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BIPOLAR SOFT CONTINUITY ON BIPOLAR SOFT TOPOLOGICAL SPACES

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Abstract. The amazing idea of soft sets was first claimed by Molodtsov [18], a new mathematical tool for dealing with uncertainties free from the other theories' limitations. After the advent of soft set theory, bipolar soft sets, as a generalization of soft sets, a new model of uncertain information, were introduced by Shabir and Naz [21]. The primary purpose of this paper is to introduce and investigate the structures of bipolar soft continuity, bipolar soft openness, bipolar soft closedness and bipolar soft homeomorphism.

1. Introduction

Mainly since the problems in many vital areas of our lives, such as economics, environment, and engineering, cannot be solved due to the inherent difficulties of conventional methods, many theories have been put forward to combat these problems. In 1999, Molodtsov [18] introduced a practical theory called soft set theory which is free from the other theories. The amazing idea of soft sets was given as a new mathematical tool for dealing with uncertainties free from the other theories' limitations. At present, many studies on soft set theory have been carried out different areas by some researchers [1, 5, 6, 9, 13, 16, 2, 3].

After the advent of soft set theory, bipolar soft sets, a new model of uncertain information, were introduced by Shabir and Naz [21]. It is known that the structure of a bipolar soft set consists of two mappings. Since these mappings explain both positive information and opposite approximation, the idea of the bipolar soft set has recently gained momentum among many researchers. Aslam et al. [4] combined the concept of a bipolar fuzzy set and a soft set. In addition, they introduced the notion of bipolar fuzzy soft set and studied fundamental properties. Naz and Shabir [22] gave algebraic structures of bipolar fuzzy soft sets. Hayat et al. [14] applied the concept of bipolar soft sets to hemirings. Karaaslan and Karataş [15] redefined the idea of bipolar soft set and bipolar soft set operations as more functional than Shabir

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and Naz's definition and operations. In 2017, Shabir and Bakhtawar [23] employed the notion of bipolar soft sets to give the concept of bipolar soft topological spaces, which are extensions of soft topological space. They gave some new important structures in bipolar soft topological spaces, such as bipolar soft connected spaces, bipolar soft disconnected spaces and bipolar soft compact spaces. Since the concept of bipolar soft topology has great importance, the topological structures of bipolar soft sets have been studied by many authors [7, 19, 20]. Gunduz et al. [10] recently defined a new bipolar soft point. By using the bipolar soft point, Gunduz et al. [12] examined some important properties of bipolar soft functions.

Since functions are one of the most critical concepts in mathematics, they have many applications. Many studies have been done on soft functions in soft topological spaces in [11, 17]. The concept of bipolar soft functions was defined in [8] as the generalization of soft functions and given the notion of bipolar soft image and inverse image.

The primary purpose of this paper is to introduce and investigate the structures of bipolar soft continuity, bipolar soft openness, bipolar soft closedness and bipolar soft homeomorphism and show these examples.

2. **Preliminaries**

Throughout this section, the symbols U, $\widetilde{E} = E \cup \neg E$ and $P(U)$ denote the initial universe, a set of parameters, and the power set of U, respectively.

Definition 2.1. [16] *Let $E = \{e_i : i = 1, 2, ..., n\}$ be a set of parameters. The not set of E, denoted by $\neg E$, is defined by $\neg E = \{\neg e_i : i = 1, 2, ..., n\}$, where $\neg e_i =$ not e_i for all i.*

Definition 2.2. [21] *A bipolar soft set $\left(F, \widetilde{E}\right)$ on U is defined as*

$$F_{\widetilde{E}} = \{(e_i, F(e_i), F(\neg e_i)) : e_i \in E\},$$

where $F : \widetilde{E} \to P(U)$ such that $F(e) \cap F(\neg e) = \varnothing$, for each $e \in E$.

In this paper, a bipolar soft set denoted by $F_{\widetilde{E}}$ instead of $\left(F, \widetilde{E}\right)$. The collection of all bipolar soft sets on U is denoted by $BS\left(U_{\widetilde{E}}\right)$.

Definition 2.3. [21] *Let $F_{\widetilde{E}}, G_{\widetilde{E}} \in BS\left(U_{\widetilde{E}}\right)$. Then,*

1. $F_{\widetilde{E}} \widetilde{\subseteq} G_{\widetilde{E}}$, if $F(e) \subset G(e)$ and $G(\neg e) \subset F(\neg e)$, for each $e \in E$.
2. $F_{\widetilde{E}} = G_{\widetilde{E}}$, if $F_{\widetilde{E}} \widetilde{\subseteq} G_{\widetilde{E}}$ and $G_{\widetilde{E}} \widetilde{\subseteq} F_{\widetilde{E}}$.
3. $F_{\widetilde{E}} \widetilde{\cup} G_{\widetilde{E}} = H_{\widetilde{E}}$ where $H(e) = (F \cup G)(e) = F(e) \cup G(e)$ and $H(\neg e) = (F \cup G)(\neg e) = F(\neg e) \cap G(\neg e)$, for each $e \in E$.
4. $F_{\widetilde{E}} \widetilde{\cap} G_{\widetilde{E}} = Z_{\widetilde{E}}$ where $Z(e) = (F \cap G)(e) = F(e) \cap G(e)$ and $Z(\neg e) = (F \cap G)(\neg e) = F(\neg e) \cup G(\neg e)$, for each $e \in E$.

Definition 2.4. [21] *The bipolar soft complement of $F_{\widetilde{E}}$, denoted by $F^c_{\widetilde{E}}$, where $F^c : \widetilde{E} \to P(U)$ is a mapping defined by $F^c(e) = F(\neg e)$ and $F^c(\neg e) = F(e)$, for each $e \in E$.*

Definition 2.5. [21] *If $F(e) = \varnothing$ and $F(\neg e) = U$ for each $e \in E$,$\Phi_{\widetilde{E}}$ is called a null bipolar soft set. Also, if $F(e) = U$ and $F(\neg e) = \varnothing$ for each $e \in E$, $\widetilde{U}_{\widetilde{E}}$ is called an absolute bipolar soft set.*

Symbolically, $\Phi_{\widetilde{E}} = \{(e_i, \varnothing, U) : e_i \in E\}$ and $\widetilde{U}_{\widetilde{E}} = \{(e_i, U, \varnothing) : e_i \in E\}$.

Definition 2.6. [23] *Let $\widetilde{\tau} \subset BS\left(U_{\widetilde{E}}\right)$. $\widetilde{\tau}$ is said to be a bipolar soft topology on U, if $\widetilde{\tau}$ confirms the following conditions:*
(1) $\Phi_{\widetilde{E}}, \widetilde{U}_{\widetilde{E}} \in \widetilde{\tau}$.
(2) The union of any number of bipolar soft sets in $\widetilde{\tau}$ belongs to $\widetilde{\tau}$.
(3) The intersection of any two bipolar soft sets in $\widetilde{\tau}$ belongs to $\widetilde{\tau}$.
$\left(U, \widetilde{\tau}, \widetilde{E}\right)$ is called bipolar soft topological space and the family of all bipolar soft topological space on U denoted as $BSTS$.

Definition 2.7. [23] *Let $\left(U, \widetilde{\tau}, \widetilde{E}\right)$ be a bipolar soft topological space on U and $F_{\widetilde{E}} \in BS\left(U_{\widetilde{E}}\right)$. $F_{\widetilde{E}}$ is called*

(1) a bipolar soft open set, if it belongs to $\widetilde{\tau}$.
(2) a bipolar soft closed set, if $F^c_{\widetilde{E}}$ belongs to $\widetilde{\tau}$.

Definition 2.8. [7] *Let $\left(U, \widetilde{\tau}, \widetilde{E}\right)$ be a $BSTS$ and $F_{\widetilde{E}} \in BS\left(U_{\widetilde{E}}\right)$. Then, the bipolar soft interior of $F_{\widetilde{E}}$, denoted by $F^{\circ}_{\widetilde{E}}$, is the union of all bipolar soft open subsets of $F_{\widetilde{E}}$.*

Definition 2.9. [7] *Let $\left(U, \widetilde{\tau}, \widetilde{E}\right)$ be a $BSTS$ and $F_{\widetilde{E}} \in BS\left(U_{\widetilde{E}}\right)$. Then, the bipolar soft closure of $F_{\widetilde{E}}$, denoted by $\overline{F_{\widetilde{E}}}$, is the intersection of all bipolar soft closed sets containing $F_{\widetilde{E}}$.*

Theorem 2.1. [7] *Let $\left(U, \widetilde{\tau}, \widetilde{E}\right)$ be a $BSTS$ and $F_{\widetilde{E}} \in BS\left(U_{\widetilde{E}}\right)$. Then, $\left[\overline{F_{\widetilde{E}}}\right]^c = \left(F^c_{\widetilde{E}}\right)^{\circ}$.*

Theorem 2.2. [7] *Let $\left(U, \widetilde{\tau}, \widetilde{E}\right)$ be a $BSTS$ and $F_{\widetilde{E}} \in BS\left(U_{\widetilde{E}}\right)$. Then, $\overline{\left[F^c_{\widetilde{E}}\right]} = \left[F^{\circ}_{\widetilde{E}}\right]^c$.*

3. Continuous functions on bipolar soft topological spaces

We mention, in this section, some important concepts, such as bipolar soft image and bipolar soft pre-image for the bipolar soft sets on U whose set of parameters is a subset of E. In addition, we recall the relationship between the image and the inverse image of bipolar soft sets. This is followed by the definition of bipolar soft continuous function associated with some of its results. Later, we will give a detailed investigation of bipolar soft continuous functions.

E and E', respectively, stand for the sets of parameters of U and V; $\varnothing \neq E_1, E_2, E_3 \subset E$ and $\varnothing \neq E'_1, E'_2 \subset E'$.

Definition 3.1. [8] *Let $f : U \to V$ be an injective function, $\varphi : E \to E'$ and $\vartheta : \neg E \to \neg E'$ be two functions where $\vartheta(\neg e) = \neg\varphi(e)$, for all $\neg e \in \neg E$. Then, $\psi_{f\varphi\vartheta} : BS\left(U_{\widetilde{E}}\right) \to BS\left(V_{\widetilde{E'}}\right)$ is called a bipolar soft function.*

Definition 3.2. [8] *Let $\psi_{f\varphi\vartheta} : BS\left(U_{\widetilde{E}}\right) \to BS\left(V_{\widetilde{E'}}\right)$ be a bipolar soft function and $F_{\widetilde{E}} \in BS\left(U_{\widetilde{E}}\right)$. Then, the image of $F_{\widetilde{E}}$ under $\psi_{f\varphi\vartheta}$,*

$$\psi_{f\varphi\vartheta}\left(F_{\widetilde{E}}\right) = \left(\psi_{f\varphi\vartheta}\left(F(e)\right), \psi_{f\varphi\vartheta}\left(F(\neg e)\right), E'\right)$$

is defined as follows: for all $e' \in E'$,

$$\psi_{f\varphi\vartheta}(F)(e') = \begin{cases} f\left(\underset{e\in\ \varphi^{-1}(e')\cap E_1}{\cup} F(e)\right), & \text{if } \varphi^{-1}(e')\cap E_1 \neq \varnothing, \\ \varnothing, & \text{otherwise.} \end{cases}$$

and

$$\psi_{f\varphi\vartheta}(F)(\neg e') = \begin{cases} f\left(\underset{\neg e\in\ \vartheta^{-1}(\neg e')\cap\neg E_1}{\cap} F(\neg e)\right), & \text{if } \vartheta^{-1}(\neg e')\cap\neg E_1 \neq \varnothing, \\ V, & \text{otherwise.} \end{cases}$$

Remark. *The condition that the function f is an injective function is essential.*

Proposition 3.1. *Let $\psi_{f\varphi\vartheta} : BS\left(U_{\tilde{E}}\right) \to BS\left(V_{\widetilde{E'}}\right)$ be a bipolar soft function and $F_{\widetilde{E}} \in BS\left(U_{\tilde{E}}\right)$. Then, $\psi_{f\varphi\vartheta}\left(F_{\widetilde{E}}\right)$ is a bipolar soft set in $BS\left(V_{\widetilde{E'}}\right)$.*

Proof. For all $e' \in E'$,

$$\begin{aligned} \psi_{f\varphi\vartheta}(F)(e') \cap \psi_{f\varphi\vartheta}(F)(\neg e') &= f\left(\underset{e\in\ \varphi^{-1}(e')\cap E}{\cup} F(e)\right) \cap f\left(\underset{\neg e\in\ \vartheta^{-1}(\neg e')\cap\neg E}{\cap} F(\neg e)\right) \\ &= \varnothing \end{aligned}$$

Then, $\psi_{f\varphi\vartheta}\left(F_{\widetilde{E}}\right)$ is a bipolar soft set in $BS\left(V_{\widetilde{E'}}\right)$. □

Definition 3.3. [8] *Let $\psi_{f\varphi\vartheta} : BS\left(U_{\tilde{E}}\right) \to BS\left(V_{\widetilde{E'}}\right)$ be a bipolar soft function. Then,*

(1) If f and φ are surjective functions, then $\psi_{f\varphi\vartheta}$ is called a bipolar soft surjective function.
(2) If f and φ are injective functions, then $\psi_{f\varphi\vartheta}$ is called a bipolar soft injective function.
(3) If f and φ are bijective functions, then $\psi_{f\varphi\vartheta}$ is called a bipolar soft bijective function.

Remark. [8] *It is clear that $\psi_{f\varphi\vartheta}$ is a bipolar soft surjective if and only if $\psi_{f\varphi\vartheta}\left(\widetilde{U}_{\widetilde{E}}\right) = \widetilde{V}_{\widetilde{E'}}$.*

Remark. [8] *If the bipolar soft sets have the same set of parameters, for each $F_{\widetilde{E_1}}, G_{\widetilde{E_1}} \in BS\left(U_{\tilde{E}}\right)$, when $\psi_{f\varphi\vartheta}\left(F_{\widetilde{E_1}}\right) = \psi_{f\varphi\vartheta}\left(G_{\widetilde{E_1}}\right)$, we obtain $F_{\widetilde{E_1}} = G_{\widetilde{E_1}}$, i.e. $\psi_{f\varphi\vartheta}$ is a bipolar soft injective function.*

Theorem 3.2. [8] *Let $\psi_{f\varphi\vartheta} : BS\left(U_{\tilde{E}}\right) \to BS\left(V_{\widetilde{E'}}\right)$ be a bipolar soft function. If $F_{\widetilde{E_1}}, G_{\widetilde{E_2}} \in BS\left(U_{\tilde{E}}\right)$, then*

(1) $\psi_{f\varphi\vartheta}\left(\Phi_{\widetilde{E}}\right) \widetilde{\supseteq} \left(\Phi_{\widetilde{E'}}\right)$. If f is surjective, then the equality holds.
(2) $\psi_{f\varphi\vartheta}\left(\widetilde{U}_{\widetilde{E}}\right) \widetilde{\subseteq} \left(\widetilde{V}_{\widetilde{E'}}\right)$.
(3) $F_{\widetilde{E_1}} \widetilde{\subseteq} G_{\widetilde{E_2}} \Rightarrow \psi_{f\varphi\vartheta}\left(F_{\widetilde{E_1}}\right) \subseteq \psi_{f\varphi\vartheta}\left(G_{\widetilde{E_2}}\right)$.
(4) $\psi_{f\varphi\vartheta}\left(F_{\widetilde{E_1}} \widetilde{\cup} G_{\widetilde{E_2}}\right) = \psi_{f\varphi\vartheta}\left(F_{\widetilde{E_1}}\right) \widetilde{\cup}\, \psi_{f\varphi\vartheta}\left(G_{\widetilde{E_2}}\right)$.
(5) $\psi_{f\varphi\vartheta}\left(F_{\widetilde{E_1}} \widetilde{\cap} G_{\widetilde{E_2}}\right) \subseteq \psi_{f\varphi\vartheta}\left(F_{\widetilde{E_1}}\right) \widetilde{\cap}\, \psi_{f\varphi\vartheta}\left(G_{\widetilde{E_2}}\right)$. If $\psi_{f\varphi\vartheta}$ is a bipolar soft injective function, then the equality holds.

Definition 3.4. [8] *Let* $\psi_{f\varphi\vartheta} : BS\left(U_{\widetilde{E}}\right) \to BS\left(V_{\widetilde{E'}}\right)$ *be a bipolar soft function. The inverse image of the bipolar soft set* $H_{\widetilde{E'_1}} \in BS\left(V_{\widetilde{E'}}\right)$ *under* $\psi_{f\varphi\vartheta}$,

$$\psi^{-1}_{f\varphi\vartheta}\left(H_{\widetilde{E'_1}}\right) = \left(\psi^{-1}_{f\varphi\vartheta}\left(H\left(e\right)\right), \psi^{-1}_{f\varphi\vartheta}\left(H\left(\neg e\right)\right), E\right)$$

is given as follows: for all $e \in E$,

$$\begin{aligned}
\psi^{-1}_{f\varphi\vartheta}\left(H\left(e\right)\right) &= \begin{cases} f^{-1}\left(H\left(\varphi\left(e\right)\right)\right), & \text{if } \varphi\left(e\right) \in E'_1, \\ \varnothing, & \text{if } \varphi\left(e\right) \notin E'_1. \end{cases} \\
\psi^{-1}_{f\varphi\vartheta}\left(H\left(\neg e\right)\right) &= \begin{cases} f^{-1}\left(H\left(\vartheta\left(\neg e\right)\right)\right), & \text{if } \vartheta\left(\neg e\right) \in \neg E'_1, \\ U, & \text{if } \vartheta\left(\neg e\right) \notin \neg E'_1. \end{cases}
\end{aligned}$$

Proposition 3.3. *Let* $\psi_{f\varphi\vartheta} : BS\left(U_{\widetilde{E}}\right) \to BS\left(V_{\widetilde{E'}}\right)$ *be a bipolar soft function and* $H_{\widetilde{E'_1}} \in BS\left(V_{\widetilde{E'}}\right)$. *Then,* $\psi^{-1}_{f\varphi\vartheta}\left(H_{\widetilde{E'_1}}\right)$ *is a bipolar soft set in* $BS\left(U_{\widetilde{E}}\right)$.

Proof. For all $e \in E$,

$$\begin{aligned}
\psi^{-1}_{f\varphi\vartheta}\left(H\left(e\right)\right) \cap \psi^{-1}_{f\varphi\vartheta}\left(H\left(\neg e\right)\right) &= f^{-1}\left(H\left(\varphi\left(e\right)\right)\right) \cap f^{-1}\left(H\left(\vartheta\left(\neg e\right)\right)\right) \\
&= f^{-1}\left(\left(H\left(\varphi\left(e\right)\right) \cap H\left(\vartheta\left(\neg e\right)\right)\right)\right) \\
&= f^{-1}\left(\varnothing\right) \\
&= \varnothing
\end{aligned}$$

Thus, $\psi^{-1}_{f\varphi\vartheta}\left(H_{\widetilde{E'_1}}\right)$ is a bipolar soft set in $BS\left(U_{\widetilde{E}}\right)$. ☐

Remark. *Although the image of the inverse image of a set for any classical function is a subset of this set, this is not true in bipolar soft functions. A condition must be added to enable this property.*

Theorem 3.4. [8] *Let* $\psi_{f\varphi\vartheta} : BS\left(U_{\widetilde{E}}\right) \to BS\left(V_{\widetilde{E'}}\right)$ *be a bipolar soft function. If* $H_{\widetilde{E'_1}}, Q_{\widetilde{E'_2}} \in BS\left(V_{\widetilde{E'}}\right)$, *then*

(1) $\psi^{-1}_{f\varphi\vartheta}\left(\widetilde{V}_{\widetilde{E'}}\right) = \widetilde{U}_{\widetilde{E}}$.
(2) $\psi^{-1}_{f\varphi\vartheta}\left(\Phi_{\widetilde{E'}}\right) = \Phi_{\widetilde{E}}$.
(3) $H_{\widetilde{E'_1}} \widetilde{\subseteq} Q_{\widetilde{E'_2}} \Rightarrow \psi^{-1}_{f\varphi\vartheta}\left(H_{\widetilde{E'_1}}\right) \widetilde{\subseteq} \psi^{-1}_{f\varphi\vartheta}\left(Q_{\widetilde{E'_2}}\right)$.
(4) $\psi^{-1}_{f\varphi\vartheta}\left(H_{\widetilde{E'_1}} \widetilde{\cup} Q_{\widetilde{E'_2}}\right) = \psi^{-1}_{f\varphi\vartheta}\left(H_{\widetilde{E'_1}}\right) \widetilde{\cup} \psi^{-1}_{f\varphi\vartheta}\left(Q_{\widetilde{E'_2}}\right)$.
(5) $\psi^{-1}_{f\varphi\vartheta}\left(H_{\widetilde{E'_1}} \widetilde{\cap} Q_{\widetilde{E'_2}}\right) = \psi^{-1}_{f\varphi\vartheta}\left(H_{\widetilde{E'_1}}\right) \widetilde{\cap} \psi^{-1}_{f\varphi\vartheta}\left(Q_{\widetilde{E'_2}}\right)$.
(6) $\psi^{-1}_{f\varphi\vartheta}\left(H^c_{\widetilde{E'_1}}\right) = \left(\psi^{-1}_{f\varphi\vartheta}\left(H_{\widetilde{E'_1}}\right)\right)^c$.

Now, we consider the relationships between the image and inverse image of bipolar soft sets.

Theorem 3.5. [8] *Let* $\psi_{f\varphi\vartheta} : BS\left(U_{\widetilde{E}}\right) \to BS\left(V_{\widetilde{E'}}\right)$ *be a bipolar soft function,* $F_{\widetilde{E_1}} \in BS\left(U_{\widetilde{E}}\right)$ *and* $H_{\widetilde{E'}} \in BS\left(V_{\widetilde{E'}}\right)$. *Then,*

(1) $F_{\widetilde{E_1}} \widetilde{\subseteq}\ \psi^{-1}_{f\varphi\vartheta}\left(\psi_{f\varphi\vartheta}\left(F_{\widetilde{E_1}}\right)\right)$. If $E_1 = E$ and $\psi_{f\varphi\phi}$ is a bipolar soft injective function, then the equality holds.

(2) If f is a surjective function, then $\psi_{f\varphi\vartheta}\left(\psi^{-1}_{f\varphi\vartheta}H_{\widetilde{E'}}\right)\widetilde{\subseteq}H_{\widetilde{E'}}$. If $\psi_{f\varphi\vartheta}$ is a bipolar soft surjective function, the equality holds.

Now, we consider the concept of bipolar soft point followed by some relations between them.

Definition 3.5. [10] *A bipolar soft subset $F_{\widetilde{E}}$ of $\widetilde{U}_{\widetilde{E}}$ is called a bipolar soft point if there exist $x, y \in U$,($x \neq y$ need not be true) $e \in E$ satisfying*

$$x_e^y(e_1) = \begin{cases} \varnothing, & \text{if } e \neq e_1, \\ \{x\}, & \text{if } e = e_1. \end{cases}$$

and

$$x_e^y(\neg e_1) = \begin{cases} U, & \text{if } e \neq e_1, \\ U - \{x, y\}, & \text{if } e = e_1. \end{cases}$$

The bipolar soft point will be shortly denoted by x_e^y.

Definition 3.6. *Let x_e^y and $x_{1e_1}^{y_1}$ be two bipolar soft points over U. Then, x_e^y and $x_{1e_1}^{y_1}$ are called different bipolar soft points, if $x \neq x_1$ or $e \neq e_1$.*

Definition 3.7. [20] *Let x_e^y be a bipolar soft point and $F_{\widetilde{E}} \in BS\left(U_{\widetilde{E}}\right)$. We said that x_e^y belongs to the bipolar soft set $F_{\widetilde{E}}$, denoted by $x_e^y \widetilde{\in} F_{\widetilde{E}}$, if $x_e^y(e) = \{x\} \subset F(e)$ and $x_e^y(\neg e) \supset F(\neg e)$.*

Remark. *Every bipolar soft set can be written as a union of its bipolar soft points.*

Example 3.1. *Let $U = \{x_1, x_2, x_3, x_4\}$, $E = \{e_1, e_2\}$ and*

$$F_{\widetilde{E}} = \{(e_1, \{x_1, x_2\}, \{x_4\}), (e_2, \{x_2, x_3\}, \{x_1, x_4\})\}.$$

Then, we can write $F_{\widetilde{E}}$ as a union of some bipolar soft points. Indeed, for $e_1, e_2 \in E$,

$$\begin{aligned} F(e_1) &= \left(x_{1e_1}^{x_2} \cup x_{1e_1}^{x_3} \cup x_{2e_1}^{x_1} \cup x_{2e_1}^{x_3}\right)(e_1), \\ F(\neg e_1) &= \left(x_{1e_1}^{x_2} \cap x_{1e_1}^{x_3} \cap x_{2e_1}^{x_1} \cap x_{2e_1}^{x_3}\right)(\neg e_1), \\ F(e_2) &= \left(x_{2e_2}^{x_3} \cup x_{3e_2}^{x_2}\right)(e_2), \\ F(\neg e_2) &= \left(x_{2e_2}^{x_3} \cap x_{3e_2}^{x_2}\right)(\neg e_2), \end{aligned}$$

where

$$\begin{aligned} x_{1e_1}^{x_2}(e_1) &= \{x_1\},\ x_{1e_1}^{x_2}(\neg e_1) = \{x_3, x_4\}, \\ x_{1e_1}^{x_3}(e_1) &= \{x_1\},\ x_{1e_1}^{x_3}(\neg e_1) = \{x_2, x_4\}, \\ x_{2e_1}^{x_1}(e_1) &= \{x_2\},\ x_{2e_1}^{x_1}(\neg e_1) = \{x_3, x_4\}, \\ x_{2e_1}^{x_3}(e_1) &= \{x_2\},\ x_{2e_1}^{x_3}(\neg e_1) = \{x_1, x_4\}, \\ x_{2e_2}^{x_3}(e_2) &= \{x_2\},\ x_{2e_2}^{x_3}(\neg e_2) = \{x_1, x_4\}, \\ x_{3e_2}^{x_2}(e_2) &= \{x_3\},\ x_{3e_2}^{x_2}(\neg e_2) = \{x_1, x_4\}. \end{aligned}$$

Definition 3.8. [20] *Let $\left(U, \widetilde{\tau}, \widetilde{E}\right)$ be a $BSTS$, $F_{\widetilde{E}} \in BS\left(U_{\widetilde{E}}\right)$ and x_e^y be a bipolar soft point in U. Then, $F_{\widetilde{E}}$ is said to be a bipolar soft neighbourhood of x_e^y, if there exists a bipolar soft open set $G_{\widetilde{E}}$ such that $x_e^y \widetilde{\in} G_{\widetilde{E}} \widetilde{\subseteq} F_{\widetilde{E}}$.*

Definition 3.9. *Let* $\left(U,\widetilde{\tau},\widetilde{E}\right)$ *and* $\left(V,\widetilde{\tau'},\widetilde{E'}\right)$ *be two bipolar soft topological spaces over* U *and* V*, respectively and* $\psi_{f\varphi\vartheta}:\left(U,\widetilde{\tau},\widetilde{E}\right)\to\left(V,\widetilde{\tau'},\widetilde{E'}\right)$ *be a function. For each bipolar soft neighbourhood* $H_{\widetilde{E'}}$ *of* $\psi_{f\varphi\vartheta}\left(x_e^y\right)$*, if there exists a bipolar soft neighbourhood* $F_{\widetilde{E}}$ *of* x_e^y *such that* $\psi_{f\varphi\vartheta}\left(F_{\widetilde{E}}\right)\widetilde{\subseteq}H_{\widetilde{E'}}$*, then* $\psi_{f\varphi\vartheta}$ *is called a bipolar soft continuous function at* x_e^y*.*

Moreover, $\psi_{f\varphi\vartheta}$ is called bipolar soft continuous function on U if $\psi_{f\varphi\vartheta}$ is a bipolar soft continuous function for all x_e^y.

Theorem 3.6. *Let* $\left(U,\widetilde{\tau},\widetilde{E}\right)$ *and* $\left(V,\widetilde{\tau'},\widetilde{E'}\right)$ *be two bipolar soft topological spaces over* U *and* V*, respectively and* $\psi_{f\varphi\vartheta}:\left(U,\widetilde{\tau},\widetilde{E}\right)\to\left(V,\widetilde{\tau'},\widetilde{E'}\right)$ *be a mapping. Then, the following conditions are equivalent:*

(1) $\psi_{f\varphi\vartheta}:\left(U,\widetilde{\tau},\widetilde{E}\right)\to\left(V,\widetilde{\tau'},\widetilde{E'}\right)$ is a bipolar soft continuous function,
(2) For each $G_{\widetilde{E'}}\in\widetilde{\tau'}$, $\psi_{f\varphi\vartheta}^{-1}\left(G_{\widetilde{E'}}\right)\in\widetilde{\tau}$,
(3) For each bipolar soft closed set $H_{\widetilde{E'}}$ over V, $\psi_{f\varphi\vartheta}^{-1}\left(H_{\widetilde{E'}}\right)$ is a bipolar soft closed set over U,
(4) For each $F_{\widetilde{E}}\in BS\left(U_{\widetilde{E}}\right)$, $\psi_{f\varphi\vartheta}\left(\overline{F_{\widetilde{E}}}\right)\widetilde{\subseteq}\overline{\psi_{f\varphi\vartheta}\left(F_{\widetilde{E}}\right)}$,
(5) For each $D_{\widetilde{E'}}\in BS\left(V_{\widetilde{E'}}\right)$, $\overline{\psi_{f\varphi\vartheta}^{-1}\left(D_{\widetilde{E'}}\right)}\widetilde{\subseteq}\ \psi_{f\varphi\vartheta}^{-1}\left(\overline{D_{\widetilde{E'}}}\right)$,
(6) For each $D_{\widetilde{E'}}\in BS\left(V_{\widetilde{E'}}\right)$, $\psi_{f\varphi\vartheta}^{-1}\left(D_{\widetilde{E'}}^{\circ}\right)\widetilde{\subseteq}\left(\psi_{f\varphi\vartheta}\left(D_{\widetilde{E'}}\right)\right)^{\circ}$.

Proof. (1) ⇒ (2) Let $G_{\widetilde{E'}}\in\widetilde{\tau'}$ and $x_e^y\widetilde{\in}\psi_{f\varphi\vartheta}^{-1}\left(G_{\widetilde{E'}}\right)$. Then, $\psi_{f\varphi\vartheta}\left(x_e^y\right)\widetilde{\in}G_{\widetilde{E'}}$. Since $\psi_{f\varphi\vartheta}:\left(U,\widetilde{\tau},\widetilde{E}\right)\to\left(V,\widetilde{\tau'},\widetilde{E'}\right)$ is a bipolar soft continuous mapping, there is $x_e^y\widetilde{\in}F_{\widetilde{E}}\widetilde{\in}\widetilde{\tau}$ such that $\left(\psi_{f\varphi\vartheta}\right)\left(F_{\widetilde{E}}\right)\widetilde{\subseteq}G_{\widetilde{E'}}$. Hence, $x_e^y\ \widetilde{\in}F_{\widetilde{E}}\widetilde{\subseteq}\psi_{f\varphi\vartheta}^{-1}\left(G_{\widetilde{E'}}\right)$. This implies that $\psi_{f\varphi\vartheta}^{-1}\left(G_{\widetilde{E'}}\right)$ is a bipolar soft open set over U.

(2) ⇒ (1) Let x_e^y be any bipolar soft point and $\left(\psi_{f\varphi\vartheta}\right)\left(x_e^y\right)\widetilde{\in}G_{\widetilde{E'}}$ be an arbitrary bipolar soft neighbourhood of $\psi_{f\varphi\vartheta}\left(x_e^y\right)$. Then, $x_e^y\ \widetilde{\in}\psi_{f\varphi\vartheta}^{-1}\left(G_{\widetilde{E'}}\right)$ is a bipolar soft neighbourhood and $\left(\psi_{f\varphi\vartheta}\right)\left(\psi_{f\varphi\vartheta}^{-1}\left(G_{\widetilde{E'}}\right)\right)\widetilde{\subseteq}G_{\widetilde{E'}}$.

(2) ⇒ (3) From the definition of complement of bipolar soft set, it is obtained.

(3) ⇒ (4) Let $F_{\widetilde{E}}\in BS\left(U_{\widetilde{E}}\right)$. Since

$$\left(\psi_{f\varphi\vartheta}\right)\left(F_{\widetilde{E}}\right)\widetilde{\subseteq}\overline{\left(\psi_{f\varphi\vartheta}\right)\left(F_{\widetilde{E}}\right)},$$

$F_{\widetilde{E}}\widetilde{\subseteq}\psi_{f\varphi\vartheta}^{-1}\overline{\left(\psi_{f\varphi\vartheta}\right)\left(F_{\widetilde{E}}\right)}$ is obtained. By part (3), since $\psi_{f\varphi\vartheta}^{-1}\overline{\left(\psi_{f\varphi\vartheta}\right)\left(F_{\widetilde{E}}\right)}$ is a bipolar soft closed set over U, $\overline{F_{\widetilde{E}}}\widetilde{\subseteq}\psi_{f\varphi\vartheta}^{-1}\overline{\left(\psi_{f\varphi\vartheta}\right)\left(F_{\widetilde{E}}\right)}$. Thus, $\left(\psi_{f\varphi\vartheta}\right)\left(\overline{F_{\widetilde{E}}}\right)\widetilde{\subseteq}\overline{\left(\psi_{f\varphi\vartheta}\right)\left(F_{\widetilde{E}}\right)}$ is satisfied.

(4) ⇒ (5) Let $D_{\widetilde{E'}}\in BS\left(V_{\widetilde{E'}}\right)$ and $\left(\psi_{f\varphi\phi}\right)^{-1}\left(D_{\widetilde{E'}}\right)=F_{\widetilde{E}}$. By part (4),

$$\begin{aligned}\left(\psi_{f\varphi\vartheta}\right)\left(\overline{F_{\widetilde{E}}}\right) &= \left(\psi_{f\varphi\vartheta}\right)\left(\overline{\psi_{f\varphi\vartheta}^{-1}\left(D_{\widetilde{E'}}\right)}\right)\\ &\widetilde{\subseteq}\overline{\left(\psi_{f\varphi\vartheta}\right)\left(\psi_{f\varphi\vartheta}^{-1}\left(D_{\widetilde{E'}}\right)\right)}\widetilde{\subseteq}\overline{D_{\widetilde{E'}}}.\end{aligned}$$

Then, $\left(\overline{\psi_{f\varphi\vartheta}^{-1}\left(D_{\widetilde{E'}}\right)}\right)=\overline{\left(F_{\widetilde{E}}\right)}\widetilde{\subseteq}\psi_{f\varphi\vartheta}^{-1}\left(\overline{D_{\widetilde{E'}}}\right)$ is obtained.

(5) $\Rightarrow$ (6) Let $D_{\widetilde{E'}} \in BS\left(V_{\widetilde{E'}}\right)$. Substituting $D^c_{\widetilde{E'}}$ for condition in (5). Then, $\overline{\psi^{-1}_{f\varphi\vartheta}\left(D^c_{\widetilde{E'}}\right)} \widetilde{\subseteq} \psi^{-1}_{f\varphi\vartheta}\left(\overline{D^c_{\widetilde{E'}}}\right)$. Since $D^{\circ}_{\widetilde{E'}} = \left(\overline{D^c_{\widetilde{E'}}}\right)^c$, then

$$\begin{array}{rcl} \psi^{-1}_{f\varphi\vartheta}\left(D^{\circ}_{\widetilde{E'}}\right) & = & \psi^{-1}_{f\varphi\vartheta}\left(\left(\overline{D^c_{\widetilde{E'}}}\right)^c\right) \\ & = & \left(\left(\psi^{-1}_{f\varphi\vartheta}\left(\overline{D^c_{\widetilde{E'}}}\right)\right)\right)^c \\ & \widetilde{\subseteq} & \left(\overline{\psi^{-1}_{f\varphi\vartheta}\left(D^c_{\widetilde{E'}}\right)}\right)^c \\ & = & \left(\overline{\left(\psi^{-1}_{f\varphi\vartheta}\left(D_{\widetilde{E'}}\right)\right)^c}\right)^c \\ & = & \left(\psi^{-1}_{f\varphi\vartheta}\left(D_{\widetilde{E'}}\right)\right)^{\circ}. \end{array}$$

(6) $\Rightarrow$ (2) Let $D_{\widetilde{E'}} \in \widetilde{\tau}'$. Since

$$\begin{array}{rcl} \left(\psi^{-1}_{f\varphi\vartheta}\left(D_{\widetilde{E'}}\right)\right)^{\circ} & \widetilde{\subseteq} & \left(\psi^{-1}_{f\varphi\vartheta}\left(D_{\widetilde{E'}}\right)\right) \\ & = & \psi^{-1}_{f\varphi\vartheta}\left(\left(D_{\widetilde{E'}}\right)^{\circ}\right) \\ & \widetilde{\subseteq} & \left(\psi^{-1}_{f\varphi\vartheta}\left(D_{\widetilde{E'}}\right)\right)^{\circ}, \end{array}$$

then $\psi^{-1}_{f\varphi\vartheta}\left(D_{\widetilde{E'}}\right) \in \widetilde{\tau}$. $\square$

Example 3.2. *Let $U = \{x_1, x_2, x_3, x_4\}$, $V = \{y_1, y_2, y_3, y_4\}$ be two sets and $E = E' = \{e_1, e_2\}$ be two sets of parameters. Then,*

$$\widetilde{\tau} = \left\{\Phi_{\widetilde{E}}, \widetilde{U}_{\widetilde{E}}, F_{1_{\widetilde{E}}}, F_{2_{\widetilde{E}}}, F_{3_{\widetilde{E}}}, F_{4_{\widetilde{E}}}\right\}$$

is a bipolar soft topology over U and

$$\widetilde{\tau}' = \left\{\Phi_{\widetilde{E}}, \widetilde{V}_{\widetilde{E}}, G_{1_{\widetilde{E}}}, G_{2_{\widetilde{E}}}, G_{3_{\widetilde{E}}}, G_{4_{\widetilde{E}}}\right\}$$

is a bipolar soft topology over V, where

$$\begin{array}{rcl} F_{1_{\widetilde{E}}} & = & \{(e_1, \{x_1, x_2\}, \{x_3\}), (e_2, \{x_1, x_3\}, \{x_2, x_4\})\}, \\ F_{2_{\widetilde{E}}} & = & \{(e_1, \{x_1, x_4\}, \{x_2, x_3\}), (e_2, \{x_2, x_3\}, \{x_4\})\}, \\ F_{3_{\widetilde{E}}} & = & \{(e_1, \{x_1, x_2, x_4\}, \{x_3\}), (e_2, \{x_1, x_2, x_3\}, \{x_4\})\}, \\ F_{4_{\widetilde{E}}} & = & \{(e_1, \{x_1\}, \{x_2, x_3\}), (e_2, \{x_3\}, \{x_2, x_4\})\} \end{array}$$

and

$$\begin{array}{rcl} G_{1_{\widetilde{E}}} & = & \{(e_1, \{y_1, y_2\}, \{y_3\}), (e_2, \{y_1, y_3\}, \{y_2, y_4\})\}, \\ G_{2_{\widetilde{E}}} & = & \{(e_1, \{y_1, y_4\}, \{y_2, y_3\}), (e_2, \{y_2, y_3\}, \{y_4\})\}, \\ G_{3_{\widetilde{E}}} & = & \{(e_1, \{y_1, y_2, y_4\}, \{y_3\}), (e_2, \{y_1, y_2, y_3\}, \{y_4\})\}, \\ G_{4_{\widetilde{E}}} & = & \{(e_1, \{y_1\}, \{y_2, y_3\}), (e_2, \{y_3\}, \{y_2, y_4\})\}. \end{array}$$

Let $f : U \to V$ be a function defined as $f(x_i) = y_i$, $i = 1, 2, 3, 4$, the function $\varphi : E \to E$ be defined as $\varphi(e_i) = e_i$ and the function $\vartheta : \neg E \to \neg E$ be defined as $\vartheta(\neg e_i) = \neg\varphi(e_i)$, $i = 1, 2$. Then, $\psi_{f\varphi\vartheta} : \left(U, \widetilde{\tau}, \widetilde{E}\right) \to \left(V, \widetilde{\tau}', \widetilde{E'}\right)$ is a bipolar soft continuous function.

Theorem 3.7. *Let* $\psi_{f\varphi\vartheta} : \left(U, \widetilde{\tau}, \widetilde{E}\right) \to \left(V, \widetilde{\tau'}, \widetilde{E'}\right)$ *be a bipolar soft continuous function. Then, the functions* $f_\varphi : (U, \tau, E) \to (V, \tau', E')$ *and* $f_\varphi : (U, \tau, \neg E) \to (V, \tau', \neg E')$ *are soft continuous functions.*

Proof. The proof is clear. □

Remark. *If* $\psi_{f\varphi\vartheta} : \left(U, \widetilde{\tau}, \widetilde{E}\right) \to \left(V, \widetilde{\tau'}, \widetilde{E'}\right)$ *is a bipolar soft continuous function, then* $f_e : (U, \tau_e) \to \left(V, \tau'_{\varphi(e)}\right)$ *and* $f_e : (U, \tau_{\neg e}) \to \left(V, \tau'_{\neg\varphi(e)}\right)$ *are continuous function on topological spaces, for each* $e \in E$.

Example 3.3. *Consider the bipolar soft function defined previous example. Then, for the parameter* $e_i \in E, i = 1, 2,$

$$\begin{aligned}
\tau_{e_1} &= \{\varnothing, U, \{x_1, x_2\}, \{x_1, x_4\}, \{x_1, x_2, x_4\}, \{x_1\}\},\ \tau_{\neg e_1} = \{\varnothing, U, \{x_3\}, \{x_2, x_3\}\}, \\
\tau_{e_2} &= \{\varnothing, U, \{x_1, x_3\}, \{x_2, x_3\}, \{x_1, x_2, x_3\}, \{x_3\}\},\ \tau_{\neg e_2} = \{\varnothing, U, \{x_2, x_4\}, \{x_4\}\}, \\
\tau'_{e_1} &= \{\varnothing, V, \{y_1\}, \{y_1, y_2\}, \{y_1, y_4\}, \{y_1, y_2, y_4\}\},\ \tau'_{\neg e_1} = \{\varnothing, V, \{y_3\}, \{y_2, y_3\}\}, \\
\tau'_{e_2} &= \{\varnothing, V, \{y_3\}, \{y_1, y_3\}, \{y_2, y_3\}, \{y_1, y_2, y_3\}\},\ \tau'_{\neg e_2} = \{\varnothing, V, \{y_4\}, \{y_2, y_4\}\}.
\end{aligned}$$

Therefore, $f_{e_i} : (U, \tau_{e_i}) \to \left(V, \tau'_{e_i}\right)$ *and* $f_{e_i} : (U, \tau_{\neg e_i}) \to \left(V, \tau'_{\neg e_i}\right)$ *are continuous functions on topological spaces, for* $i = 1, 2$.

Example 3.4. *Let* $U = \{x_1, x_2, x_3\}$, $V = \{y_1, y_2, y_3\}$ *and* $E = E' = \{e_1, e_2\}$. *Then,* $\widetilde{\tau} = \left\{\Phi_{\widetilde{E}}, \widetilde{U}_{\widetilde{E}}, F1_{\widetilde{E}}, F2_{\widetilde{E}}, F3_{\widetilde{E}}, F4_{\widetilde{E}}\right\}$ *is a bipolar soft topology over* U *and* $\widetilde{\tau'} = \left\{\Phi_{\widetilde{E}}, \widetilde{V}_{\widetilde{E}}, G_{\widetilde{E}}\right\}$ *is a bipolar soft topology over* V, *where*

$$\begin{aligned}
F1_{\widetilde{E}} &= \{(e_1, \{x_1, x_3\}, \{x_2\}), (e_2, \{x_1\}, \{x_2, x_3\})\}, \\
F2_{\widetilde{E}} &= \{(e_1, \{x_2, x_3\}, \{x_1\}), (e_2, \{x_1, x_3\}, \{x_2\})\}, \\
F3_{\widetilde{E}} &= \{(e_1, U, \varnothing), (e_2, \{x_1, x_3\}, \{x_2\})\}, \\
F4_{\widetilde{E}} &= \{(e_1, \{x_3\}, \{x_1, x_2\}), (e_2, \{x_1\}, \{x_2, x_3\})\},
\end{aligned}$$

and

$$G_{\widetilde{E}} = \{(e_1, \{y_1\}, \{y_2, y_3\}), (e_2, \{y_1, y_2\}, \{y_3\})\}.$$

Let $f : U \to V$ *be a function defined as* $f(x_1) = y_2$, $f(x_2) = y_1$, $f(x_3) = y_3$, *the function* $\varphi : E \to E$ *be defined as* $\varphi(e_i) = e_i$ *and the function* $\vartheta : \neg E \to \neg E$ *be defined as* $\vartheta(\neg e_i) = \neg\varphi(e_i)$, $i = 1, 2$. *Since* $\psi^{-1}_{f\varphi\vartheta}\left(G_{\widetilde{E}}\right) \notin \widetilde{\tau}$, $\psi_{f\varphi\vartheta} : \left(U, \widetilde{\tau}, \widetilde{E}\right) \to \left(V, \widetilde{\tau'}, \widetilde{E'}\right)$ *is not a bipolar soft continuous function. Here,*

$$\begin{aligned}
\psi^{-1}_{f\varphi\vartheta}(G)(e_1) &= f^{-1}(G(\varphi(e_1))) = \{x_2\}, \\
\psi^{-1}_{f\varphi\vartheta}(G)(\neg e_1) &= f^{-1}(G(\vartheta(\neg e_1))) = \{x_1, x_3\}, \\
\psi^{-1}_{f\varphi\vartheta}(G)(e_2) &= f^{-1}(G(\varphi(e_2))) = \{x_1, x_2\}, \\
\psi^{-1}_{f\varphi\vartheta}(G)(\neg e_2) &= f^{-1}(G(\vartheta(\neg e_2))) = \{x_3\}.
\end{aligned}$$

Theorem 3.8. *Let* $\left(U, \widetilde{\tau}, \widetilde{E}\right), \left(V, \widetilde{\tau'}, \widetilde{E'}\right)$ *and* $\left(W, \widetilde{\tau'}, \widetilde{E^*}\right)$ *be bipolar soft topological spaces over* U, V *and* W, *respectively. If* $\psi_{f\varphi\vartheta} : \left(U, \widetilde{\tau}, \widetilde{E}\right) \to \left(V, \widetilde{\tau'}, \widetilde{E'}\right)$ *and* $\omega_{g\varphi_1\vartheta_1} : \left(V, \widetilde{\tau'}, \widetilde{E'}\right) \to \left(W, \widetilde{\tau'}, \widetilde{E^*}\right)$ *are bipolar soft continuous functions, then* $\left(\omega_{g\varphi_1\vartheta_1} \circ \psi_{f\varphi\vartheta}\right) : \left(U, \widetilde{\tau}, \widetilde{E}\right) \to \left(W, \widetilde{\tau'}, \widetilde{E^*}\right)$ *is a bipolar soft continuous function.*

Proof. Let $K_{\widetilde{E^{*}}} \in \widetilde{\tau^{*}}$. Let us show that $\left(\omega_{g\varphi_1\vartheta_1} \circ \psi_{f\varphi\vartheta}\right)^{-1} K_{\widetilde{E^{*}}} \in \widetilde{\tau}$. Since

$$\begin{aligned}
\left(\omega_{g\varphi_1\vartheta_1} \circ \psi_{f\varphi\vartheta}\right)^{-1}(K)(e) &= f^{-1}\left(g^{-1}\left(K\left(\left(\varphi_1 \circ \varphi\right)(e)\right)\right)\right), \\
\left(\omega_{g\varphi_1\vartheta_1} \circ \psi_{f\varphi\vartheta}\right)^{-1}(K)(\neg e) &= f^{-1}\left(g^{-1}\left(K\left(\left(\vartheta_1 \circ \vartheta\right)(\neg e)\right)\right)\right),
\end{aligned}$$

and $\omega_{g\varphi_1\vartheta_1}$ is a bipolar soft continuous function, then $g^{-1}\left(K\left(\left(\varphi_1 \circ \varphi\right)(e)\right)\right) \in \widetilde{\tau'}$. Also, since $\psi_{f\varphi\vartheta}$ is a bipolar soft continuous function, then $f^{-1}\left(g^{-1}\left(K\left(\left(\varphi_1 \circ \varphi\right)(e)\right)\right)\right) \in \widetilde{\tau}$. That is,

$$\left(\omega_{g\varphi_1\vartheta_1} \circ \psi_{f\varphi\vartheta}\right)^{-1} K_{\widetilde{E^{*}}} \in \widetilde{\tau}$$

and $\left(\omega_{g\varphi_1\vartheta_1} \circ \psi_{f\varphi\vartheta}\right)$ is a bipolar soft continuous function. □

Definition 3.10. *Let* $\left(U, \widetilde{\tau}, \widetilde{E}\right)$ *and* $\left(V, \widetilde{\tau'}, \widetilde{E'}\right)$ *be two bipolar soft topological spaces and* $\psi_{f\varphi\vartheta} : \left(U, \widetilde{\tau}, \widetilde{E}\right) \rightarrow \left(V, \widetilde{\tau'}, \widetilde{E'}\right)$ *be a bipolar soft function.*

(1) If the image $\psi_{f\varphi\vartheta}\left(F_{\widetilde{E}}\right) \in \widetilde{\tau'}$ for any $F_{\widetilde{E}} \in \widetilde{\tau}$, then $\psi_{f\varphi\vartheta}$ is called a bipolar soft open function,

(2) If the image $\psi_{f\varphi\vartheta}\left(G_{\widetilde{E}}\right)$ is a bipolar soft closed set in V for any bipolar soft closed set $G_{\widetilde{E}}$ in U, then $\psi_{f\varphi\vartheta}$ is called a bipolar soft closed function.

Proposition 3.9. *Let* $\psi_{f\varphi\vartheta} : \left(U, \widetilde{\tau}, \widetilde{E}\right) \rightarrow \left(V, \widetilde{\tau'}, \widetilde{E'}\right)$ *be a bipolar soft open (closed) function. Then, the functions* $f_{\varphi} : (U, \tau, E) \rightarrow (V, \tau', E')$ *and* $f_{\varphi} : (U, \tau, \neg E) \rightarrow (V, \tau', \neg E')$ *are soft open (closed) functions.*

Proof. The proof is obtained from the definition of bipolar soft open (closed) function. □

Proposition 3.10. *If* $\psi_{f\varphi\vartheta} : \left(U, \widetilde{\tau}, \widetilde{E}\right) \rightarrow \left(V, \widetilde{\tau'}, \widetilde{E'}\right)$ *be a bipolar soft open (closed) function, for each* $e \in E$, *then* $f_e : (U, \tau_e) \rightarrow \left(V, \tau'_{\varphi(e)}\right)$ *is a open (closed) function on topological spaces, for each* $e \in E$.

Proof. The proof of the proposition is straightforward. □

Example 3.5. *Let* $U = \{x_1, x_2\}$, $V = \{y_1, y_2\}$ *and* $E = E' = \{e_1, e_2\}$. *Then* $\widetilde{\tau} = \left\{\Phi_{\widetilde{E}}, \widetilde{U}_{\widetilde{E}}, F_{\widetilde{E}}\right\}$ *is a bipolar soft topology over* U *and* $\widetilde{\tau'} = \left\{\Phi_{\widetilde{E}}, \widetilde{V}_{\widetilde{E}}, G_{1_{\widetilde{E}}}, G_{2_{\widetilde{E}}}, G_{3_{\widetilde{E}}}, G_{4_{\widetilde{E}}}\right\}$ *is a bipolar soft topology over* V, *where*

$$F_{\widetilde{E}} = \left\{\left(e_1, \{x_1\}, \{x_2\}\right), \left(e_2, \{x_2\}, \varnothing\right)\right\},$$

and

$$\begin{aligned}
G_{1_{\widetilde{E}}} &= \left\{\left(e_1, \{y_1\}, \{y_2\}\right), \left(e_2, \{y_2\}, \varnothing\right)\right\}, \\
G_{2_{\widetilde{E}}} &= \left\{\left(e_1, \{y_1\}, \{y_2\}\right), \left(e_2, \{y_1\}, \varnothing\right)\right\}, \\
G_{3_{\widetilde{E}}} &= \left\{\left(e_1, \{y_1\}, \{y_2\}\right), \left(e_2, \{y_1, y_2\}, \varnothing\right)\right\}, \\
G_{4_{\widetilde{E}}} &= \left\{\left(e_1, \{y_1\}, \{y_2\}\right), \left(e_2, \varnothing, \varnothing\right)\right\}.
\end{aligned}$$

Let $f : U \rightarrow V$ *be a function defined as* $f(x_1) = y_1$, $f(x_2) = y_2$, *the function* $\varphi : E \rightarrow E$ *be defined as* $\varphi(e_i) = e_i$ *and the function* $\vartheta : \neg E \rightarrow \neg E$ *be defined* *as* $\vartheta(\neg e_i) = \neg\varphi(e_i)$, $i = 1,2$. *Then, since* $\psi^{-1}_{f\varphi\vartheta}\left(G_{2_{\widetilde{E}}}\right) \notin \widetilde{\tau}$, $\psi_{f\varphi\vartheta} : \left(U, \widetilde{\tau}, \widetilde{E}\right) \to \left(V, \widetilde{\tau'}, \widetilde{E'}\right)$ *is not a bipolar soft continuous function. Here,*

$$\begin{aligned}
\psi^{-1}_{f\varphi\vartheta}(G_2)(e_1) &= f^{-1}\left(G_2^{+}(\varphi(e_1))\right) = \{x_1\}, \\
\psi^{-1}_{f\varphi\vartheta}(G_2)(\neg e_1) &= f^{-1}\left(G_2^{-}(\vartheta(\neg e_1))\right) = \{x_2\}, \\
\psi^{-1}_{f\varphi\vartheta}(G_2)(e_2) &= f^{-1}\left(G_2^{+}(\varphi(e_2))\right) = \{x_1\}, \\
\psi^{-1}_{f\varphi\vartheta}(G_2)(\neg e_2) &= f^{-1}\left(G_2^{-}(\vartheta(\neg e_2))\right) = \varnothing.
\end{aligned}$$

Since $\psi_{f\varphi\vartheta}\left(F_{\widetilde{E}}\right) = G_{1_{\widetilde{E}}} \in \widetilde{\tau'}$, $\psi_{f\varphi\vartheta} : \left(U, \widetilde{\tau}, \widetilde{E}\right) \to \left(V, \widetilde{\tau'}, \widetilde{E'}\right)$ *is a bipolar soft open function.*

Theorem 3.11. *Let* $\left(U, \widetilde{\tau}, \widetilde{E}\right)$ *and* $\left(V, \widetilde{\tau'}, \widetilde{E'}\right)$ *be two bipolar soft topological spaces and* $\psi_{f\varphi\vartheta} : \left(U, \widetilde{\tau}, \widetilde{E}\right) \to \left(V, \widetilde{\tau'}, \widetilde{E'}\right)$ *be a bipolar soft function.*

(1) $\psi_{f\varphi\vartheta}$ is a bipolar soft open function if and only if for any $F_{\widetilde{E}} \in BS\left(U_{\widetilde{E}}\right)$,

$$\psi_{f\varphi\vartheta}\left(F^{\circ}_{\widetilde{E}}\right) \widetilde{\subseteq} \left(\psi_{f\varphi\vartheta}\left(F_{\widetilde{E}}\right)\right)^{\circ}.$$

(2) $\psi_{f\varphi\vartheta}$ is a bipolar soft closed function if and only if for any $F_{\widetilde{E}} \in BS\left(U_{\widetilde{E}}\right)$,

$$\overline{\left(\psi_{f\varphi\vartheta}\left(F_{\widetilde{E}}\right)\right)} \widetilde{\subseteq} \psi_{f\varphi\vartheta}\overline{\left(F_{\widetilde{E}}\right)}.$$

Proof. (1) Let $\psi_{f\varphi\vartheta}$ be a bipolar soft open function and $F_{\widetilde{E}} \in BS\left(U_{\widetilde{E}}\right)$. Then, $F^{\circ}_{\widetilde{E}} \in \widetilde{\tau}$ and $F^{\circ}_{\widetilde{E}} \widetilde{\subseteq} F_{\widetilde{E}}$. Since $\psi_{f\varphi\vartheta}$ is a bipolar soft open function, $\psi_{f\varphi\vartheta}\left(F^{\circ}_{\widetilde{E}}\right) \in \widetilde{\tau'}$ and $\psi_{f\varphi\vartheta}\left(F^{\circ}_{\widetilde{E}}\right) \widetilde{\subseteq} \psi_{f\varphi\vartheta}\left(F_{\widetilde{E}}\right)$. Thus,

$$\psi_{f\varphi\vartheta}\left(F^{\circ}_{\widetilde{E}}\right) \widetilde{\subseteq} \left(\psi_{f\varphi\vartheta}\left(F_{\widetilde{E}}\right)\right)^{\circ}$$

is obtained.

Conversely, let $F_{\widetilde{E}} \in \widetilde{\tau}$. Then, $F_{\widetilde{E}} = F^{\circ}_{\widetilde{E}}$. From the condition of theorem,

$$\psi_{f\varphi\vartheta}\left(F^{\circ}_{\widetilde{E}}\right) \widetilde{\subseteq} \left(\psi_{f\varphi\vartheta}\left(F_{\widetilde{E}}\right)\right)^{\circ}.$$

Hence,

$$\psi_{f\varphi\vartheta}\left(F_{\widetilde{E}}\right) = \psi_{f\varphi\vartheta}\left(F^{\circ}_{\widetilde{E}}\right) \widetilde{\subseteq} \left(\psi_{f\varphi\vartheta}\left(F_{\widetilde{E}}\right)\right)^{\circ} \widetilde{\subseteq} \psi_{f\varphi\vartheta}\left(F_{\widetilde{E}}\right).$$

It is clear that $\psi_{f\varphi\vartheta}\left(F_{\widetilde{E}}\right) = \left(\psi_{f\varphi\vartheta}\left(F_{\widetilde{E}}\right)\right)^{\circ}$, i.e. $\psi_{f\varphi\vartheta}$ is a bipolar soft open function.

(2) Let $\psi_{f\varphi\vartheta}$ be a bipolar soft closed function and $F_{\widetilde{E}} \in BS\left(U_{\widetilde{E}}\right)$. Since $\psi_{f\varphi\vartheta}$ is a bipolar soft closed function, $\psi_{f\varphi\vartheta}\left(\overline{F_{\widetilde{E}}}\right)$ is a bipolar soft closed set in V and $\psi_{f\varphi\vartheta}\left(F_{\widetilde{E}}\right) \widetilde{\subseteq} \psi_{f\varphi\vartheta}\left(\overline{F_{\widetilde{E}}}\right)$. Henceforth $\overline{\psi_{f\varphi\vartheta}\left(F_{\widetilde{E}}\right)} \widetilde{\subseteq} \psi_{f\varphi\vartheta}\left(\overline{F_{\widetilde{E}}}\right)$ is obtained.

Conversely, now let $F_{\widetilde{E}}$ be any bipolar soft closed set over U. Then, $F_{\widetilde{E}} = \overline{F_{\widetilde{E}}}$. From the condition of theorem, $\overline{\left(\psi_{f\varphi\vartheta}\left(F_{\widetilde{E}}\right)\right)} \widetilde{\subseteq} \psi_{f\varphi\vartheta}\overline{\left(F_{\widetilde{E}}\right)} = \psi_{f\varphi\vartheta}\left(F_{\widetilde{E}}\right) \widetilde{\subseteq} \overline{\left(\psi_{f\varphi\vartheta}\left(F_{\widetilde{E}}\right)\right)}$. It is clear that $\psi_{f\varphi\vartheta}\left(F_{\widetilde{E}}\right) = \overline{\left(\psi_{f\varphi\vartheta}\left(F_{\widetilde{E}}\right)\right)}$, i.e. $\psi_{f\varphi\vartheta}$ is a bipolar soft closed function. □

Definition 3.11. *Let* $\left(U,\widetilde{\tau},\widetilde{E}\right)$ *and* $\left(V,\widetilde{\tau'},\widetilde{E'}\right)$ *be two bipolar soft topological spaces and* $\psi_{f\varphi\vartheta}:\left(U,\widetilde{\tau},\widetilde{E}\right)\rightarrow\left(V,\widetilde{\tau'},\widetilde{E'}\right)$ *be a bipolar soft function.* $\psi_{f\varphi\phi}$ *is called a bipolar soft homeomorphism from* $\left(U,\widetilde{\tau},\widetilde{E}\right)$ *to* $\left(V,\widetilde{\tau'},\widetilde{E'}\right)$, *if* $\psi_{f\varphi\vartheta}$ *is both bipolar soft bijective and bipolar soft continuous,* $\psi^{-1}_{f\varphi\vartheta}$ *is a bipolar soft continuous function.*

Theorem 3.12. *Let* $\left(U,\widetilde{\tau},\widetilde{E}\right)$ *and* $\left(V,\widetilde{\tau'},\widetilde{E'}\right)$ *be two bipolar soft topological spaces and* $\psi_{f\varphi\vartheta}:\left(U,\widetilde{\tau},\widetilde{E}\right)\rightarrow\left(V,\widetilde{\tau'},\widetilde{E'}\right)$ *be a bipolar soft function. Then, the following conditions are equivalent:*

(1) $\psi_{f\varphi\vartheta}$ is a bipolar soft homeomorphism,
(2) $\psi_{f\varphi\vartheta}$ is both a bipolar soft continuous and bipolar soft closed function,
(3) $\psi_{f\varphi\vartheta}$ is both a bipolar soft continuous and bipolar soft open function.

Proof. The proof is clear. □

4. Conclusion

Since we have defined the concepts of bipolar soft continuity, bipolar soft openness, bipolar soft closedness and bipolar soft homeomorphism, this paper contributes to the topology field from a bipolar view. Theorems and examples support the concepts given. In forthcoming works, we aim to give the concept bipolar infra soft topology and examine some structures such as connectedness and different separation axioms.

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