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Research Article

A Geometrical Interpretation of Production Functions in Economics in terms of Second Fundamental Form

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Abstract

The two well-known production models in microeconomics are Cobb-Douglas and ACMS production functions. We study such production functions with 2-inputs in terms of the differential-geometrical properties of their graphs. In particular, we investigate the production functions when their graphs have the second fundamental forms of constant length. We obtain that the ACMS production surface has the second fundamental forms of constant length if and only if the ACMS production function is a perfect substitute. Furthermore, in the case of Cobb-Douglas production function, we provide a non-existence result.

Keywords: Cobb-Douglas production function, ACMS production function, production surface, second fundamental form

1. INTRODUCTION

One of the arguments for modelling the production process in economics (on the firm or the aggregate levels) is the production function, expressing the relations between inputs (the factors contributing to the process) and level of output. It is based on the presumed choosing of technologically the best one among the techniques used in the production process. This is so that only if at least one of the inputs increases will the level of output increase [1,2].

Let $\mathbb{R}_*$ be the set of all the positive real numbers. Then, mathematically, a *production function* is a mapping [3]

$$f: \mathbb{R}_* \to \mathbb{R}_*^n, \ \ (x_1, \dots, x_n) \mapsto f(x_1, \dots, x_n), \tag{1}$$

where $x_1, \dots, x_n$ are the inputs, f the level of output and $n \geq 2$ the number of the inputs. In principle, one assumes that f holds the following conditions: (i) $f \in C^2$, i.e. it has twice continuous partial derivatives; (ii) $f_{,i} = \partial f / \partial x_i > 0$, for every $i \in \{1, \dots, n\}$; (iii) f is p-homogeneous, i.e.

$$f(tx_1, \dots, tx_n) = t^p f(x_1, \dots, x_n),$$

for all $(x_1, \dots, x_n) \in \mathbb{R}_*^n$ and $t > 0$.

Notice that constant inputs are omitted. With the homogeneity condition of a production function, it is able to classify the property of return to scale. More clearly, we call that a production function has *constant return to scale* if the degree of homogeneity p is 1. In addition, one has *decreasing* (*increasing*) *return to scale* if p is less than (greater than) 1. For instance, fix $p > 1$ (resp. $p < 1$). Hence, when increased every input by 10 percent, then the level of output shows a rise of greater than (resp. less than) 10 percent. Fix now $p = 1$. Then, the level of output shows a rise of 10 percent when every input is increased by 10 percent.

The production functions played a key role in calculating the social-economy damage and in solving production scarcity originated by Covid-19 breaking out in late 2019 [4–6]. Furthermore, they have numereous uses outside of economics, including biology [7,8], education [9,10], in engineering [11,12].

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In 2011, an interesting perspective on production functions was exhibited by Vilcu [3] where a production function is analysed by the differential-geometrical properties of its own graph. More clearly, the graph of the production function f given by (1) is actually a hypersurface M^n of the Euclidean space $\mathbb{R}_*^{n+1}$ parametrized by (see [13])

$$\varphi : \mathbb{R}_*^n \to \mathbb{R}_*^{n+1}$$
$$(x_1, \dots, x_n) \mapsto \varphi(x_1, \dots, x_n) = \big(x_1, \dots, x_n, f(x_1, \dots, x_n)\big)$$

which we call *production hypersurface*. Hence, Vilcu's perspective uses an argument characterizing the economic properties (i.e. the elasticity of production, the marginal rate of technical substitution, elasticity of substitution and etc.) of f in terms of the basic curvature invariants of M^n.

We emphasize that although the first studies in this direction were published in early 2000s (see [14,15]), the serious results were firstly obtained in [3,13,16]. Afterwards, Chen published a series of papers (including homogeneous, weighted homogeneous, homothetic, quasi-sum, quasi-product production functions) jointly with Alodan, Deshmukh, Vilcu and Vilcu [17–26]. In addition, the first author of the present paper has some studies jointly with Ergut, Mihai, Yılmaz and Gülşen [27–30], relating to quasi-sum, quasi-product, homothetic production functions. We also refer to [31,32].

In the cited papers above, the production functions have been analysed in terms of the Gauss-Kronecker curvature, the mean curvature, the sectional curvature, the Ricci curvature and the Riemann curvature tensor (for details, see [33]) as curvature invariants of the associated production hypersurfaces. As a new idea in this field, we analyse Cobb-Douglas and ACMS production functions and the associated production hypersurfaces in terms of the length of the second fundamental form. More explicitly, we do an analysis of Cobb-Douglas and ACMS production functions with 2-inputs and their production surfaces when the second fundamental forms have constant length. We obtain that the ACMS production surface has a second fundamental form of constant length if and only if the ACMS production function is a perfect substitute. We also state that the Cobb-Douglas production surface cannot satisfy this property.

The motivation of our problem is as follows: Let $S \subset \mathbb{R}^3$ be a regular surface and κ_1, κ_2 its principal curvatures. We call S *Weingarten surface* if there is a relation given by [34],

$$\mathcal{W}(\kappa_1, \kappa_2) = 0. \quad (2)$$

Notice that the relation (2) is equivalent to $\mathcal{U}(K, H) = 0$, where K and H are the Gaussian and mean curvatures of S. Hence, as trivial examples of Weingarten surfaces are when K or H are constants. Studying differential-geometrical properties of such surfaces is a classical subject, see [35].

Denote by h and $\|h\|^2$ the second fundamental form of S and its squared length. On one hand, $\|h\|^2$ is equivalent to the trace of A^2, where A is the shape operator of S, yielding

$$\|h\|^2 = \kappa_1^2 + \kappa_2^2. \quad (3)$$

Therefore, the condition that $\|h\|^2$ is constant turns the surface S to be a Weingarten surface. On the other hand, the expression in the right-hand side of (3) is actually twice the so-called *Casorati curvature* [36], which is an important curvature invariant, see [37,38].

2. MATERIAL AND METHODS

2.1 Curvature Invariants of Surfaces

Let $\mathbb{R}^3$ be the Euclidean space of dimension 3 endowed with the Euclidean inner product $\langle \cdot,\cdot \rangle$ and $\|\cdot\|$ the induced norm. Given a regular surface S in $\mathbb{R}^3$ and $U \subset \mathbb{R}^2$ an open subset. Denote by φ a local parametrization on S such that

$$\varphi : U \to \mathbb{R}^3$$
$$(x, y) \mapsto \varphi(x, y).$$

The normal vector field ξ is

$$\xi = \frac{\varphi_x \times \varphi_y}{\|\varphi_x \times \varphi_y\|},$$

where φ_x is the partial derivative of φ with respect to x and $\times$ the cross product in $\mathbb{R}^3$. The components of the first fundamental form g (i.e. the induced metric from $\mathbb{R}^3$) are

$$E = \langle \varphi_x, \varphi_x \rangle, \ \ F = \langle \varphi_x, \varphi_y \rangle, \ \ G = \langle \varphi_y, \varphi_y \rangle$$

and those of the second fundamental form h

$$L = \langle \varphi_{xx}, \xi \rangle, \ \ M = \langle \varphi_{xy}, \xi \rangle, \ \ N = \langle \varphi_{yy}, \xi \rangle.$$

Let ∇ be the Levi-Civita connection on $\mathbb{R}^3$. Then, for any vector field X tangent to S, we have the formula of Weingarten

$$\nabla_X \xi = -A(X),$$

where A is the shape operator. Then, the matrix of the shape operator $[A] = [g]^{-1}[h]$, for the corresponding matrices $[g]$ and $[h]$ of the bilinear forms g and h, respectively. Let $\{e_1, e_2\}$ be an orthonormal basis of S diagonalizing A. Hence, $[A] = \mathrm{diag}[\kappa_1 \ \kappa_2]$, where κ_1 and κ_2 are the principal curvatures.

The Gaussian curvature K and the mean curvature H are

$$K = \det(A) = \kappa_1 \kappa_2$$

and

$$H = \mathrm{trace}(A) = \frac{\kappa_1 + \kappa_2}{2}.$$

One directly follows $\kappa_1^2 + \kappa_2^2 = 4H^2 - 2K$. In terms of the components of g and h,

$$K = \frac{LN - M^2}{EG - F^2}$$

and

$$H = \frac{EN - 2FM + GL}{2(EG - F^2)}.$$

Now fix $c \in \mathbb{R}, c > 0$, and assume that

$$\kappa_1^2 + \kappa_2^2 = c. \tag{4}$$

Three immediate examples appear when the condition (4) holds: The first is a plane with $\kappa_1 = 0 = \kappa_2$; the second is a circular cylinder whose one principal curvature is 0 and the other one nonzero constant; the third is a sphere whose both principal curvatures are nonzero constant. Those are known as *isoparametric surfaces*.

Let $f(x,y)$ be a smooth real-valued function. We may locally assume the surface S as a graph of f. Then,

$$K = \frac{f_{xx} f_{yy} - f_{xy}^2}{\left(1 + f_x^2 + f_y^2\right)^2}$$

and

$$H = \frac{(1 + f_x^2) f_{yy} - 2 f_x f_y f_{xy} + \left(1 + f_y^2\right) f_{xx}}{2\left(1 + f_x^2 + f_y^2\right)^{3/2}}.$$

If the graph holds (4), then one gives the following partial differential equation (PDE),

$$\left\{(1 + f_x^2) f_{yy} - 2 f_x f_y f_{xy} + \left(1 + f_y^2\right) f_{xx}\right\}^2 - 2\left(1 + f_x^2 + f_y^2\right)\left\{f_{xx} f_{yy} - f_{xy}^2 + \frac{c}{2}\left(1 + f_x^2 + f_y^2\right)^2\right\} = 0. \tag{5}$$

Hence, we may conclude that, in some sense, finding a surface with $\kappa_1^2 + \kappa_2^2 = c$ is equivalent to obtain the solutions of the PDE given by (5).

3. RESULTS AND DISCUSSION

3.1 Generalized Cobb-Douglas Production Functions

Let k, m, n be some nonzero constants with $k > 0$. A *generalized Cobb-Douglas production function* with 2-inputs is defined by

$$f: \mathbb{R}_* \to \mathbb{R}_*^2$$
$$(x, y) \mapsto f(x, y) = kx^m y^n, \tag{6}$$

where $p = m + n$ is the homogeneity degree of f [39]. We call that the graph of f is a *Cobb-Douglas surface*.

In what follows, we will discuss the Cobb-Douglas surface with the condition (4). But, as a prior result, we give the following proposition.

Proposition 1. The second fundamental form of the graph of the form (6) has a constant length if and only if it is a plane.

Proof. By a direct computation, we have from (6) that

$$f_x = \frac{1}{x} mf,$$
$$f_y = \frac{1}{y} nf,$$
$$f_{xx} = \frac{1}{x^2} m(m-1)f,$$
$$f_{xy} = \frac{1}{xy} mnf,$$
$$f_{yy} = \frac{1}{y^2} n(n-1)f.$$

We substitute these partial derivatives above into (5) and we write

$$(xyf)^2\{n(n-1)x^2 + m(m-1)y^2 - mn(m+n)f^2\}^2 -$$
$$2mn(1-m-n)(xyf)^2((xy)^2 + (m^2y^2 + n^2x^2)f^2) + c((xy)^2 + (m^2y^2 + n^2x^2)f^2)^3 = 0. \tag{7}$$

Notice here that the roles of the independent variables x and y are symmetric. Namely, when a statement holds with respect to one, so directly does with respect to the other one. Hence, (7) can be viewed as a sum of the powers of x (or y). For example, (7) can be written as

$$A_1(c,k,m,n,y)x^6 + A_2(c,k,m,n,y)x^{2m+2} + A_3(c,k,m,n,y)x^{2m+4} + A_4(c,k,m,n,y)x^{2m+6} + A_5(c,k,m,n,y)x^{2m+8}$$
$$+A_6(c,k,m,n,y)x^{4m+2} + A_7(c,k,m,n,y)x^{4m+4} + A_8(c,k,m,n,y)x^{4m+6} +$$
$$A_9(c,k,m,n,y)x^{6m} + A_{10}(c,k,m,n,y)x^{6m+2} + A_{11}(c,k,m,n,y)x^{6m+4} + A_{12}(c,k,m,n,y)x^{6m+6} = 0,$$

where $A_i(c,k,m,n,y)$, $i = 1, \dots, 12$, are the coefficient functions depending on the parameters c, k, m, n and y. Since x is an independent variable and the powers of x are linearly independent, every $A_i(c,k,m,n,y)$, $i = 1, \dots, 12$, must be zero. Hence, we may conclude as:

$$A_{12}(c,k,m,n,y) = c(kn)^6 y^{6n} = 0,$$

yielding either $c = 0$ or $k = 0$ or $n = 0$ because y is an independent variable. The second possibility is obviously eliminated. If $c = 0$, then we understand from (4) that the principal curvatures κ_1 and κ_2 vanish, implying that the graph is a plane. If $n = 0$, then we may parametrize the graph as

$$(x, y) \mapsto (x, y, kx^m) = (x, 0, kx^m) + y(0,1,0),$$

which is a cylindrical surface whose rulings are parallel to (0,1,0). However, in this case, one principal curvature vanishes and the other one is

$$\frac{km(m-1)x^{m+1}}{(x^2 + (kmx^m)^2)^{3/2}},$$

which is obviously not constant. Hence, this is not our case, completing the proof.

We notice that Proposition 1 is a pure differential-geometrical result and not valid from the microeconomic perspective. Because, the original definition of a Cobb-Douglas production function allows none of k, m, n to vanish. Therefore, we have the following non-existence result.

Theorem 1. A Cobb-Douglas surface with second fundamental form of constant length does not exist.

3.2 Generalized ACMS Production Functions

Let a, b, k, m, p be some nonzero constants with $k > 0$. A *generalized ACMS production function* with 2-inputs is defined by

$$f: \mathbb{R}_* \to \mathbb{R}_*^2$$
$$(x, y) \mapsto f(x, y) = k((mx)^q + (ny)^q)^{p/q}, \tag{8}$$

where p is the homogeneity degree of f [40]. We call that the graph of f is an *ACMS surface*.

We will discuss the ACMS surface with the condition (4). We first give the following proposition.

Proposition 2. The second fundamental form of the graph of the form (8) has a constant length if and only if it is a plane.

Proof. We set $B = B(x, y) = (mx)^q + (ny)^q$. Then, by (8), a direct computation follows as:

$$\begin{aligned} f_x &= a_1 B^{\frac{p}{q}-1}, \\ f_y &= a_2 B^{\frac{p}{q}-1}, \\ f_{xx} &= a_3 B^{\frac{p}{q}-1} + a_4 B^{\frac{p}{q}-2}, \\ f_{xy} &= a_5 B^{\frac{p}{q}-2}, \\ f_{yy} &= a_6 B^{\frac{p}{q}-1} + a_7 B^{\frac{p}{q}-2}, \end{aligned}$$

where

$$\begin{aligned} a_1 &= km^q p x^{q-1}, \\ a_2 &= kn^q p y^{q-1}, \\ a_3 &= km^q p(q-1)x^{q-2}, \\ a_4 &= km^{2q} p(p-q)x^{2q-2}, \\ a_5 &= km^q n^q p(p-q)(xy)^{q-1}, \\ a_6 &= kn^q p(q-1)y^{q-2}, \\ a_7 &= kn^{2q} p(p-q)y^{2q-2}. \end{aligned}$$

We now substitute the partial derivatives into (5), then we obtain:

$$\begin{aligned} &\left\{(a_3 + a_6)B^{\frac{p}{q}-1} + (a_4 + a_7)B^{\frac{p}{q}-2} + (a_1^2 a_6 + a_2^2 a_3)B^{\frac{3p}{q}-3} + (a_1^2 a_7 - 2a_1 a_2 a_5 + a_2^2 a_4)B^{\frac{3p}{q}-4}\right\}^2 \\ &\quad -2\left\{1 + (a_1^2 + a_2^2)B^{\frac{2p}{q}-2}\right\}\left\{(a_3 a_6 + a_4 a_7)B^{\frac{2p}{q}-2} + (a_3 a_7 + a_4 a_6)B^{\frac{2p}{q}-3}\right\} \\ &\quad -c\left\{1 + (a_1^2 + a_2^2)B^{\frac{2p}{q}-2}\right\}^3 = 0. \end{aligned} \tag{9}$$

As in the proof of Proposition 1, (9) can be viewed as an equation in B. Since it can be written as a sum of the powers of B where the terms are linearly independent, all the coefficients must be zero. We distinguish two cases:

Case 1. Assume that $p \neq q$. Hence, the constant term, i.e. $c = B^0$, of this equation is 0, implying that the graph is a plane.

Case 2. Assume that $p = q$. If also $p = 1$, then the function (8) is linear in both x and y and so its graph directly becomes a plane. In this case, there is nothing to prove. Otherwise, (9) reduces to

$$(a_3(1 + a_2^2) + a_6(1 + a_1^2))^2 - 2\{1 + (a_1^2 + a_2^2)\}(a_3 a_6) - c\{1 + (a_1^2 + a_2^2)\}^3 = 0,$$

which can be viewed as a sum of the powers of x. Again, the coefficients of the powers of x are all zero. The coefficient of the term x^{6q-6} is $c(km^p p)^6$, which must be zero. Similar to the proof of Proposition 1, the only possibility is that $c = 0$. This completes the proof.

On the other hand, in economics, goods that are completely substitutable with each other are called perfect substitutes. Also, we call a production function $f(x, y)$ as a *perfect substitute* if it is a linear function in both x and y. Hence, with Proposition 2 we have the following result.

Theorem 2. The second fundamental form of the ACMS surface has a constant length if and only if the ACMS production function is a perfect substitute.

4. CONCLUSION

In this paper, we studied two well-known production models in microeconomics, Cobb-Douglas and ACMS production functions, in terms of the differential-geometrical properties of their graph surfaces. Having their graph surfaces a second fundamental form of constant length, we proved that a generalized ACMS production function is a perfect substitute. In the case of a Cobb-Douglas production function, a non-existence result was given. Although, as a first stage, we were only interested in the production functions with 2-inputs, our results are open to extension to higher dimensions.

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AUTHOR'S CONTRIBUTIONS

MEA: Conceptualization, investigation, methodology, supervision. MBA: Investigation, methodology.

CONFLICTS OF INTEREST

The authors declare no conflict of interest.

RESEARCH AND PUBLICATION ETHICS

The authors declare that this study complies with Research and Publication Ethics.

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