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Research Article

The Alpha Distance Formulae

Harun Barış Çolakoğlu *
Akdeniz University, Technical Sciences Vocational
School Department of Computer Technologies, Antalya, Türkiye

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Abstract: In this study, we define the concept of tangent in two and three-dimensional alpha spaces concerning the alpha circles and the alpha spheres. Then using this concept, we derive the alpha distance formulae between points, a point and a line, between two lines and a point and a plane of the alpha spaces. Finally, we give simple area and volume formulas in the three dimensional space in terms of the alpha distances.

Keywords: Alpha metric, distance formula, metric geometry.

1. Introduction

Metrics with their special properties have been very important keys for many application areas during the recent years. There are many metrics used in the mathematics (see [8]) to measure the distance (similarity or dissimilarity) between points (or vectors). These measurements are important for determining how closely related two pieces of data are in statistical analysis. The alpha metric (α-metric) is a generalization of two famous metrics known as the taxicab and the Chinese checker metric which are used in such applications. They are very suitable for new studies since it includes infinitely many metrics in which the alpha can be considered as a weight that can reflect relative importance of different criteria or dimensions. On the other hand, the derived conclusions are rather wide (for example see [6, 7, 9]).

On the road to the alpha metric, first Menger introduced the taxicab geometry using the taxicab metric [14], and Krause took the first steps to develop it [13]. The taxicab metric is the special case of the l_p-metric for $p = 1$. In [13], Krause asked how to develop a distance function from a point to another which measures the length of ways mimicking the movements of the Chinese checkers in the Cartesian coordinate plane. Then, Chen answered this question defining the Chinese checker metric [2]. After a while, the α-metric for $\alpha \in \left[0, \frac{\pi}{4}\right]$, which includes the taxicab and Chinese checker metrics as special cases for $\alpha = 0$ and $\alpha = \frac{\pi}{4}$, defined by Tian [15].

*Correspondence: hbcolakoglu@akdeniz.edu.tr
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Later, Gelişgen and Kaya gave n-dimensional α-metric [11, 12]. Finally, Çolakoğlu expanded the interval $\alpha \in \left[0, \frac{\pi}{4}\right]$ to $\alpha \in \left[0, \frac{\pi}{2}\right)$ for the α-metric [3].

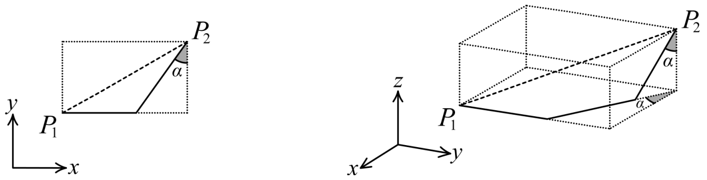

Figure 1: The distances between two points

Geometrically, the α-distance between two points in the plane, is the sum of Euclidean lengths of line segments joining the points, one of which is parallel to a coordinate axis and the other one is parallel to a line making angle α with the other coordinate axis (see Figure 1). So far many studies have been done on this topic (see [1, 4–7, 9, 10]). In this study, we determine the α-distance formulae between two basic elements such as points, lines and planes, whose Euclidean analogs are well-known already, and give simple area and volume formulas in the three dimensional alpha space.

2. Preliminaries

For the positive real number $\lambda(\alpha) = (\sec\alpha - \tan\alpha)$, where $\alpha \in \left[0, \frac{\pi}{2}\right)$, the α-distance between points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ in the plane is

$$d_\alpha(P_1, P_2) = \max\{|x_1 - x_2|, |y_1 - y_2|\} + \lambda(\alpha)\min\{|x_1 - x_2|, |y_1 - y_2|\}. \tag{1}$$

Clearly, the unit α-circle has the following equation:

$$\max\{|x|, |y|\} + \lambda(\alpha)\min\{|x|, |y|\} = 1. \tag{2}$$

One can see that in the plane; if $\alpha \in \left(0, \frac{\pi}{2}\right)$, the unit α-circle is an octagon (see Figure 2) having corners $C_1 = (1, 0)$, $C_2 = \left(\frac{1}{\tau}, \frac{1}{\tau}\right)$, $C_3 = (0, 1)$, $C_4 = \left(\frac{-1}{\tau}, \frac{1}{\tau}\right)$, C_1', C_2', C_3', C_4', where C_i' are the symmetric points of C_i about the origin and $\tau = 1 + \lambda(\alpha)$. In addition, if $\alpha = \frac{\pi}{4}$ then the unit α-circle is a regular octagon with the same vertices, and if $\alpha = 0$, the unit α-circle is a square having corners C_1, C_1', C_3, C_3'.

Similarly, the α-distance between points $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$ in the three dimensional space is

$$d_\alpha(P_1, P_2) = \Delta_{P_1P_2} + \lambda(\alpha)\delta_{P_1P_2}, \tag{3}$$

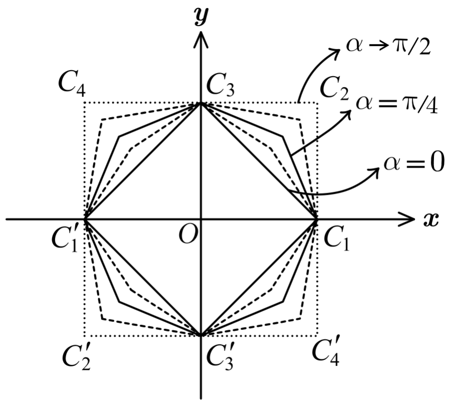

Figure 2: The unit α-circles for $\alpha = 0$, $\alpha = \frac{\pi}{4}$ and $\alpha \to \frac{\pi}{2}$

where

$$\Delta_{P_1P_2} = \max\{|x_1 - x_2|, |y_1 - y_2|, |z_1 - z_2|\}, \tag{4}$$

$$\delta_{P_1P_2} = \min\{|x_1 - x_2| + |y_1 - y_2|, |x_1 - x_2| + |z_1 - z_2|, |y_1 - y_2| + |z_1 - z_2|\}, \tag{5}$$

and the unit α-sphere has the following equation:

$$\max\{|x|, |y|, |z|\} + \lambda(\alpha)\min\{|x| + |y|, |x| + |z|, |y| + |z|\} = 1. \tag{6}$$

One can also see that in three dimensional space; if $\alpha \in \left(0, \frac{\pi}{2}\right)$ then the unit α-sphere is deltoidal icositetrahedron (see Figure 3) having corners $S_1 = (1, 0, 0)$, $S_2 = (0, 1, 0)$, $S_3 = (0, 0, 1)$, $S_4 = \left(\frac{1}{\tau}, \frac{1}{\tau}, 0\right)$, $S_5 = \left(\frac{-1}{\tau}, \frac{1}{\tau}, 0\right)$, $S_6 = \left(\frac{1}{\tau}, 0, \frac{1}{\tau}\right)$, $S_7 = \left(\frac{-1}{\tau}, 0, \frac{1}{\tau}\right)$, $S_8 = \left(0, \frac{1}{\tau}, \frac{1}{\tau}\right)$, $S_9 = \left(0, \frac{-1}{\tau}, \frac{1}{\tau}\right)$, $S_{10} = \left(\frac{1}{\tau}, \frac{1}{\tau}, \frac{1}{\tau}\right)$, $S_{11} = \left(\frac{-1}{\tau}, \frac{1}{\tau}, \frac{1}{\tau}\right)$, $S_{12} = \left(\frac{-1}{\tau}, \frac{-1}{\tau}, \frac{1}{\tau}\right)$, $S_{13} = \left(\frac{1}{\tau}, \frac{-1}{\tau}, \frac{1}{\tau}\right)$, S_1', S_2', S_3', S_4', S_5', S_6', S_7', S_8', S_9', S_{10}', S_{11}', S_{12}', S_{13}', where S_i' are the symmetric points of S_i about the origin and $\tau = 1 + \lambda(\alpha)$, if $\alpha \in \left(0, \frac{\pi}{2}\right)$ then the unit α-sphere is a regular octahedron.

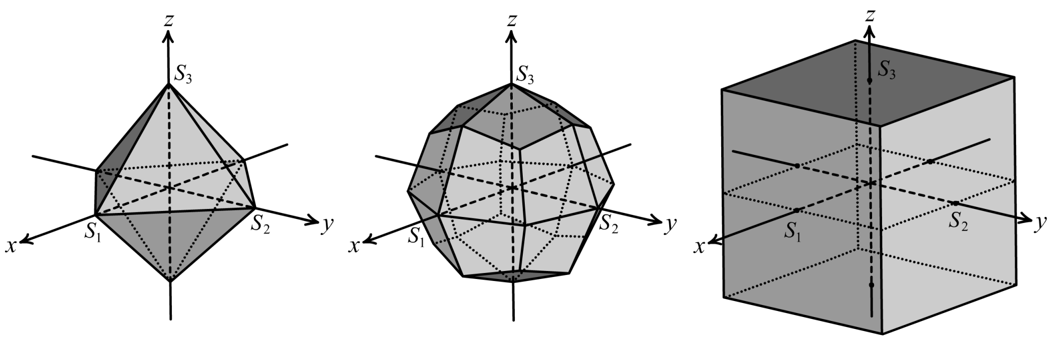

Figure 3: The unit α-spheres for $\alpha = 0$, $\alpha = \frac{\pi}{4}$ and $\alpha \to \frac{\pi}{2}$

3. Main Results

We use the tangent notion to determine α-distance formulae. As a natural analog of the tangent notion in the Euclidean geometry, a line whose α-distance from the center of a given α-circle is the radius of the α-circle, is called a tangent to the α-circle, and a line or a plane whose α-distance from the center of a given α-sphere is the radius of the α-sphere, is called a tangent line or tangent plane to the α-sphere. For instance, in Figure 4, while the lines l_1 and l_2 are tangent to the α-circle with center P_1; the lines l_3, l_4, and the planes Ω_1, Ω_2 are tangent to the α-sphere with center P_2.

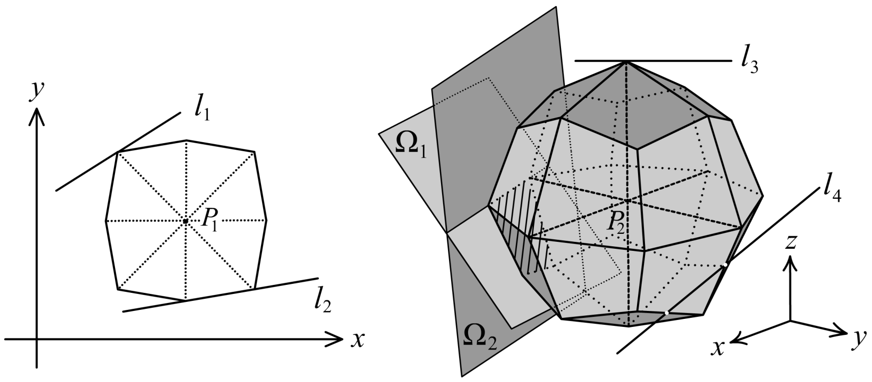

Figure 4: Tangent lines and planes to a α-circle and a α-sphere

Before we start determining the α-distance formulae, let us define three following vector sets that we will use in the proofs as V_1, V_3, V_2, respectively:

$$\{(1,0),(0,1),(1,1),(-1,1)\},$$
$$\{(1,0,0),(0,1,0),(0,0,1),(1,1,0),(-1,1,0),(1,0,1),(-1,0,1),(0,1,1),(0,-1,1)\},$$
$$V_3 \cup \{(1,1,1),(-1,1,1),(1,-1,1),(1,1,-1)\}.$$

The formula of the α-distance between a point and a line is given by the following proposition:

Proposition 3.1 *The α-distance between a point $P=(x_0,y_0)$ and a line $l: Ax+By+C=0$ in $\mathbb{R}^2$ is*

$$d_\alpha(P,l)=\frac{|Ax_0+By_0+C|}{\max\left\{|A|,|B|,\frac{|A\mp B|}{1+\lambda(\alpha)}\right\}}. \tag{7}$$

Proof It is clear that

$$d_\alpha(P,l)=\min\{d_\alpha(P,X): X\in l\},$$

which is equal to the radius of the α-circle with center P, that is tangent to the line l. So, a corner of the α-circle is on l and one of the lines $l_{\mathbf{v}}:\beta(t_{\mathbf{v}})=(x_0,y_0)+\mathbf{v}t_{\mathbf{v}}$ passing the point P

having the direction vector $\mathbf{v} \in V_1$. Therefore, at least one of the points $Q_{\mathbf{v}} = l \cap l_{\mathbf{v}}$ exists, and l is tangent to the α-circle at one of them. So, we have

$$d_\alpha(P,l) = \min\{d_\alpha(P,Q_{\mathbf{v}}) : \mathbf{v} \in V_1)\}.$$

One can find that

$$Q_{(1,0)} = \left(x_0 + t_{(1,0)}, y_0\right), Q_{(0,1)} = \left(x_0, y_0 + t_{(0,1)}\right),$$

$$Q_{(1,1)} = \left(x_0 + t_{(1,1)}, y_0 + t_{(1,1)}\right), Q_{(-1,1)} = \left(x_0 + t_{(-1,1)}, y_0 + t_{(-1,1)}\right),$$

where $t_{\mathbf{v}} = \frac{-Ax_0 - By_0 - C}{\langle (A,B), \mathbf{v}\rangle}$.

If the line l is not parallel to the lines $l_{\mathbf{v}}$, where $\mathbf{v} \in V_1$, then all of the points $Q_{\mathbf{v}}$ can be obtained and one gets

$$d_\alpha(P,Q_{(1,0)}) = \left|t_{(1,0)}\right| = \frac{|Ax_0 + By_0 + C|}{|A|},$$

$$d_\alpha(P,Q_{(0,1)}) = \left|t_{(0,1)}\right| = \frac{|Ax_0 + By_0 + C|}{|B|},$$

$$d_\alpha(P,Q_{(1,1)}) = \left|t_{(1,1)}\right|(1 + \lambda(\alpha)) = \frac{|Ax_0 + By_0 + C|}{|A + B|/(1 + \lambda(\alpha))},$$

$$d_\alpha(P,Q_{(-1,1)}) = \left|t_{(-1,1)}\right|(1 + \lambda(\alpha)) = \frac{|Ax_0 + By_0 + C|}{|-A + B|/(1 + \lambda(\alpha))}.$$

Then one has

$$d_\alpha(P,l) = \min\left\{\frac{|Ax_0+By_0+C|}{|A|}, \frac{|Ax_0+By_0+C|}{|B|}, \frac{|Ax_0+By_0+C|}{|A+B|/(1+\lambda(\alpha))}, \frac{|Ax_0+By_0+C|}{|A-B|/(1+\lambda(\alpha))}\right\},$$

and

$$d_\alpha(P,l) = \frac{|Ax_0 + By_0 + C|}{\max\left\{|A|, |B|, \frac{|A\mp B|}{1+\lambda(\alpha)}\right\}}.$$

Other conditions do not change the result. □

The α-distance between two parallel lines in the plane can be determined by the following formula:

Corollary 3.2 *The α-distance between $l_1 : Ax + By + C_1 = 0$ and $l_2 : Ax + By + C_2 = 0$ in $\mathbb{R}^2$ is*

$$d_\alpha(l_1,l_2) = \frac{|C_1 - C_2|}{\max\left\{|A|, |B|, \frac{|A\mp B|}{1+\lambda(\alpha)}\right\}}. \tag{8}$$

The three dimensional case is similar. One can consider an α-sphere instead of an α-circle. The α-distance from a point to a plane or a line is equal to the radius of the widening α-sphere when the plane or the line becomes tangent to the α-sphere. The following proposition states a formula for the α-distance between a point and a plane in three-dimensional space:

Proposition 3.3 *The α-distance between the point $P = (x_0, y_0, z_0)$ and the plane $\Omega : Ax + By + Cz + D = 0$ in $\mathbb{R}^3$ is*

$$d_\alpha(P,\Omega) = \frac{|Ax_0 + By_0 + Cz_0 + D|}{\max\left\{|K|, \frac{|K\mp L|}{1+\lambda(\alpha)}, \frac{|K+L\mp M|}{1+2\lambda(\alpha)}\right\}}, \tag{9}$$

where $K, L, M \in \{A, B, C\}$ and $K \neq L \neq M \neq K$.

Proof It is obvious that

$$d_\alpha(P,\Omega) = \min\{d_\alpha(P,X) : X \in \Omega\},$$

which is equal to the radius of the α-sphere with center P, that is tangent to the plane Ω. So, at least one vertex of the α-sphere is on Ω and one of the lines $l_\mathbf{v} : \beta(t_\mathbf{v}) = (x_0, y_0, z_0) + \mathbf{v}t_\mathbf{v}$ passing the point P having direction vector $\mathbf{v} \in V_2$. Therefore, at least one of the points $Q_\mathbf{v} = \Omega \cap l_\mathbf{v}$ exists, and Ω is tangent to the α-sphere at one of them. So, we have

$$d_\alpha(P,\Omega) = \min\{d_\alpha(P,Q_v) : v \in V_2\}.$$

One can find that

$$Q_{(1,0,0)} = \left(x_0 + t_{(1,0,0)}, y_0, z_0\right),\ Q_{(0,1,0)} = \left(x_0, y_0 + t_{(0,1,0)}, z_0\right),\ Q_{(0,0,1)} = \left(x_0, y_0, z_0 + t_{(0,0,1)}\right),$$
$$Q_{(1,1,0)} = \left(x_0 + t_{(1,1,0)}, y_0 + t_{(1,1,0)}, z_0\right),\ Q_{(-1,1,0)} = \left(x_0 - t_{(-1,1,0)}, y_0 + t_{(-1,1,0)}, z_0\right),$$
$$Q_{(1,0,1)} = \left(x_0 + t_{(1,0,1)}, y_0, z_0 + t_{(1,0,1)}\right),\ Q_{(-1,0,1)} = \left(x_0 - t_{(-1,0,1)}, y_0, z_0 + t_{(-1,0,1)}\right),$$
$$Q_{(0,1,1)} = \left(x_0, y_0 + t_{(0,1,1)}, z_0 + t_{(0,1,1)}\right),\ Q_{(0,-1,1)} = \left(x_0, y_0 - t_{(0,-1,1)}, z_0 + t_{(0,-1,1)}\right),$$
$$Q_{(1,1,1)} = \left(x_0 + t_{(1,1,1)}, y_0 + t_{(1,1,1)}, z_0 + t_{(1,1,1)}\right),$$
$$Q_{(-1,1,1)} = \left(x_0 - t_{(-1,1,1)}, y_0 + t_{(-1,1,1)}, z_0 + t_{(-1,1,1)}\right),$$
$$Q_{(1,-1,1)} = \left(x_0 + t_{(1,-1,1)}, y_0 - t_{(1,-1,1)}, z_0 + t_{(1,-1,1)}\right),$$
$$Q_{(1,1,-1)} = \left(x_0 + t_{(1,1,-1)}, y_0 + t_{(1,1,-1)}, z_0 - t_{(1,1,-1)}\right),$$

where $t_\mathbf{v} = \frac{-Ax_0 - By_0 - Cz_0 - D}{\langle (A,B,C), \mathbf{v}\rangle}$.

Thus, if the plane Ω is not parallel to the lines $l_\mathbf{v}$ where $\mathbf{v} \in V_2$, then all of the points $Q_\mathbf{v}$

exist and we obtain

$$\begin{aligned}
d_\alpha(P,Q_{(1,0,0)}) &= \left|t_{(1,0,0)}\right| = \tfrac{|Ax_0+By_0+Cz_0+D|}{|A|}, \\
d_\alpha(P,Q_{(1,0,0)}) &= \left|t_{(0,1,0)}\right| = \tfrac{|Ax_0+By_0+Cz_0+D|}{|B|}, \\
d_\alpha(P,Q_{(1,0,0)}) &= \left|t_{(0,0,1)}\right| = \tfrac{|Ax_0+By_0+Cz_0+D|}{|C|}, \\
d_\alpha(P,Q_{(1,1,0)}) &= \left|t_{(1,1,0)}\right|(1+\lambda(\alpha)) = \tfrac{|Ax_0+By_0+Cz_0+D|}{|A+B|/(1+\lambda(\alpha))},
\end{aligned}$$

$$\begin{aligned}
d_\alpha(P,Q_{(-1,1,0)}) &= \left|t_{(-1,1,0)}\right|(1+\lambda(\alpha)) = \tfrac{|Ax_0+By_0+Cz_0+D|}{|-A+B|/(1+\lambda(\alpha))}, \\
d_\alpha(P,Q_{(1,0,1)}) &= \left|t_{(1,0,1)}\right|(1+\lambda(\alpha)) = \tfrac{|Ax_0+By_0+Cz_0+D|}{|A+C|/(1+\lambda(\alpha))}, \\
d_\alpha(P,Q_{(-1,0,1)}) &= \left|t_{(-1,0,1)}\right|(1+\lambda(\alpha)) = \tfrac{|Ax_0+By_0+Cz_0+D|}{|-A+C|/(1+\lambda(\alpha))}, \\
d_\alpha(P,Q_{(0,1,1)}) &= \left|t_{(0,1,1)}\right|(1+\lambda(\alpha)) = \tfrac{|Ax_0+By_0+Cz_0+D|}{|B+C|/(1+\lambda(\alpha))}, \\
d_\alpha(P,Q_{(0,-1,1)}) &= \left|t_{(0,-1,1)}\right|(1+\lambda(\alpha)) = \tfrac{|Ax_0+By_0+Cz_0+D|}{|-B+C|/(1+\lambda(\alpha))}, \\
d_\alpha(P,Q_{(1,1,1)}) &= \left|t_{(1,1,1)}\right|(1+2\lambda(\alpha)) = \tfrac{|Ax_0+By_0+Cz_0+D|}{|A+B+C|/(1+2\lambda(\alpha))},
\end{aligned}$$

$$\begin{aligned}
d_\alpha(P,Q_{(-1,1,1)}) &= \left|t_{(-1,1,1)}\right|(1+2\lambda(\alpha)) = \tfrac{|Ax_0+By_0+Cz_0+D|}{|-A+B+C|/(1+2\lambda(\alpha))}, \\
d_\alpha(P,Q_{(1,-1,1)}) &= \left|t_{(1,-1,1)}\right|(1+2\lambda(\alpha)) = \tfrac{|Ax_0+By_0+Cz_0+D|}{|A-B+C|/(1+2\lambda(\alpha))}, \\
d_\alpha(P,Q_{(1,1,-1)}) &= \left|t_{(1,1,-1)}\right|(1+2\lambda(\alpha)) = \tfrac{|Ax_0+By_0+Cz_0+D|}{|A+B-C|/(1+2\lambda(\alpha))}.
\end{aligned}$$

Therefore, we get

$$d_\alpha(P,\Omega) = \min\left\{\tfrac{|Ax_0+By_0+Cz_0+D|}{|K|}, \tfrac{|Ax_0+By_0+Cz_0+D|}{|K\mp L|/(1+\lambda(\alpha))}, \tfrac{|Ax_0+By_0+Cz_0+D|}{|K+L\mp M|/(1+2\lambda(\alpha))}\right\}$$

and so

$$d_\alpha(P,\Omega) = \frac{|Ax_0+By_0+Cz_0+D|}{\max\left\{|K|, \frac{|K\mp L|}{1+\lambda(\alpha)}, \frac{|K+L\mp M|}{1+2\lambda(\alpha)}\right\}},$$

where $K,L,M \in \{A,B,C\}$ and $K \neq L \neq M \neq K$. Other conditions do not change the result. □

The α-distance between two parallel planes in three-dimensional space can be given as follows:

Corollary 3.4 *The α-distance between $\Omega_1 : Ax+By+Cz+D_1 = 0$ and $\Omega_2 : Ax+By+Cz+D_2 = 0$ in $\mathbb{R}^3$ is*

$$d_\alpha(\Omega_1,\Omega_2) = \frac{|D_1-D_2|}{\max\left\{|K|, \frac{|K\mp L|}{1+\lambda(\alpha)}, \frac{|K+L\mp M|}{1+2\lambda(\alpha)}\right\}}, \tag{10}$$

where $K, L, M \in \{A, B, C\}$ *and* $K \neq L \neq M \neq K$.

The α-distance between a point and a line in three-dimensional space can be computed by the formula given in the following proposition:

Proposition 3.5 *The* α *-distance between the point* $P = (x_0, y_0, z_0)$ *and the line* l *passing through the point* $P_1 = (x_1, y_1, z_1)$ *, with the direction vector* $\mathbf{u} = (u_1, u_2, u_3)$ *in* $\mathbb{R}^3$ *is*

$$d_\alpha(P, l) = \min_{\mathbf{v} \in V_3} \left\{ \max \left\{ \left| \rho_i - \frac{u_i \langle \rho, \mathbf{v} \rangle}{\langle \mathbf{u}, \mathbf{v} \rangle} \right| \right\} + \lambda(\alpha) \min \left\{ \left| \rho_j - \frac{u_j \langle \rho, \mathbf{v} \rangle}{\langle \mathbf{u}, \mathbf{v} \rangle} \right| + \left| \rho_k - \frac{u_k \langle \rho, \mathbf{v} \rangle}{\langle \mathbf{u}, \mathbf{v} \rangle} \right| \right\} \right\}, \tag{11}$$

where $\rho = (\rho_1, \rho_2, \rho_3) = (x_1 - x_2, y_1 - y_2, z_1 - z_2)$ *, and* $i, j, k \in \{1, 2, 3\}$ *for* $i \neq j \neq k \neq i$ *.*

Proof We get that

$$d_\alpha(P, l) = \min \{ d_\alpha(P, X) : X \in l \},$$

which is equal to the radius of the α-sphere with center P, that is tangent to the line l. One can see that if the line l tangent to this α-sphere, at least one point on an edge of the sphere is on both the line l and one of the planes $\Omega_\mathbf{v}$ passing P having the normal vector $\mathbf{v} \in V_3$. Therefore, at least one of the points $R_\mathbf{v} = l \cap \Omega_\mathbf{v}$ exists, and l is tangent to the α-sphere at one of them. So, we have

$$d_\alpha(P, l) = \min\{ d_\alpha(P, R_\mathbf{v}) : \mathbf{v} \in V_3 \}.$$

Considering $l : \beta(t) = (x_1 + tu_1, y_1 + tu_2, z_1 + tu_3)$ and $\Omega_\mathbf{v}$, one can find that

$$R_\mathbf{v} = (x_1 + u_1 t_\mathbf{v}, y_1 + u_2 t_\mathbf{v}, z_1 + u_3 t_\mathbf{v}),$$

where $t_v = \frac{\langle \rho, \mathbf{v} \rangle}{\langle \mathbf{u}, \mathbf{v} \rangle}$, $\rho = (\rho_1, \rho_2, \rho_3) = (x_0 - x_1, y_0 - y_1, z_0 - z_1)$ and $\mathbf{v} \in V_3$. If the line l is not parallel to the planes $\Omega_{\mathbf{v}}$, where $\mathbf{v} \in V_3$, then all of the points $R_{\mathbf{v}}$ exist and we obtain

$$\begin{aligned}
d_\alpha(P, R_{(1,0,0)}) &= \max\left\{\left|\rho_2 - \tfrac{u_2}{u_1}\rho_1\right|\left|\rho_3 - \tfrac{u_3}{u_1}\rho_1\right|\right\} + \lambda(\alpha)\min\left\{\left|\rho_2 - \tfrac{u_2}{u_1}\rho_1\right|\left|\rho_3 - \tfrac{u_3}{u_1}\rho_1\right|\right\}, \\
d_\alpha(P, R_{(0,1,0)}) &= \max\left\{\left|\rho_1 - \tfrac{u_1}{u_2}\rho_2\right|\left|\rho_3 - \tfrac{u_3}{u_2}\rho_2\right|\right\} + \lambda(\alpha)\min\left\{\left|\rho_1 - \tfrac{u_1}{u_2}\rho_2\right|\left|\rho_3 - \tfrac{u_3}{u_2}\rho_2\right|\right\}, \\
d_\alpha(P, R_{(0,0,1)}) &= \max\left\{\left|\rho_1 - \tfrac{u_1}{u_3}\rho_3\right|\left|\rho_2 - \tfrac{u_2}{u_3}\rho_3\right|\right\} + \lambda(\alpha)\min\left\{\left|\rho_1 - \tfrac{u_1}{u_3}\rho_3\right|\left|\rho_2 - \tfrac{u_2}{u_3}\rho_3\right|\right\}, \\
d_\alpha(P, R_{(1,1,0)}) &= \max\{k_1, k_2, k_3\} + \lambda(\alpha)\min\{k_1+k_2, k_1+k_3, k_2+k_3\}, \text{ where } k_i = \left|\rho_i - \tfrac{u_i(\rho_1+\rho_2)}{u_1+u_2}\right| \\
d_\alpha(P, R_{(-1,1,0)}) &= \max\{k_1, k_2, k_3\} + \lambda(\alpha)\min\{k_1+k_2, k_1+k_3, k_2+k_3\}, \text{ where } k_i = \left|\rho_i - \tfrac{u_i(-\rho_1+\rho_2)}{-u_1+u_2}\right| \\
d_\alpha(P, R_{(1,0,1)}) &= \max\{k_1, k_2, k_3\} + \lambda(\alpha)\min\{k_1+k_2, k_1+k_3, k_2+k_3\}, \text{ where } k_i = \left|\rho_i - \tfrac{u_i(\rho_1+\rho_3)}{u_1+u_3}\right| \\
d_\alpha(P, R_{(-1,0,1)}) &= \max\{k_1, k_2, k_3\} + \lambda(\alpha)\min\{k_1+k_2, k_1+k_3, k_2+k_3\}, \text{ where } k_i = \left|\rho_i - \tfrac{u_i(-\rho_1+\rho_3)}{-u_1+u_3}\right| \\
d_\alpha(P, R_{(0,1,1)}) &= \max\{k_1, k_2, k_3\} + \lambda(\alpha)\min\{k_1+k_2, k_1+k_3, k_2+k_3\}, \text{ where } k_i = \left|\rho_i - \tfrac{u_i(\rho_2+\rho_3)}{u_2+u_3}\right| \\
d_\alpha(P, R_{(0,-1,1)}) &= \max\{k_1, k_2, k_3\} + \lambda(\alpha)\min\{k_1+k_2, k_1+k_3, k_2+k_3\}, \text{ where } k_i = \left|\rho_i - \tfrac{u_i(-\rho_2+\rho_3)}{-u_2+u_3}\right|
\end{aligned}$$

Therefore, we get

$$d_\alpha(P, l) = \min_{\mathbf{v} \in V_3}\left\{\max\left\{\left|\rho_i - \frac{u_i\langle\rho, \mathbf{v}\rangle}{\langle\mathbf{u}, \mathbf{v}\rangle}\right|\right\} + \lambda(\alpha)\min\left\{\left|\rho_j - \frac{u_j\langle\rho, \mathbf{v}\rangle}{\langle\mathbf{u}, \mathbf{v}\rangle}\right| + \left|\rho_k - \frac{u_k\langle\rho, \mathbf{v}\rangle}{\langle\mathbf{u}, \mathbf{v}\rangle}\right|\right\}\right\},$$

where $\rho = (\rho_1, \rho_2, \rho_3) = (x_1 - x_2, y_1 - y_2, z_1 - z_2)$, and $i, j, k \in \{1, 2, 3\}$ for $i \neq j \neq k \neq i$. Other conditions do not change the result. □

The α-distance between two skew lines in three dimensional space can be determined by the following proposition:

Proposition 3.6 *Let*

$$\begin{aligned}
l_1 &: \beta_1(t) = (x_1, y_1, z_1) + t(u_1, u_2, u_3), \\
l_2 &: \beta_2(t) = (x_2, y_2, z_2) + t(v_1, v_2, v_3)
\end{aligned}$$

be two skew lines. Then the α*-distance between* l_1 *and* l_2 *is*

$$d_\alpha(l_1, l_2) = \frac{\left|(x_1 - x_2)\mu_{(2,3)} + (y_1 - y_2)\mu_{(3,1)} + (z_1 - z_2)\mu_{(1,2)}\right|}{\max\left\{\left|\mu_{(2,3)}/\lambda_1\right|, \left|\mu_{(3,1)}/\lambda_2\right|, \left|\mu_{(1,2)}/\lambda_3\right|\right\}} \tag{12}$$

with $\mu_{(m,n)} = u_m v_n - u_n v_m$.

Proof Since the lines l_1 and l_2 are skew, there is only one plane Ω through l_2, parallel to l_1. Then we have

$$d_\alpha(l_1, \Omega) = d_\alpha(P_1, \Omega)$$

for any point P_1 on l_1. Thus we get

$$d_\alpha(l_1, l_2) = d_\alpha(P_1, \Omega)$$

since there is an α-sphere whose center at l_1 and radius $d_\alpha(P_1, \Omega)$, that is tangent to l_2. So, since

$$\langle P_2X, (u_1, u_2, u_3) \times (v_1, v_2, v_3) \rangle = 0$$

for $X = (x, y, z)$ and $P_2 = (x_2, y_2, z_2)$ on Ω, we get

$$(x - x_2)\mu_{(2,3)} + (y - y_2)\mu_{(3,1)} + (z - z_2)\mu_{(1,2)} = 0,$$

where $\mu_{(m,n)} = u_m v_n - u_n v_m$, for the equation of the plane Ω. Therefore, by Proposition 3.3, one gets

$$d_\alpha(l_1, l_2) = d_\alpha(P_1, \Omega) = \frac{\left|(x_1 - x_2)\mu_{(2,3)} + (y_1 - y_2)\mu_{(3,1)} + (z_1 - z_2)\mu_{(1,2)}\right|}{\max\left\{|K|, \frac{|K \mp L|}{1+\lambda(\alpha)}, \frac{|K+L\mp M|}{1+2\lambda(\alpha)}\right\}}$$

with $K, L, M \in \{\mu_{(1,2)}, \mu_{(3,1)}, \mu_{(2,3)}\}$ and $K \neq L \neq M \neq K$. □

Clearly, the distance formulae derived here give also the taxicab and Chinese checker distance formulae when $\alpha = 0$ and $\alpha = \frac{\pi}{4}$, respectively (see [4, 10]).

4. Area and Volume in Terms of the Alpha Distance

Here, we give an alpha version of the area and volume formulas in terms of the alpha distance using the following equation which relates the Euclidean distance to the alpha distance between two points in the three dimensional space.

Proposition 4.1 *For any two points P_1 and P_2 in $\mathbb{R}^3$, if $\mathbf{u} = (u_1, u_2, u_3)$ is a direction vector of the line through P_1 and P_2, then*

$$d_E(P_1, P_2) = \rho(\mathbf{u}) d_\alpha(P_1, P_2), \tag{13}$$

where $\rho(\mathbf{u}) = (u_1^2 + u_2^2 + u_3^2)^{1/2} / (\max\{|u_1|, |u_2|, |u_3|\} + \lambda(\alpha)\min\{|u_1 + u_2|, |u_1 + u_3|, |u_2 + u_3|\})$.

Proof Let $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$. Then $\mathbf{u} = k(x_1 - x_2, y_1 - y_2, z_1 - z_2)$ for some $k \in \mathbb{R}^*$. Since

$$\frac{d_E(P_1, P_2)}{d_\alpha(P_1, P_2)} = \frac{\|\mathbf{u}\|_E}{\|\mathbf{u}\|_\alpha},$$

we have

$$d_E(P_1, P_2) = \rho(\mathbf{u}) d_\alpha(P_1, P_2),$$

where

$$\rho(\mathbf{u})=\frac{\|\mathbf{u}\|_E}{\|\mathbf{u}\|_\alpha}=\frac{(u_1^2+u_2^2+u_3^2)^{1/2}}{\max\{|u_1|,|u_2|,|u_3|\}+\lambda(\alpha)\min\{|u_1+u_2|,|u_1+u_3|,|u_2+u_3|\}}.$$

□

The following corollaries gives alpha versions of the standard area and volume formulas in terms of alpha distances. The proofs are straightforward.

Corollary 4.2 *Let PQR be a triangle with the area $\mathcal{A}$ in the three dimensional alpha space, and let $a=d_\alpha(Q,R)$ and $h=d_\alpha(P,H)$, where H is the Euclidean orthogonal projection of the point P on the line QR. If $\mathbf{u}$ and $\mathbf{v}$ are direction vectors of the lines QR and PH, respectively, then*

$$\mathcal{A}=ah/2\rho(\mathbf{u})\rho(\mathbf{v}).$$

Corollary 4.3 *Let $PQRS$ be a tetrahedron having the base QRS in the plane $Ax+By+Cz+D=0$, and let $h=d_\alpha(P,H)$, where H is the Euclidean orthogonal projection of the point P to the base. If the area of the triangle QRS is $\mathcal{A}$, then the volume of the tetrahedron is*

$$\mathcal{V}=\mathcal{A}h/3\rho(A,B,C).$$

Declaration of Ethical Standards

The author declares that the materials and methods used in his study do not require ethical committee and/or legal special permission.

Conflicts of Interest

The author declares no conflict of interest.

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