In our study we do not focus on such matters but instead are interested in a lens as a means to materialize a real valued function. Moreover, as we shall see, we are also not interested in which way the incoming ray of light goes after it leaves the lens. Ref. [1] is the closest work to ours that we could find in the literature, however it is not directly about *materialization* of a function, rather about the problem of finding the geometry of lens surface that produces a specific illumination pattern far from the lens. As far as we know, our study is a new contribution to the literature in this regard.

The organization of the paper is as follows: In Section 2 we introduce the model and related definitions, in Section 3 we provide information on how to engineer a lens to model a function $F(x)$, in Section 4 we give four different lens surfaces for various functions and finally in Section 5 we conclude the paper.

## 2. The model

In this Section we introduce the model and perform calculations that will be useful later.

### 2.1 Variables and assumptions

We consider a lens between the curves $y = f(x)$ and $y = 0$. We suppose rays hit the lens from above, in the $-\hat{y}$ direction. For simplicity we consider the whole real line, but in applications one may put a cutoff for minimum and maximum $x$ values, because obviously a lens cannot occupy an infinite extent in applications. Moreover, we suppose that $f(x) > 0$ for all $x \in \mathbb{R}$.

**Table 1.** A list of definitions of variables we will use in the rest of the paper. You may see Figure 2 for a graphical illustration of some of the variables defined in this table.

| | |
|---|---|
| $r_1$ | the position where the incoming ray meets the lens |
| $r_2$ | the position where the outgoing ray leaves the lens |
| $\hat{n}_1$ | normal vector of upper lens surface |
| $\hat{n}_2$ | normal vector of lower lens surface |
| $\hat{\delta}_1$ | direction of the ray in the lens |
| $\hat{\delta}_2$ | direction of the ray when it leaves the lens |
| $\theta_1$ | the angle that the incoming ray makes with $\hat{n}_1$ |
| $\theta_2$ | the angle that ray in the lens makes with $-\hat{n}_1$ |
| $\theta_3$ | the angle that ray in the lens makes with $-\hat{n}_2$ |
| $\theta_4$ | the angle that ray that leaves lens makes with $\hat{n}_2$ |
| $R(\theta)$ | matrix that rotates a vector counter-clockwise by an angle $\theta$ |

The incoming ray is characterized solely by an $x$ value. The list of definitions of variables we use are found in Table 1 and illustrated in Figure 2.

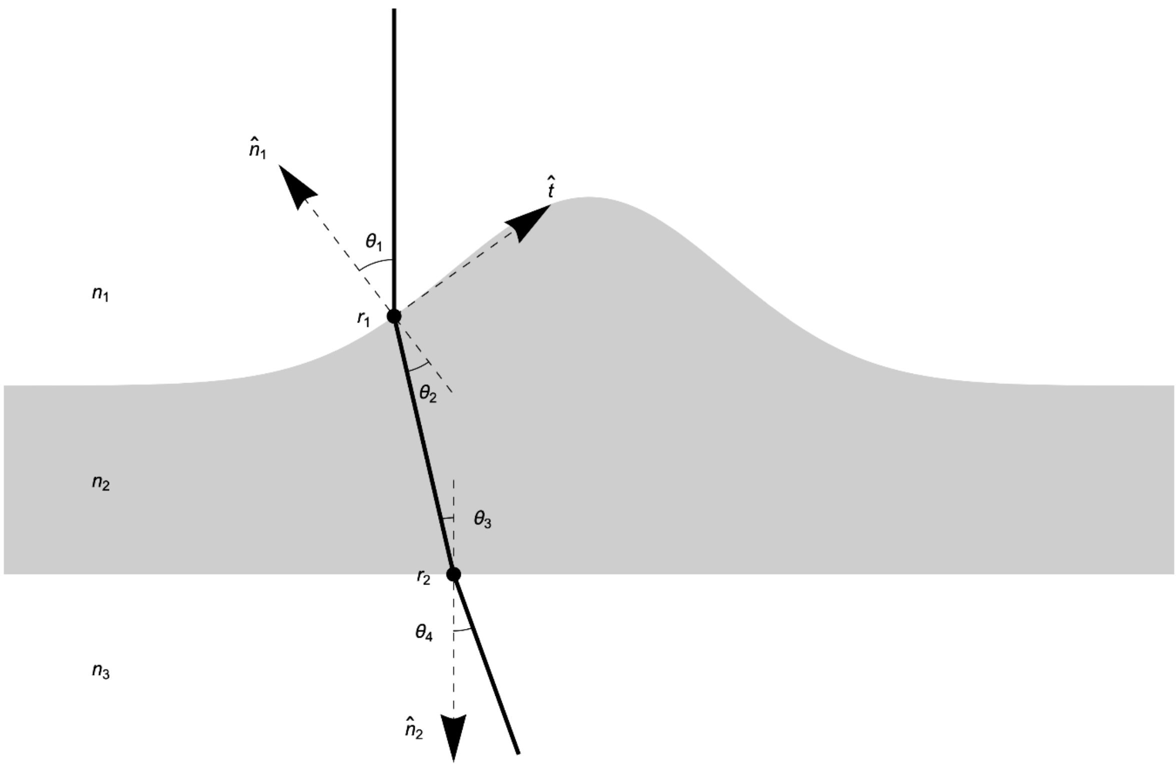


**Figure 2.** An example of a lens where $f(x) = 1 + e^{-x^2}$ . The medium that the original rays come from has refractive index $n_1$, the lens $n_2$, and the last medium has $n_3$. The variables are shown on the figure. Directions of the ray, $\hat{\delta}_1$ and $\hat{\delta}_2$, follow the ray downwards. For their definitions see Table 1.

In the context of this paper, we are mainly concerned with $r_2$, where the ray leaves the lens. This is the function $F$ we introduced in the Introduction. Our aim is to engineer the lens surface such that the function $r_2$ is realized. There is one assumption we use, in order to make the calculations easier, that the incoming ray hits the surface $f(x)$ then hits the $y = 0$ surface and then leaves. We do not consider the cases where path of the ray intersects the lens more than twice.

### 2.2 Calculations

The incoming ray comes from above, vertically, with horizontal coordinate \$x\$. The point where it hits the lens, is easy to calculate:

$$r_1 = (x, f(x)) \tag{2}$$

The angle that the incoming ray makes with the surface normal ($\hat{n}_1$) is $\theta_1$. So the ray, when it goes inside the lens, makes an angle $\theta_2$ with $-\hat{n}_1$. This angle is calculated by the Snell's Law:

$$\theta_2 = \arcsin\left(\frac{n_1}{n_2}\sin\theta_1\right) \tag{3}$$

Here one should not confuse $n_1$ (the refractive index of the first medium) with the normal vector $\hat{n}_1$. Now let us find the direction ($\hat{\delta}_1$) that the ray has in the lens:

$$\hat{\delta}_1 = -R(-sgn(f')\theta_2)\hat{n}_1 \quad (4)$$

So, the equation of the ray inside the lens can be written as follows:

$$(x, y) = \hat{\delta}_1 \lambda + r_1 \quad (5)$$

where $\lambda$ is a free parameter. The ray will leave the lens from the $y = 0$ surface. So when we solve this equation for $y = 0$, we find $r_2$ (the $x$ value of intersection point) as:

$$r_2(x) = r_{1,x}(x) - \frac{\hat{\delta}_{1,x}}{\hat{\delta}_{1,y}} r_{1,y}(x) \quad (6)$$

Using the explicit formula of $r_1(x)$ in Equation (2) we obtain:

$$r_2(x) = x - \frac{\hat{\delta}_{1,x}}{\hat{\delta}_{1,y}} f(x) \quad (7)$$

We will concentrate on $r_2$ later. Let us complete this Subsection by calculating the angle that the ray makes with the $y = 0$ surface, when it leaves the lens. The surface normal in this case is $\hat{n}_2 = (0, -1)$. The ray in the lens makes an angle $\theta_3$. The angle the outgoing ray makes with the surface normal is $\theta_4$ which can be given through the Snell's Law:

$$\theta_4 = \arcsin\left(\frac{n_2}{n_3} \sin\theta_3\right) \quad (8)$$

We now have enough tools to move on to specifying the lens surface in the next Section.

## 3. Engineering the lens surface

In this Section, we focus on the effect $f(x)$ has on $r_2$ , or in other words $F$ which we will use from now on. However, let us start by giving an explicit form of $R(\theta)$ which is a matrix that rotates the vector in the counter-clockwise direction by an angle $\theta$. In order to make the calculations simpler, we will use complex numbers in this Section. Hence:

$$R(\theta) = e^{i\theta} \quad (9)$$

In Equation (4) the surface normal ($\hat{n}_1$) appears. We need to write it as a complex number. If $\hat{t}(x)$ is the tangent vector of $f(x)$ at the point $x$, we can write $\hat{n}_1 = e^{i\,\pi/2}\hat{t}$. The tangent vector can be written as follows:

$$\hat{t} = \exp(i \arctan(f')) \quad (10)$$

and we can write:

$$\hat{n}_1 = i \exp(i \arctan(f')) \quad (11)$$

Using this information in Equation (4) we can write:

$$\hat{\delta}_1 = -i \exp[i \arctan(f') - sgn(f')\theta_2] \quad (12)$$

The direction of the incoming ray is $(0,-1) = -i$ and the surface normal ( $\hat{n}_1$) is given above in Equation (11). The angle in-between, $\theta_1$ is found as (we put the absolute value function in order to make sure that the angle is non-negative):

$$\theta_1 = \arctan(|f'|) \tag{13}$$

Using this information in Equation (3) we obtain:

$$\theta_2 = \arcsin\left(\frac{n_1}{n_2}\sin(\arctan(|f'|))\right) \tag{14}$$

Let us simply define $n$ via $1/n = n_1/n_2$.

$$\theta_2 = \arcsin\left(\frac{\sin(\arctan(|f'|))}{n}\right) \tag{15}$$

Returning back to Equation (12) we have:

$$\hat{\delta}_1 = -i\exp[i(\arctan(f') - sgn(f')\theta_2)] \tag{16}$$
$$= -i\exp\left[i\left(\arctan(f') - sng(f')\arcsin\left(\frac{\sin(\arctan(|f'|))}{n}\right)\right)\right] \tag{17}$$

In order to find $F$, see Equation (7), we need to calculate the ratio of $x, y$ components of $\hat{\delta}_1$. We find it as:

$$\frac{\hat{\delta}_{1,x}}{\hat{\delta}_{1,y}} = -\tan\left[\arctan(f') - sgn(f')\arcsin\left(\frac{\sin(\arctan(|f'|))}{n}\right)\right] \tag{18}$$

Finally, we obtain a relation between $f(x)$ and $F(x)$:

$$F(x) = x + f(x)\tan\left[\arctan(f') - sgn(f')\arcsin\left(\frac{\sin(\arctan(|f'|))}{n}\right)\right] \tag{19}$$

This is a highly nonlinear, 1st order ordinary differential equation. What we would like to find is $f(x)$ in terms of $F(x)$. On the other hand, we can do one more simplification, using the identity $|a| = sgn(a)\,a$. Using the fact that arcsin,sin,arctan are odd functions, we rewrite Equation (19) as:

$$F(x) = x + f(x)\tan\left[\arctan(f') - \arcsin\left(\frac{\sin(\arctan(f'))}{n}\right)\right] \tag{20}$$

With one more simplification, we obtain:

$$F(x) = x + f(x)\tan\left[\arctan(f'(x)) - \arcsin\left(\frac{f'(x)}{n\sqrt{f'(x)^2+1}}\right)\right] \tag{21}$$

## 4. Examples of engineered lens surfaces

In this Section, we give one analytical solution for a very simple case and the rest of our results are numerical solutions for some functions $F(x)$. For the ease of illustration, we consider *odd* functions $F(x)$ and this makes $f(x)$ function an *even* function. Due to this assumption, $F(0) = 0$ is obtained and we can set the initial value $f'(0) = 0$. Moreover, we can restrict ourselves with $x \in \mathbb{R}^{\geq 0}$. Since we will provide numerical solutions, we will consider $1/n = n_1/n_2 = 1/1.53$, which is equivalent to saying that the first medium is air and the second medium (the lens) is glass.

Our first example is the identity function, $F(x) = x$. This is the easiest case. Any $f(x) = c$ where $c > 0$ describes the identity function $F(x) = x$. Let us now consider $F(x) = ax$ for some $a > 0$. These functions correspond to zooming-in and zooming-out operations. For illustrative purposes we will consider $F(x) = 2x$ and $F(x) = x/2$. Numerical solutions (they are *not* unique) are shown in Figure 3. The lens that does the $F(x) = 2x$ scaling can be extended to infinity, however the lens designed for $F(x) = x/2$ cannot be extended to infinity since the lens surface is *not* convex. One may consider more complicated functions. For example, functions of the form $F(x) = x^2$ and $F(x) = x + \sin(x)/10$. The resulting lenses are plotted in Figure 3.

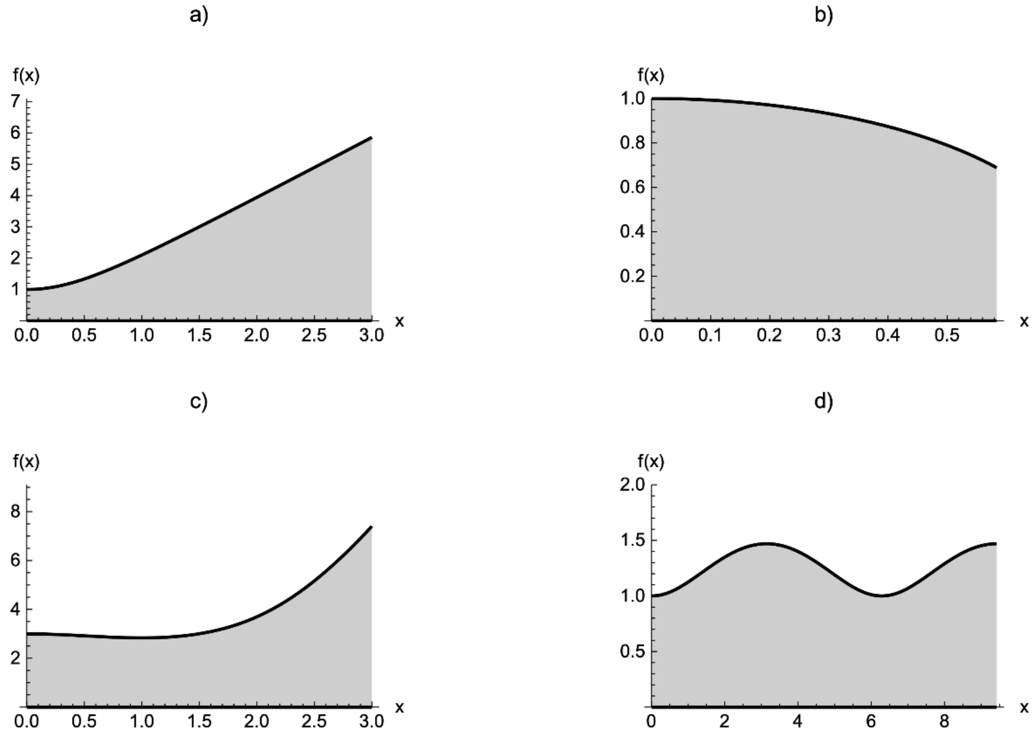


**Figure 3.** Illustration of four lenses that *materializes* different functions, $F(x)$. a) A lens design that corresponds to $F(x) = 2x$ (with initial conditions $f(0) = 1, f'(0) = 0$), b) A lens design that corresponds to $F(x) = x/2$ (with initial conditions $f(0) = 1, f'(0) = 0$), c) A lens design that corresponds to $F(x) = x^2$ (with initial conditions $f(0) = 3, f'(0) = 0$), and d) A lens design that corresponds to $F(x) = x + \sin(x)/10$ (with initial conditions $f(0) = 1, f'(0) = 0$).

## 5. Conclusion

In this study, we considered vertically incoming light rays towards a lens where the bottom surface is flat and the upper surface is specifically designed. Our main focus has been to model a real valued function via freeform lens design, such that the upper surface--characterized by a function $f(x)$--is designed in a manner to map $x$ to $F(x)$ where $F(x)$ is the function that we model and $x$ is the x-coordinate of the incoming light ray. Here $F(x)$ is the x-coordinate of the light ray, when it leaves the bottom surface of the lens. Apart from that, we have not been interested in what happens to rays after they leave the lens. Although we have seen that there are a few studies similar to our own, the closest being Ref. [1], their focus has been observed to be different. In that respect, as far as we are aware, our study is a new contribution to the literature.

## Acknowledgements

All of the figures in this work have been drawn with Mathematica 13.2 [3].

## References

[1] Doskolovich, L.L., Mingazov, A.A., Bykov, D.A., Bezus, E.A., "Formulation of the inverse problem of calculating the optical surface for an illuminating beam with a plane wavefront as the Monge Kantrovich problem", Computer Optics 43(5) (2019) : 705-713.

[2] Fang, F.Z., Zhang, X.D., Weckenmann, A., Zhang, G.X., Evans, C., "Manufacturing and measurement of freeform optics", CIRP Annals 62(2) (2013) : 823-846.

[3] Wolfram Research, Inc. Mathematica, Version 13.2. Champaign, IL. (2022)

[4] Kumar, S., Tong, Z., Jiang, X., "Advances in the design and manufacturing of novel freeform optics", International Journal of Extreme Manufacturing 4(3) (2022) : 032004.

[5] Valencia-Estrada, J.C., Garcia-Marquez, J., "Freeform geometrical optics i: principles", Applied Optics 58(34) (2019) : 9455-9464.

[6] van Roosmalen, A.H., Anthonissen, M.J.H., Ijzerman, W.L., ten Thije Boonkkamp, J.H.M., "Fresnel reflections in inverse double freeform lens design", Journal of the Optical Society of America A 40(7) (2023) : 1310-1318.

[7] Wu, R., Chang, S., Zheng, Z., Zhao, L., Liu, X., "Formulating the design of two freeform lens surfaces for point-like light sources." Optics Letters 43(7) (2018) : 1619-1622.

[8] Yang, L., Liu, Y., Ding, Z., Zhang, J, Tao, X., Zheng, Z.R., Wu, R., "Design of freeform lenses for illuminating hard-to-reach areas through a light-guiding system", Optics Express 28(25) (2020) : 38155-38168.