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Normal Paracontact Metric Space Forms Admitting Almost $\eta-$Ricci Solitons

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Abstract

In this paper, we have considered normal paracontact metric space forms admitting almost $\eta-$Ricci solitons in some curvature tensors. Ricci pseudosymmetry concepts of normal paracontact metric space forms admitting $\eta-$Ricci soliton have introduced according to the choosing of some special curvature tensors such as Riemann, concircular, projective and W_1 curvature tensor. After then, according to the choice of the curvature tensors, necessary conditions are given for normal paracontact metric space form admitting $\eta-$Ricci soliton to be Ricci semisymmetric. Then some characterizations are obtained and some classifications have made under the some conditions.

Keywords: *Ricci-pseudosymmetric Manifold, $\eta-$Ricci Soliton, Normal Paracontact Metric Space Form*
2010 Mathematics Subject Classification: *53C15; 53C44, 53D10*

1. Introduction

The study of paracontact geometry was initiated by Kenayuki and Williams [1]. Zamkovoy studied paracontact metric manifolds and their subclasses [2]. Recently, Welyczko studied curvature and torsion of Frenet Legendre curves in 3-dimensional normal paracontact metric manifolds [3],[4]. Recently, contact metric manifolds and their curvature properties have been studied by many authors in [5],[6],[7].
The notion of Ricci flow was introduced by Hamilton in 1982. With the help of this concept, Hamilton found the canonical metric on a smooth manifold. Then Ricci flow has become a powerful tool for the study of Riemannianian manifolds, especially for those manifolds with positive curvature. Perelman used Ricci flow and it surgery to prove Poncare conjecture in [8],[9]. The Ricci flow is an flow is an evolution equation for metrics on a Riemannianian manifold defined as follows:

$$\frac{\partial}{\partial t}g(t) = -2S(g(t)).$$

A Ricci soliton emerges as the limit of the solitons of the Ricci flow. A solution to the Ricci flow is called Ricci soliton if it moves only by a one parameter group of diffeomorphism and scaling.
During the last two decades, the geometry of Ricci solitons has been the focus of attention of many mathematicians. In particular, it has become more important after Perelman applied Ricci solitons to solve the long standing Poincare conjecture posed in 1904. In [10], Sharma studied the Ricci solitons in contact geometry. Thereafter, Ricci solitons in contact metric manifolds have been studied by various authors such as Bagewadi et al. in [11, 12, 13, 14], Bejan and Crasmareanu in [15], Blaga in [16], Chandra et al. in [17], Chen and Deshmukh in [18], Deshmukh et al. in [19], He and Zhu in [20], Atçeken et al. in [21], Nagaraja and Premalatta in [22], Tripathi in [23] and many others in [24, 25, 26, 27].
In this paper, we have considered normal paracontact metric space forms admitting almost $\eta-$Ricci solitons in some curvature tensors. Ricci pseudosymmetry concepts of normal paracontact metric space forms admitting $\eta-$Ricci soliton have been introduced according to the choosing of some special curvature tensors such as Riemannian, concircular, projective and W_1 curvature tensor. After then, according to the choice of the curvature tensors, necessary conditions are given for normal paracontact metric space form admitting $\eta-$Ricci soliton to be Ricci semisymmetric. Then some characterizations are obtained and some classifications have been made under the some conditions.
For simplicity's sake, the normal paracontact metric space form expression will be expressed as *NPMS*-form after this part of the article. Similarly, for brevity, after this part of the article, $\eta-$Ricci soliton expressions will be shown as $\eta-RS$, Ricci pseudosymmetric as $Ricci-P$, and Ricci semisymetric as $Ricci-S$.

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2. Preliminaries

Let's take an $n-$dimensional differentiable Φ manifold. If it admits a tensor field ϕ of type $(1,1)$, a contravariant vector field ξ and a 1-form η satisfying the following conditions;

$$\phi^2\varepsilon_1 = \varepsilon_1 - \eta(\varepsilon_1)\xi,\ \phi\xi = 0,\ \eta(\phi\varepsilon_1) = 0, \eta(\xi) = 1, \tag{1}$$

and

$$g(\phi\varepsilon_1, \phi\varepsilon_2) = g(\varepsilon_1, \varepsilon_2) - \eta(\varepsilon_1)\eta(\varepsilon_2),\ g(\varepsilon_1, \xi) = \eta(\varepsilon_1), \tag{2}$$

for all $\varepsilon_1, \varepsilon_2, \xi \in \chi(\Phi)$, (ϕ, ξ, η) is called almost paracontact structure and (Φ, ϕ, ξ, η) is called almost paracontact metric manifold. If the covariant derivative of ϕ satisfies

$$(\nabla_{\varepsilon_1}\phi)\varepsilon_2 = -g(\varepsilon_1, \varepsilon_2)\xi - \eta(\varepsilon_2)\varepsilon_1 + 2\eta(\varepsilon_1)\eta(\varepsilon_2)\xi, \tag{3}$$

then, Φ is called a normal paracontact metric manifold, where ∇ is Levi-Civita connection. From (3), we can easily to see that

$$\phi\varepsilon_1 = \nabla_{\varepsilon_1}\xi, \tag{4}$$

for any $\varepsilon_1 \in \chi(\Phi)$ [1].
Moreover, if such a manifold has constant sectional curvature equal to c, then it is the Riemannian curvature tensor is R given by

$$\begin{aligned} R(\varepsilon_1, \varepsilon_2)\varepsilon_3 &= \tfrac{c+3}{4}[g(\varepsilon_2, \varepsilon_3)\varepsilon_1 - g(\varepsilon_1, \varepsilon_3)\varepsilon_2] + \tfrac{c-1}{4}[\eta(\varepsilon_1)\eta(\varepsilon_3)\varepsilon_2 \\ &- \eta(\varepsilon_2)\eta(\varepsilon_3)\varepsilon_1 + g(\varepsilon_1, \varepsilon_3)\eta(\varepsilon_2)\xi - g(\varepsilon_2, \varepsilon_3)\eta(\varepsilon_1)\xi + g(\phi\varepsilon_2, \varepsilon_3)\phi\varepsilon_1 \\ &- g(\phi\varepsilon_1, \varepsilon_3)\phi\varepsilon_2 - 2g(\phi\varepsilon_1, \varepsilon_2)\phi\varepsilon_3], \end{aligned} \tag{5}$$

for any vector fields $\varepsilon_1, \varepsilon_2, \varepsilon_3 \in \chi(\Phi)$ [5].
In a $NPMS-$form by direct calculations, we can easily to see that

$$S(\varepsilon_1, \varepsilon_2) = \frac{c(n-5) + 3n + 1}{4}g(\varepsilon_1, \varepsilon_2) + \frac{(c-1)(5-n)}{4}\eta(\varepsilon_1)\eta(\varepsilon_2), \tag{6}$$

from which

$$Q\varepsilon_1 = \frac{c(n-5) + 4n + 1}{4}\varepsilon_1 + \frac{(c-1)(5-n)}{4}\eta(\varepsilon_1)\xi, \tag{7}$$

for any $\varepsilon_1, \varepsilon_2 \in \chi(\Phi)$, where Q is the Ricci operator and S is the Ricci tensor of Φ.

Lemma 2.1. *Let Φ be a n-dimensional NPMS-form. In this case, the following equations hold.*

$$R(\xi, \varepsilon_1)\varepsilon_2 = g(\varepsilon_1, \varepsilon_2)\xi - \eta(\varepsilon_2)\varepsilon_1, \tag{8}$$

$$R(\varepsilon_1, \xi)\varepsilon_2 = -g(\varepsilon_1, \varepsilon_2)\xi + \eta(\varepsilon_2)\varepsilon_1, \tag{9}$$

$$R(\varepsilon_1, \varepsilon_2)\xi = \eta(\varepsilon_2)\varepsilon_1 - \eta(\varepsilon_1)\varepsilon_2, \tag{10}$$

$$\eta(R(\varepsilon_1, \varepsilon_2)\varepsilon_3) = g(\eta(\varepsilon_1)\varepsilon_2 - \eta(\varepsilon_2)\varepsilon_1, \varepsilon_3) \tag{11}$$

$$S(\varepsilon_1, \xi) = (n-1)\eta(\varepsilon_1), \tag{12}$$

$$Q\xi = (n-1)\xi, \tag{13}$$

where R, S and Q are the Riemann curvature tensor, Ricci curvature tensor and Ricci operator, respectively.

Let Φ be a Riemannian manifold, T is $(0,k)-$type tensor field and A is $(0,2)-$type tensor field. In this case, Tachibana tensor field $Q(A,T)$ is defined as

$$Q(A,T)(X_1,...,X_k;\varepsilon_1,\varepsilon_2)=-T((\varepsilon_1\wedge_A\varepsilon_2)X_1,...,X_k)-...-T(X_1,...,X_{k-1},(\varepsilon_1\wedge_A\varepsilon_2)X_k), \tag{14}$$

where,

$$(\varepsilon_1\wedge_A\varepsilon_2)\varepsilon_3=A(\varepsilon_2,\varepsilon_3)\varepsilon_1-A(\varepsilon_1,\varepsilon_3)\varepsilon_2, \tag{15}$$

$k\geq 1, X_1,X_2,...,X_k,\varepsilon_1,\varepsilon_2\in\Gamma(T\Phi)$.
Precisely, a Ricci soliton on a Riemannian manifold (Φ,g) is defined as a triple (g,ξ,λ) on Φ satisfying

$$L_\xi g+2S+2\lambda g=0, \tag{16}$$

where L_ξ is the Lie derivative operator along the vector field ξ and λ is a real constant. We note that if ξ is a Killing vector field, then the Ricci soliton reduces to an Einstein metric (g,λ). Futhermore, in [28], generalization is the notion of $\eta-RS$ defined by J.T. Cho and M. Kimura as a quadruple (g,ξ,λ,μ) satisfying

$$L_\xi g+2S+2\lambda g+2\mu\eta\otimes\eta=0, \tag{17}$$

where λ and μ are real constants and η is the dual of ξ and S denotes the Ricci tensor of Φ. Furthermore if λ and μ are smooth functions on Φ, then it called almost $\eta-RS$ on Φ [28].
Suppose the quartet (g,ξ,λ,μ) is almost $\eta-RS$ on manifold Φ. Then, classification is as follows.
· If $\lambda<0$, then Φ is shrinking.
· If $\lambda=0$, then Φ is steady.
· If $\lambda>0$, then Φ is expanding.

3. Almost $\eta-$Ricci Solitons on Ricci Pseudosymmetric and Ricci Semisymmetric Normal Paracontact Metric Space Forms

Now let (g,ξ,λ,μ) be almost $\eta-RS$ on $NPMS-$form. Then we have

$$\left(L_\xi g\right)(\varepsilon_1,\varepsilon_2)=L_\xi g(\varepsilon_1,\varepsilon_2)-g\left(L_\xi\varepsilon_1,\varepsilon_2\right)-g\left(\varepsilon_1,L_\xi\varepsilon_2\right)$$

$$=\xi g(\varepsilon_1,\varepsilon_2)-g([\xi,\varepsilon_1],\varepsilon_2)-g(\varepsilon_1,[\xi,\varepsilon_2])$$

$$=g\left(\nabla_\xi\varepsilon_1,\varepsilon_2\right)+g\left(\varepsilon_1,\nabla_\xi\varepsilon_2\right)-g\left(\nabla_\xi\varepsilon_1,\varepsilon_2\right)$$

$$+g(\nabla_{\varepsilon_1}\xi,\varepsilon_2)-g\left(\nabla_\xi\varepsilon_2,\varepsilon_1\right)+g(\varepsilon_1,\nabla_{\varepsilon_2}\xi),$$

for all $\varepsilon_1,\varepsilon_2\in\Gamma(T\Phi)$. By using ϕ is symmetric, we have

$$\left(L_\xi g\right)(\varepsilon_1,\varepsilon_2)=2g(\phi\varepsilon_1,\varepsilon_2). \tag{18}$$

Thus, in a $NPMS-$forms, from (17) and (18), we have

$$S(\varepsilon_1,\varepsilon_2)+g(\phi\varepsilon_1,\varepsilon_2)+\lambda g(\varepsilon_1,\varepsilon_2)+\mu\eta(\varepsilon_1)\eta(\varepsilon_2)=0. \tag{19}$$

For $\varepsilon_2=\xi$ in (19), this implies that

$$S(\xi,\varepsilon_1)=-(\lambda+\mu)\eta(\varepsilon_1). \tag{20}$$

Taking into account of (12) and (20), we conclude that

$$\lambda+\mu=1-n. \tag{21}$$

Definition 3.1. *Let Φ be an $n-$dimensional $NPMS-$form. If there exists a function H_1 on Φ such that*

$$R\cdot S=H_1Q(g,S),$$

then the Φ is called $Ricci-P$.

Also, if $H_1=0$, the Φ is called $Ricci-S$.
Let us now investigate the $Ricci-P$ case of the $n-$dimensional $NPMS-$forms.

Theorem 3.2. *Let Φ be a $NPMS-$forms and (g,ξ,λ,μ) be almost $\eta-RS$ on Φ. If Φ is a $Ricci-P$, then Φ is either a shrinking or $H_1=1$.*

Proof. Let's assume that $NPMS-$form Φ be $Ricci-P$ and (g,ξ,λ,μ) be almost $\eta-RS$ on Φ. That's mean

$$(R(\varepsilon_1,\varepsilon_2)\cdot S)(\varepsilon_4,\varepsilon_5)=H_1Q(g,S)(\varepsilon_4,\varepsilon_5;\varepsilon_1,\varepsilon_2),$$

for all $\varepsilon_1,\varepsilon_2,\varepsilon_4,\varepsilon_5\in\Gamma(T\Phi)$. From the last equation, we can easily write

$$S(R(\varepsilon_1,\varepsilon_2)\varepsilon_4,\varepsilon_5)+S(\varepsilon_4,R(\varepsilon_1,\varepsilon_2)\varepsilon_5)=H_1\left\{S\left(\left(\varepsilon_1\wedge_g\varepsilon_2\right)\varepsilon_4,\varepsilon_5\right)+S\left(\varepsilon_4,\left(\varepsilon_1\wedge_g\varepsilon_2\right)\varepsilon_5\right)\right\}. \tag{22}$$

If we choose $\varepsilon_5=\xi$ in (22), we get

$$S(R(\varepsilon_1,\varepsilon_2)\varepsilon_4,\xi)+S(\varepsilon_4,R(\varepsilon_1,\varepsilon_2)\xi)=H_1\{S(g(\varepsilon_2,\varepsilon_4)\varepsilon_1-g(\varepsilon_1,\varepsilon_4)\varepsilon_2,\xi)\ +S(\varepsilon_4,\eta(\varepsilon_2)\varepsilon_1-\eta(\varepsilon_1)\varepsilon_2)\}. \tag{23}$$

Putting (10) and (20) in (23), we have

$$\begin{aligned}&S(\varepsilon_4,\eta(\varepsilon_2)\varepsilon_1-\eta(\varepsilon_1)\varepsilon_2)-(\lambda+\mu)\eta(R(\varepsilon_1,\varepsilon_2)\varepsilon_4)\\&=H_1\{-(\lambda+\mu)g(\eta(\varepsilon_1)\varepsilon_2-\eta(\varepsilon_2)\varepsilon_1,\varepsilon_4)\ +S(\varepsilon_4,\eta(\varepsilon_2)\varepsilon_1-\eta(\varepsilon_1)\varepsilon_2)\}.\end{aligned} \tag{24}$$

If we use (11) in (24), we get

$$\begin{aligned}&-(\lambda+\mu)g(\eta(\varepsilon_1)\varepsilon_2-\eta(\varepsilon_2)\varepsilon_1,\varepsilon_4)+S(\eta(\varepsilon_2)\varepsilon_1-\eta(\varepsilon_1)\varepsilon_2,\varepsilon_4)\\&=H_1\{-(\lambda+\mu)g(\eta(\varepsilon_1)\varepsilon_2-\eta(\varepsilon_2)\varepsilon_1,\varepsilon_4)\ +S(\varepsilon_4,\eta(\varepsilon_2)\varepsilon_1-\eta(\varepsilon_1)\varepsilon_2)\}.\end{aligned} \tag{25}$$

If we use (19) in the (25), we can write

$$\mu(1-H_1)g(\eta(\varepsilon_2)\varepsilon_1-\eta(\varepsilon_1)\varepsilon_2,\varepsilon_4)-(1-H_1)g(\eta(\varepsilon_2)\varepsilon_1-\eta(\varepsilon_1)\varepsilon_2,\phi\varepsilon_4)=0. \tag{26}$$

If we write $\phi\varepsilon_4$ instead of ε_4 in (26) and make use of (1), we obtain

$$-(1-H_1)g(\eta(\varepsilon_2)\varepsilon_1-\eta(\varepsilon_1)\varepsilon_2,\varepsilon_4)+\mu(1-H_1)g(\eta(\varepsilon_2)\varepsilon_1-\eta(\varepsilon_1)\varepsilon_2,\phi\varepsilon_4)=0. \tag{27}$$

It is clear from (26) and (27), we get

$$(1-H_1)^2\left(1-\mu^2\right)g(\eta(\varepsilon_2)\varepsilon_1-\eta(\varepsilon_1)\varepsilon_2,\varepsilon_4)=0.$$

This completes the proof of Theorem. □

Corollary 3.3. *Let Φ be $NPMS-$form and (g,ξ,λ,μ) be almost $\eta-RS$ on Φ. If Φ is a $Ricci-S$, then Φ is a shrinking.*

For an $n-$dimensional semi-Riemannian manifold Φ, the concircular curvature tensor is defined as

$$C(\varepsilon_1,\varepsilon_2)\varepsilon_3=R(\varepsilon_1,\varepsilon_2)\varepsilon_3-\frac{r}{n(n-1)}[g(\varepsilon_2,\varepsilon_3)\varepsilon_1-g(\varepsilon_1,\varepsilon_3)\varepsilon_2]. \tag{28}$$

For an $n-$dimensional $NPMS-$form, if we choose $\varepsilon_3=\xi$ in (28), we can write

$$C(\varepsilon_1,\varepsilon_2)\xi=\left[1-\frac{r}{n(n-1)}\right][\eta(\varepsilon_2)\varepsilon_1-\eta(\varepsilon_1)\varepsilon_2]. \tag{29}$$

On the other hand, if we take the inner product of both sides of (26) by ξ, we get

$$\eta(C(\varepsilon_1,\varepsilon_2)\varepsilon_3)=\left[1-\frac{r}{n(n-1)}\right]g(\eta(\varepsilon_1)\varepsilon_2-\eta(\varepsilon_2)\varepsilon_1,\varepsilon_3). \tag{30}$$

Definition 3.4. *Let Φ be an $n-$dimensional $NPMS-$form. If there exists a function H_2 on Φ such that*

$$C\cdot S=H_2Q(g,S),$$

*then the Φ is called **concircular** $Ricci-P$.*

Also, if $H_2=0$, the Φ is called **concircular** $Ricci-S$.
Thus we have the following theorem.

Theorem 3.5. *Let Φ be $NPMS-$forms and (g,ξ,λ,μ) be almost $\eta-RS$ on Φ. If Φ is a concircular $Ricci-P$, then Φ is either shrinking or $H_2=\dfrac{n(n-1)-r}{n(n-1)}$.*

Proof. Let's assume that $NPMS-$form Φ be concircular $Ricci-P$ and (g,ξ,λ,μ) be almost $\eta-RS$ on Φ. This implies that

$$(C(\varepsilon_1,\varepsilon_2)\cdot S)(\varepsilon_4,\varepsilon_5)=H_2Q(g,S)(\varepsilon_4,\varepsilon_5;\varepsilon_1,\varepsilon_2),$$

for all $\varepsilon_1,\varepsilon_2,\varepsilon_4,\varepsilon_5\in\Gamma(T\Phi)$. From the last equation, we can easily write

$$S(C(\varepsilon_1,\varepsilon_2)\varepsilon_4,\varepsilon_5)+S(\varepsilon_4,C(\varepsilon_1,\varepsilon_2)\varepsilon_5)=H_2\left\{S\left(\left(\varepsilon_1\wedge_g\varepsilon_2\right)\varepsilon_4,\varepsilon_5\right)+S\left(\varepsilon_4,\left(\varepsilon_1\wedge_g\varepsilon_2\right)\varepsilon_5\right)\right\}. \tag{31}$$

If we choose $\varepsilon_5=\xi$ in (31), we get

$$S(C(\varepsilon_1,\varepsilon_2)\varepsilon_4,\xi)+S(\varepsilon_4,C(\varepsilon_1,\varepsilon_2)\xi)=H_2\{S(g(\varepsilon_2,\varepsilon_4)\varepsilon_1-g(\varepsilon_1,\varepsilon_4)\varepsilon_2,\xi)\ +S(\varepsilon_4,\eta(\varepsilon_2)\varepsilon_1-\eta(\varepsilon_1)\varepsilon_2)\}. \tag{32}$$

By using of (20) and (29) in (32), we have

$$\begin{aligned}&S(\varepsilon_4,A[\eta(\varepsilon_2)\varepsilon_1-\eta(\varepsilon_1)\varepsilon_2])-(\lambda+\mu)\eta(C(\varepsilon_1,\varepsilon_2)\varepsilon_4)\\&=H_2\{-(\lambda+\mu)g(\eta(\varepsilon_1)\varepsilon_2-\eta(\varepsilon_2)\varepsilon_1,\varepsilon_4)\ +S(\varepsilon_4,\eta(\varepsilon_2)\varepsilon_1-\eta(\varepsilon_1)\varepsilon_2)\},\end{aligned} \tag{33}$$

where $A=1-\frac{r}{n(n-1)}$. Substituting (30) into (33), we have

$$\begin{aligned}&-A(\lambda+\mu)g(\eta(\varepsilon_1)\varepsilon_2-\eta(\varepsilon_2)\varepsilon_1,\varepsilon_4)+AS(\eta(\varepsilon_2)\varepsilon_1-\eta(\varepsilon_1)\varepsilon_2,\varepsilon_4)\\&=H_2\{-(\lambda+\mu)g(\eta(\varepsilon_1)\varepsilon_2-\eta(\varepsilon_2)\varepsilon_1,\varepsilon_4)\ +S(\eta(\varepsilon_2)\varepsilon_1-\eta(\varepsilon_1)\varepsilon_2,\varepsilon_4)\}.\end{aligned} \tag{34}$$

If we use (19) in the (34), we can write

$$\mu(A-H_2)g(\eta(\varepsilon_2)\varepsilon_1-\eta(\varepsilon_1)\varepsilon_2,\varepsilon_4)-(A-H_2)g(\eta(\varepsilon_2)\varepsilon_1-\eta(\varepsilon_1)\varepsilon_2,\phi\varepsilon_4)=0. \tag{35}$$

If we write $\phi\varepsilon_4$ instead of ε_4 in (35) and make use of (1), we obtain

$$-(A-H_2)g(\eta(\varepsilon_2)\varepsilon_1-\eta(\varepsilon_1)\varepsilon_2,\varepsilon_4)+\mu(A-H_2)g(\eta(\varepsilon_2)\varepsilon_1-\eta(\varepsilon_1)\varepsilon_2,\phi\varepsilon_4)=0. \tag{36}$$

It is clear from (35) and (36),

$$(A-H_2)^2\left(1-\mu^2\right)g(\eta(\varepsilon_2)\varepsilon_1-\eta(\varepsilon_1)\varepsilon_2,\varepsilon_4)=0.$$

This completes the proof of Theorem. □

Corollary 3.6. *Let Φ be $NPMS-$forms and (g,ξ,λ,μ) be almost $\eta-RS$ on Φ. If Φ is a concircular $Ricci-S$, then Φ is either shrinking or a manifold with constant scalar curvature $r=n(n-1)$.*

For an $n-$dimensional semi-Riemannian manifold Φ, the projective curvature tensor is defined as

$$P(\varepsilon_1,\varepsilon_2)\varepsilon_3=R(\varepsilon_1,\varepsilon_2)\varepsilon_3-\frac{1}{n-1}[S(\varepsilon_2,\varepsilon_3)\varepsilon_1-S(\varepsilon_1,\varepsilon_3)\varepsilon_2]. \tag{37}$$

For an $n-$dimensional $NPMS-$form, if we choose $\varepsilon_3=\xi$ in (37), we can write

$$P(\varepsilon_1,\varepsilon_2)\xi=0, \tag{38}$$

and similarly if we take the inner product of both sides of (37) by ξ, we get

$$\eta(P(\varepsilon_1,\varepsilon_2)\varepsilon_3)=0. \tag{39}$$

Definition 3.7. *Let Φ be an $n-$dimensional $NPMS-$form. If there exists a function H_3 on Φ such that*

$$P\cdot S=H_3Q(g,S),$$

then the Φ is called **projective** $Ricci-P$**.**

Also, if $H_3=0$, the Φ is called **projective** $Ricci-S$**.**
Let us now investigate the projective $Ricci-P$ case of the $NPMS-$form.

Theorem 3.8. *Let Φ be $NPMS-$forms and (g,ξ,λ,μ) be almost $\eta-RS$ on Φ. If Φ is a projective $Ricci-P$, then Φ is either projective $Ricci-S$ or shrinking.*

Proof. Let's assume that $NPMS-$form Φ be projective $Ricci-P$ and (g,ξ,λ,μ) be almost $\eta-RS$ on Φ. Then we have

$$(P(\varepsilon_1,\varepsilon_2)\cdot S)(\varepsilon_4,\varepsilon_5)=H_3Q(g,S)(\varepsilon_4,\varepsilon_5;\varepsilon_1,\varepsilon_2),$$

for all $\varepsilon_1,\varepsilon_2,\varepsilon_4,\varepsilon_5\in\Gamma(T\Phi)$. This gives us, we can easily write

$$S(P(\varepsilon_1,\varepsilon_2)\varepsilon_4,\varepsilon_5)+S(\varepsilon_4,P(\varepsilon_1,\varepsilon_2)\varepsilon_5)=H_3\left\{S\left(\left(\varepsilon_1\wedge_g\varepsilon_2\right)\varepsilon_4,\varepsilon_5\right)+S\left(\varepsilon_4,\left(\varepsilon_1\wedge_g\varepsilon_2\right)\varepsilon_5\right)\right\}. \tag{40}$$

Putting $\varepsilon_5=\xi$ in (40), we get

$$S(P(\varepsilon_1,\varepsilon_2)\varepsilon_4,\xi)+S(\varepsilon_4,P(\varepsilon_1,\varepsilon_2)\xi)=H_3\{S(g(\varepsilon_2,\varepsilon_4)\varepsilon_1-g(\varepsilon_1,\varepsilon_4)\varepsilon_2,\xi)\ +S(\varepsilon_4,\eta(\varepsilon_2)\varepsilon_1-\eta(\varepsilon_1)\varepsilon_2)\}. \tag{41}$$

If we make use of (20) and (38) in (41), we have

$$-(\lambda+\mu)\eta(P(\varepsilon_1,\varepsilon_2)\varepsilon_4)=H_3\{-(\lambda+\mu)g(\eta(\varepsilon_1)\varepsilon_2-\eta(\varepsilon_2)\varepsilon_1,\varepsilon_4)\ +S(\varepsilon_4,\eta(\varepsilon_2)\varepsilon_1-\eta(\varepsilon_1)\varepsilon_2)\}. \tag{42}$$

If we use (39) in the (42), we get

$$H_3\{-(\lambda+\mu)g(\eta(\varepsilon_1)\varepsilon_2-\eta(\varepsilon_2)\varepsilon_1,\varepsilon_4)\ +S(\eta(\varepsilon_2)\varepsilon_1-\eta(\varepsilon_1)\varepsilon_2,\varepsilon_4)\}=0. \tag{43}$$

If we use (19) in the (43), we can write

$$H_3[\mu g(\eta(\varepsilon_2)\varepsilon_1-\eta(\varepsilon_1)\varepsilon_2,\varepsilon_4)-g(\eta(\varepsilon_2)\varepsilon_1-\eta(\varepsilon_1)\varepsilon_2,\phi\varepsilon_4)]=0. \tag{44}$$

If we write $\phi\varepsilon_4$ instead of ε_4 in (44) and make use of (1), we obtain

$$H_3[-g(\eta(\varepsilon_2)\varepsilon_1-\eta(\varepsilon_1)\varepsilon_2,\varepsilon_4)+\mu g(\eta(\varepsilon_2)\varepsilon_1-\eta(\varepsilon_1)\varepsilon_2,\phi\varepsilon_4)]=0. \tag{45}$$

It is clear from (44) and (45), we obtain

$$H_3\left(1-\mu^2\right)g(\eta(\varepsilon_2)\varepsilon_1-\eta(\varepsilon_1)\varepsilon_2,\varepsilon_4)=0.$$

This completes the proof of Theorem. □

For an $n-$dimensional semi-Riemannian manifold Φ, the W_1-curvature tensor is defined as

$$W_1(\varepsilon_1,\varepsilon_2)\varepsilon_3=R(\varepsilon_1,\varepsilon_2)\varepsilon_3+\frac{1}{n-1}[S(\varepsilon_2,\varepsilon_3)\varepsilon_1-S(\varepsilon_1,\varepsilon_3)\varepsilon_2]. \tag{46}$$

For an $n-$dimensional $NPMS-$form, if we choose $\varepsilon_3=\xi$ in (46), we can write

$$W_1(\varepsilon_1,\varepsilon_2)\xi=2[\eta(\varepsilon_2)\varepsilon_1-\eta(\varepsilon_1)\varepsilon_2], \tag{47}$$

and similarly if we take the inner product of both of sides of (46) by ξ, we get

$$\eta(W_1(\varepsilon_1,\varepsilon_2)\varepsilon_3)=2g(\eta(\varepsilon_1)\varepsilon_2-\eta(\varepsilon_2)\varepsilon_1,\varepsilon_3). \tag{48}$$

Definition 3.9. *Let Φ be an $n-$dimensional $NPMS-$form. If there exists a function H_4 on Φ such that*

$$W_1\cdot S=H_4Q(g,S),$$

then the Φ is called $W_1-Ricci-P$.

Also, if $H_4=0$, the Φ is said to be $W_1-Ricci-S$.
Let us now investigate the $W_1-Ricci-P$ case of the normal paracontact space form.

Theorem 3.10. *Let Φ be $NPMS-$forms and (g,ξ,λ,μ) be almost $\eta-RS$ on Φ. If Φ is $W_1-Ricci-P$, then Φ is either shrinking or $H_4=2$.*

Proof. Let's assume that normal paracontact space form Φ be $W_1-Ricci-P$ and (g,ξ,λ,μ) be almost $\eta-RS$ on Φ. This implies that

$$(W_1(\varepsilon_1,\varepsilon_2)\cdot S)(\varepsilon_4,\varepsilon_5)=H_4Q(g,S)(\varepsilon_4,\varepsilon_5;\varepsilon_1,\varepsilon_2),$$

for all $\varepsilon_1,\varepsilon_2,\varepsilon_4,\varepsilon_5\in\Gamma(T\Phi)$. From the last equation, we can easily write

$$S(W_1(\varepsilon_1,\varepsilon_2)\varepsilon_4,\varepsilon_5)+S(\varepsilon_4,W_1(\varepsilon_1,\varepsilon_2)\varepsilon_5)=H_4\left\{S\left((\varepsilon_1\wedge_g\varepsilon_2)\varepsilon_4,\varepsilon_5\right)+S\left(\varepsilon_4,(\varepsilon_1\wedge_g\varepsilon_2)\varepsilon_5\right)\right\}. \tag{49}$$

If we choose $\varepsilon_5=\xi$ in (49), we get

$$S(W_1(\varepsilon_1,\varepsilon_2)\varepsilon_4,\xi)+S(\varepsilon_4,W_1(\varepsilon_1,\varepsilon_2)\xi)=H_4\{S(g(\varepsilon_2,\varepsilon_4)\varepsilon_1-g(\varepsilon_1,\varepsilon_4)\varepsilon_2,\xi)\ +S(\varepsilon_4,\eta(\varepsilon_1)\varepsilon_2-\eta(\varepsilon_2)\varepsilon_1)\}. \tag{50}$$

If we make use of (20) and (47) in (50), we have

$$\begin{aligned}&2S(\varepsilon_4,\eta(\varepsilon_2)\varepsilon_1-\eta(\varepsilon_1)\varepsilon_2)-(\lambda+\mu)\eta(W_1(\varepsilon_1,\varepsilon_2)\varepsilon_4)\\&=H_4\{-(\lambda+\mu)g(\eta(\varepsilon_1)\varepsilon_2-\eta(\varepsilon_2)\varepsilon_1,\varepsilon_4)\ +S(\varepsilon_4,\eta(\varepsilon_2)\varepsilon_1-\eta(\varepsilon_1)\varepsilon_2)\}.\end{aligned} \tag{51}$$

If we use (48) in the (51), we get

$$\begin{aligned}&-2(\lambda+\mu)g(\eta(\varepsilon_1)\varepsilon_2-\eta(\varepsilon_2)\varepsilon_1,\varepsilon_4)+2S(\eta(\varepsilon_2)\varepsilon_1-\eta(\varepsilon_1)\varepsilon_2,\varepsilon_4)\\&=H_4\{-(\lambda+\mu)g(\eta(\varepsilon_1)\varepsilon_2-\eta(\varepsilon_2)\varepsilon_1,\varepsilon_4)\ +S(\eta(\varepsilon_2)\varepsilon_1-\eta(\varepsilon_1)\varepsilon_2,\varepsilon_4)\}.\end{aligned} \tag{52}$$

If we use (19) in the (52), we can write

$$\mu(2-H_4)g(\eta(\varepsilon_2)\varepsilon_1-\eta(\varepsilon_1)\varepsilon_2,\varepsilon_4)+(H_4-2)g(\eta(\varepsilon_2)\varepsilon_1-\eta(\varepsilon_1)\varepsilon_2,\phi\varepsilon_4)=0. \tag{53}$$

If we write $\phi\varepsilon_4$ instead of ε_4 in (44) and make use of (1), we obtain

$$(H_4-2)g(\eta(\varepsilon_2)\varepsilon_1-\eta(\varepsilon_1)\varepsilon_2,\varepsilon_4)+\mu(2-H_4)g(\eta(\varepsilon_2)\varepsilon_1-\eta(\varepsilon_1)\varepsilon_2,\phi\varepsilon_4)=0. \tag{54}$$

It is clear from (53) and (54), we get

$$\left(2-H_4^2\right)\left(1-\mu^2\right)g(\eta(\varepsilon_2)\varepsilon_1-\eta(\varepsilon_1)\varepsilon_2,\varepsilon_4)=0.$$

This completes the proof of Theorem. □

Corollary 3.11. *Let Φ be a $NPMS-$form and (g,ξ,λ,μ) be almost $\eta-RS$ on Φ. If Φ is a $W_1-Ricci-S$, then Φ is a shriking.*

4. Conclusion

In this paper, we have considered normal paracontact metric space forms admitting almost $\eta-$Ricci solitons in some curvature tensors. Ricci pseudosymmetry concepts of normal paracontact metric space forms admitting $\eta-$Ricci soliton have introduced according to the choosing of some special curvature tensors such as Riemann, concircular, projective and W_1 curvature tensor. After then, according to the choice of the curvature tensors, necessary conditions are given for normal paracontact metric space form admitting $\eta-$Ricci soliton to be Ricci semisymmetric. Then some characterizations are obtained and some classifications have made under the some conditions.

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References

[1] Kenayuki S. and Williams F.L., Almost paracontact and parahodge structures on manifolds, Nagoya Math. J., 99 (1985), 173-187.
[2] Zamkovoy S., Canonical connections on paracontact manifolds, Ann Glob. Anal. Geom., 36 (2009), 37-60.
[3] Welyczko J., On Legendre curvaes in 3-dimensional normal almost paracontact metric manifolds, Result. Math., 54 (2009), 377-387.
[4] Welyczko J., Slant curves in 3-dimensional normal contact metric manifolds, Mediterr. J. Math., 11 (2014), 965-978.
[5] Pandey H.B. and Kumar A., Anti invariant submanifolds of almost paracontact metric manifolds, Indian J. pure appl. math., 16(6) (1985), 586-590.
[6] Yıldırım Ü., Atçeken M. and Dirik S., A normal paracontact metric manifold satisfying some conditions on the $M-$projectivecurvature tensor, Konuralp Journal of Mathematics, 7(1) (2019), 217-221.
[7] Yıldırım Ü., Atçeken M. and Dirik S., Pseudo projective curvture tensor satisfying some properties on a normal paracontactmetric manifold, Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat., 68(1) (2019), 997-1006.
[8] Perelman G., The entropy formula for the Ricci flow and its geometric applications,http://arXiv.org/abs/math/0211159, (2002), 1–39.
[9] Perelman G., Ricci flow with surgery on three manifolds, http://arXiv.org/abs/math/0303109, (2003), 1–22.
[10] Sharma R., Certain results on k-contact and $(k,\mu)-$contact manifolds, J. Geom., 89 (2008),138–147.
[11] Ashoka S.R, Bagewadi C.S. and Ingalahalli G., Certain results on Ricci Solitons in $\alpha-$Sasakian manifolds, Hindawi Publ. Corporation, Geometry, Vol.(2013), Article ID 573925,4 Pages.
[12] Ashoka S.R., Bagewadi C.S. and Ingalahalli G., A geometry on Ricci solitons in $(LCS)_n-$manifolds, Diff. Geom.-Dynamical Systems, 16 (2014), 50–62.
[13] Bagewadi C.S. and Ingalahalli G., Ricci solitons in Lorentzian-Sasakian manifolds, Acta Math. Acad. Paeda. Nyire., 28 (2012), 59-68.
[14] Ingalahalli G. and Bagewadi C.S., Ricci solitons in $\alpha-$Sasakian manifolds, ISRN Geometry,Vol.(2012), Article ID 421384, 13 Pages.
[15] Bejan C.L. and Crasmareanu M., Ricci Solitons in manifolds with quasi-contact curvature, Publ. Math. Debrecen, 78 (2011), 235-243.
[16] Blaga A.M., $\eta-$Ricci solitons on para-kenmotsu manifolds, Balkan J. Geom. Appl., 20 (2015), 1–13.
[17] Chandra S., Hui S.K. and Shaikh A.A., Second order parallel tensors and Ricci solitons on $(LCS)_n$-manifolds, Commun. Korean Math. Soc., 30 (2015), 123–130.
[18] Chen B.Y. and Deshmukh S., Geometry of compact shrinking Ricci solitons, Balkan J. Geom. Appl., 19 (2014), 13–21.
[19] Deshmukh S., Al-Sodais H. and Alodan H., A note on Ricci solitons, Balkan J. Geom. Appl.,16 (2011), 48–55.
[20] He C. and Zhu M., Ricci solitons on Sasakian manifolds, arxiv:1109.4407V2, [Math DG], (2011).
[21] Atçeken M., Mert T. and Uygun P., Ricci-Pseudosymmetric $(LCS)_n-$manifolds admitting almost $\eta-$Ricci solitons, Asian Journal of Math. and Computer Research, 29(2), 23-32,2022.
[22] Nagaraja H. and Premalatta C.R., Ricci solitons in Kenmotsu manifolds, J. Math. Analysis, 3(2) (2012), 18–24.
[23] Tripathi M.M., Ricci solitons in contact metric manifolds, arxiv:0801,4221 V1, [Math DG], (2008).
[24] Ayar G. and Yıldırım M., $\eta-$Ricci solitons on nearly Kenmotsu manifolds, Asian-European Journal of Mathematics 12(6) (2019).
[25] Ayar G. and Yıldırım M., Ricci solitons and gradient Ricci solitons on nearly Kenmotsu manifolds, Facta Unıversitatis(Nis)Ser.Math.Inform.,34 (2019), 503-510.
[26] Yıldırım M. and Ayar G.,Ricci solitons and gradient Ricci solitons on nearly cosyplectic manifolds, Journal of Universal Mathematics, 4(2) (2021), 201-208.
[27] Ayar G. and Demirhan D., Ricci Solitons on Nearly Kenmotsu Manifolds with Semi-symmetric Metric Connection, Journal of Engineering Technology and Applied Sciences, 4(3) (2019), 131-140.
[28] Cho J.T. and Kimura M., Ricci solitons and real hypersurfaces in a complex space form, Tohoku Math. J. 61(2) (2009), 205-212.