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ON THE JACOBSTHAL NUMBERS WHICH ARE THE PRODUCT OF TWO MODIFIED PELL NUMBERS

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Abstract. This paper presents an analytic study of determining all the possible solutions of the Diophantine equations such that $q_k = J_m J_n$ and $J_k = q_m q_n$. These give intersections of the Modified Pell and Jacobsthal numbers too for the case where $m = 1$ or $n = 1$.

1. Introduction

It is well-known that the Pell, Modified Pell, and Jacobsthal numbers are defined by the recurrence relations

$$P_0 = 0,\ P_1 = 1 \text{ and } P_{n+1} = 2P_n + P_{n-1} \text{ for all } n \geqslant 2, \tag{1}$$

$$q_0 = 1,\ q_1 = 1 \text{ and } q_{n+1} = 2q_n + q_{n-1} \text{ for all } n \geqslant 2, \tag{2}$$

and

$$J_0 = 0,\ J_1 = 1 \text{ and } J_{n+1} = J_n + 2J_{n-1} \text{ for all } n \geqslant 2, \tag{3}$$

respectively. These integer sequences have very interesting characteristics. For this reason, a heavy interest has been devoted to investigation of the subject by a great number of researchers. Here, it is proposed that two fundamental books given by Vajda [1] and Koshy [2] are investigated for a piece of wide information.

As shown from Equations (1)-(3), all the desired terms of the related sequence can be computed recursively by using the respective recurrence relation. Also, as a second way, we can employ the following equations that are called Binet's formulas:

$$P_n = \frac{\gamma^n - \delta^n}{\gamma - \delta},\ q_n = \frac{\gamma^n + \delta^n}{\gamma + \delta},\ \text{and } J_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \tag{4}$$

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where γ and δ are the positive and negative roots of $x^2 - 2x - 1 = 0$, and α and β are the positive and negative roots of $x^2 - x - 2 = 0$.

Up to the present, many articles have been governed related to the identities and applications of the Modified Pell and Jacobsthal sequences. Let us briefly mention some of the relevant research. In [3], Horadam gave the definition of the Modified Pell numbers, including some elementary identities, and showed that $Q_n = 2q_n$, where Q_n is the nth Pell-Lucas numbers. In [4], the author defined the Jacobsthal numbers and presented their characteristic identities. In [5], Daşdemir developed an interesting matrix technique to find relationships between the Pell, Pell-Lucas, and Modified Pell numbers. In [6] and [7], Daşdemir brought rich elementary context related to the Jacobsthal and Jacobsthal-Lucas numbers to the available literature by using some matrix identities. In [8], Arslan and Köken presented the Jacobsthal and Jacobsthal-Lucas numbers with rational subscripts based on the idea of computing square roots of the matrices of order 2×2. In [9], Catarino and Campos introduced the Gaussian Modified Pell numbers, including Binet's formula, the generating function, and some sum formula. In [10], Radicic computed determinants, eigenvalues, and the values and boundaries of certain norms for a k-circulant matrix involving the Pell Numbers. In [11], Daşdemir expanded the usual Mersene, Jacobsthal, and Jacobsthal-Lucas numbers to the ones with negative indexes. In [12], Soykan and Göcen presented the definition, Binet formula, and generating functions of the generalized hyperbolic Pell numbers over the bi-dimensional Clifford algebra. In [13], Uygun defined the bi-periodic Jacobsthal and bi-periodic Jacobsthal-Lucas numbers and discovered some features between them.

The above brief literature survey shows that many researchers have genuinely interested in investigating the elementary identities and properties of the Modified Pell numbers and the Jacobsthal numbers with structural configurations and this trend is growing day by day. Motivated by these developments, in this paper, we consider the Diophantine equations

$$J_k = q_m q_n \tag{5}$$

and

$$q_k = J_m J_n \tag{6}$$

for any positive integer k, m, and n under $m \leq n$. The fundamental outputs of the paper are to determine the m, n, and k numbers that satisfy Equations (5) and (6).

2. Auxiliary Descriptions

The following will be used extensively in the rest of the paper

Definition 1. *Let η be an algebraic number of degree d with minimal primitive polynomial over*

$$a_0 x^d + a_1 x^{d-1} + \cdots + a_d = a_0 \prod_{i=1}^{d} \left(x - \eta^{(i)} \right),$$

where a_0 is positive and $\eta^{(i)}$ is the conjugate of η. Then,

$$h(\eta) = \frac{1}{d}\left(\log|a_0| + \sum\nolimits_{i=1}^{d} \log\left(\max\left\{\left|\eta^{(i)}\right|, 1\right\}\right)\right), \tag{7}$$

is called the logarithmic height of η.

It should be noted that this function satisfies the following properties:

$$h(\alpha \mp \beta) \leqslant h(\alpha) + h(\beta) + \log 2,\ h\left(\alpha\beta^{\mp 1}\right) \leqslant h(\alpha) + h(\beta),\ \text{and } h(\alpha^s) = sh(\alpha).$$

Theorem 1 (Matveev [14])**.** *Let $\eta_1, \eta_2, \ldots, \eta_s$ be real algebraic numbers and let $b_1, b_2, \ldots, b_s$ be nonzero rational integers. Let $d_{\mathbb{K}}$ be the degree of the number field $\mathbb{Q}(\eta_1, \eta_2, \ldots, \eta_s)$ over $\mathbb{Q}$ and let A_j be the positive real number defined by*

$$A_j \geqslant h'(\eta_j) = \max\left\{d_{\mathbb{K}} h(\eta_j), \left|\log(\eta_j)\right|, 0.16\right\} \text{ for } j = 1, 2, \ldots, l.$$

Put

$$\Lambda = \eta_1\eta_2 \ldots \eta_l - 1 \text{ and } D = \max\{|b_1|, \ldots, |b_l|\}.$$

If $\Lambda \neq 0$, then

$$\log(|\Lambda|) > -1.4 \times 30^{l+3} \times l^{4.5} \times d_{\mathbb{K}}^2 \times (1 + \log(d_{\mathbb{L}}))(1 + \log(D)) A_1 A_2 \ldots A_l.$$

Lemma 1 (Dujella and Pethö [16])**.** *Let M be a positive integer, p/q be a convergent of the continued fraction of the irrational τ such that $q > 6M$, and let A, B, τ be positive rational numbers with $A > 0$ and $B > 1$. Let $\varepsilon = \|\mu q\| - M\|\tau q\|$, where $\|\cdot\|$ is the distance from the nearest integer. If $\varepsilon > 0$, then there is no integer solution (m, n, k) of inequality*

$$0 < m\tau - n + \mu < AB^{-k}$$

with

$$m \leqslant M \text{ and } k \geqslant \frac{\log(Aq/\varepsilon)}{\log B}.$$

Lemma 2. *Let k be a positive integer and let x, y, and z be positive real numbers. Further, let $\sqrt{x}$ and $\sqrt{z}$ be irrational numbers. Then, $\sqrt{x}(y + \sqrt{z})^k$ is an irrational number.*

Proof. Introduce $A := \sqrt{x}(y + \sqrt{z})^k$. From Binomial expansion, we can write

$$\begin{aligned} A &= \sqrt{x}\sum_{i=0}^{k} \binom{k}{i} y^{k-i}\left(\sqrt{z}\right)^i \\ &= \sqrt{x}\left[\binom{k}{0} y^k + \binom{k}{1} y^{k-1}\sqrt{z} + \binom{k}{2} y^{k-2}\left(\sqrt{z}\right)^2 + \ldots + \binom{k}{k}\left(\sqrt{z}\right)^k\right]. \end{aligned}$$

If k is even, then we have

$$A = \sqrt{x}(B + C) = B\sqrt{x} + C\sqrt{x},$$

where

$$B := y^k + \binom{k}{2} y^{k-2}\left(\sqrt{z}\right)^2 + \ldots + \binom{k}{k-2} y^2\left(\sqrt{z}\right)^{k-2} + \left(\sqrt{z}\right)^k,$$
$$C := \binom{k}{1} y^{k-1}\left(\sqrt{z}\right)^1 + \ldots + \binom{k}{k-1} y\left(\sqrt{z}\right)^{k-1}.$$

Here, $B \in \mathbb{Q}$, $C \in \mathbb{R} - \mathbb{Q}$, and $B, C > 0$. $C\sqrt{x}$ can be rational or irrational depending on xz. However, $B\sqrt{x} \in \mathbb{R} - \mathbb{Q}$ due to $B \in \mathbb{Q}$. As a result, $A \in \mathbb{R} - \mathbb{Q}$. When k is odd, a similar evaluation can be done. This completes the proof. □

3. Main Results

In this section, we present the fundamental outcomes of the paper.

Theorem 2. *Let k, m, and n be any positive integers $m \leqslant n$. Then, all the solutions to Equation* (5) *are*

$$(k, m, n) \in \{(1,1,1), (2,1,1), (3,1,2), (6,2,3)\} \tag{8}$$

and the ones of Equation (6)

$$(k, m, n) \in \{(1,1,1), (1,1,2), (1,2,2), (2,1,3), (2,2,3)\}. \tag{9}$$

Proof. For validation, we apply a proof strategy of two steps. To this aim, appropriate boundaries will be computed separately for Equations (5) and (6). First, let us consider Equation (5) by considering the equations

$$\alpha^{n-2} \leqslant J_n \leqslant \alpha^{n-1} \tag{10}$$

and

$$\gamma^{n-1} \leqslant q_n \leqslant \gamma^n. \tag{11}$$

These can be proved easily by applying the induction method n. Then, we can write

$$\alpha^{k-2} \leqslant J_k = q_n.q_m \leqslant \gamma^{m+n} \text{ and } \alpha^{k-1} \geqslant J_k = q_n.q_m \geqslant \gamma^{m+n-2}.$$

and

$$1 + \frac{\log\gamma}{\log\alpha}(m+n-2) \leqslant k \leqslant 2 + \frac{\log\gamma}{\log\alpha}(m+n),$$

concluding $k < 4n$. Further, using the Binet's formulas of the Jacobsthal and Modified Pell numbers yields

$$\begin{aligned}\left|\frac{\alpha^k}{3} - \frac{\gamma^{n+m}}{4}\right| &= \left|\frac{\beta^k}{3} + \frac{\gamma^n\delta^m + \gamma^m\delta^n + \delta^{n+m}}{4}\right| < \left|\frac{\beta^k}{3} + \frac{3\gamma^{n-m}}{4}\right| \\ &< \frac{3}{2}\max\left\{|\beta|^k, \gamma^{n-m}\right\} = \frac{3\gamma^{n-m}}{2}\end{aligned}$$

or equally

$$\left|\frac{4}{3}\alpha^k\gamma^{-n-m} - 1\right| < \frac{6}{\gamma^{2m}}. \tag{12}$$

Considering the Matveev's theroem, we consider the following case:

$$\Lambda_1 = \frac{4}{3}\alpha^k\gamma^{-n-m} - 1, l = 3, \eta_1 = \frac{1}{3}, \eta_2 = \alpha, \eta_3 = \gamma, d_1 = 1, d_2 = k+2, d_3 = -n-m.$$

Here, it is easy to verify $\Lambda_1 \neq 0$. If the reverse were true anyway, $\frac{4}{3}\alpha^k = \gamma^{n+m}$ would have to be. But, while $\frac{4}{3}\alpha^k \in \mathbb{Q}$, $\gamma^{n+m} \notin \mathbb{Q}$. In this case, the assertion is true. If choosing $\eta_1, \eta_2, \eta_3 \in \mathbb{L} := \mathbb{Q}\left(\sqrt{2}\right)$, $d_L = 2$. This means that

$$h\left(\eta_1\right) = \log 3,\ h\left(\eta_2\right) = \log\alpha,\ h\left(\eta_3\right) = \frac{\log\gamma}{2},\ A_1 = 2\log 3,\ A_2 = 2\log\alpha,\ A_3 = \log\gamma$$

and $D = 4n$. As a result, we have

$$\log\left|\Lambda_1\right| > -2.61 \times 10^{12}\left(1 + \log 4n\right). \tag{13}$$

As compared Equation (12) to Equation (13), we finally get

$$m\log\gamma + \log 3 < 1.4 \times 10^{12}\left(1 + \log 4n\right). \tag{14}$$

Doing some mathematical arrangements by using the Binet's formulas in Equations (4), we compute

$$\left|\frac{2}{3q_m}\alpha^k\gamma^{-n} - 1\right| < \frac{2}{\gamma^n}. \tag{15}$$

Accordingly, from Matveev's theorem, the following equations can be obtained:

$$\Lambda_2 = \frac{2}{3q_m}\alpha^k\gamma^{-n} - 1,\ l = 3, \eta_1 = 3q_m,\ \eta_2 = \alpha,\ \eta_3 = \gamma,\ d_1 = -1,\ d_2 = k+1,$$

and, $d_3 = -n$.

$$h\left(\eta_2\right) = \log\alpha,\ h\left(\eta_3\right) = \frac{1}{2}\log\gamma\ A_2 = 2\log\alpha, \text{ and } A_3 = \log\gamma.$$

It should be noted that η_1 is also a root of the polynomial $2x^2 - 9{P_m}^2$. Then,

$$h\left(\eta_1\right) = \frac{1}{2}\left(\log 1 + \log\left|3q_m\right| + \log\left|-3q_m\right|\right) = \log q_m + \log 3 \leqslant m\log\gamma + \log 3.$$

From Equation (15), we have

$$A_1 = 2.8 \times 10^{12}\left(1 + \log 4n\right) > 2h\left(\eta_1\right).$$

Letting $D = 4n$. In this case, by Lemma 2, we can find

$$\log\left|\Lambda_2\right| > -3.32 \times 10^{24}(1 + \log 4n)^2. \tag{16}$$

and from Equation (15),

$$\log\left|\Lambda_2\right| < \log 2 - n\log\gamma. \tag{17}$$

Solving Equations (16) and (17) together, we get

$$n < 1.72 \times 10^{28} \text{ and } k < 4n. \tag{18}$$

A similar method can be applied to Equation (6). Here, to reduce the size of the current paper, we neglect an explicit proof. But, for Equation (6), we get the following boundaries:

$$n < 9 \times 10^{28} \text{ and } k < 3n. \tag{19}$$

Accordingly, from both Equations (18) and (19), our widest solution range is as follows.

$$k < 4n \text{ and } n < 9 \times 10^{28}. \tag{20}$$

Everything is ok but since our last range is not economical, investigating a solution is very difficult. Therefore, we will take an additional approach, taking into account four different situations.

Case I: For the case where $m \geqslant 2$ in Equation (12), we define

$$\Gamma_1 := k \log \alpha - (n+m) \log \gamma + \log \frac{4}{3},$$

or in another form,

$$\left|e^{\Gamma_1} - 1\right| = |\Lambda_1| < \frac{6}{\gamma^{2m}} < \frac{1}{5}, \tag{21}$$

which means that $|\Gamma_1| < \frac{1}{4}$. By the way, $|x| < \frac{1}{4}$, $|x| < \frac{3}{2}|e^x - 1|$ holds for $x \in \mathbb{R}$ without the loss of generality. For the case where $x = \Gamma_1$, we obtain

$$|\Gamma_1| < \frac{9}{\gamma^{2m}}. \tag{22}$$

Due to $\Lambda_1 \neq 0$, $\Gamma_1 \neq 0$ too. As a result, $\Gamma_1 < 0$ or $\Gamma_1 > 0$. When $\Gamma_1 > 0$,

$$0 < k\left(\frac{\log \alpha}{\log \gamma}\right) - (n+m) + \left(\frac{\log(4/3)}{\log \gamma}\right) < \frac{9}{(\log \gamma)\, \gamma^{2m}} < \frac{11}{\gamma^{2m}}.$$

According to Dujella and Pethö's lemma for $M = 3.6 \times 10^{29}$, we get

$$\tau = \frac{\log \alpha}{\log \gamma}, \ \mu = \frac{\log(4/3)}{\log \gamma}, \ A = 11, \text{ and } B = \gamma^2. \tag{23}$$

$\tau = [a_0, a_1, ...]$ is turned on to the continued fraction as follows:

$$[a_0, ..., a_{60}] = \frac{p_{60}}{q_{60}} = \frac{633284722967420948224436714420 3}{805255281332277030875937803968 5}$$

such that $6M < 2.2 \times 10^{30} < q$. As a result, $\varepsilon = \|\mu q\| - M\|\tau q\| > 0.41$. This means that $m \leqslant 42$. Also, for the case where $\Gamma_1 < 0$, a similar result can be found.

Case II: In Equation (15) for $n > 1$ under the same assumptions, we find that $n \leqslant 89$.

Case III: In Equation (6) for the case $m \geqslant 4$, we conclude that $m \leqslant 109$.

Case IV: Similarly, In Equation (6) for the case where $n > 1$, we can obtain that $n \leqslant 117$.

According to all the above results, we obtain the widest range such as $n \leqslant 117$ and $k < 469$. If checking the possible cases by using a PC algorithm composed of in Mathematica, we see the intersection set such that $\{1, 3\}$. This exhausts the proof.

□

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References

[1] Vajda, S., Fibonacci and Lucas Numbers, and the Golden Section: Theory and Applications, Courier Corporation, New York, 2008.
[2] Koshy, T., Pell and Pell-Lucas Numbers with Applications, Springer, New York, 2014.
[3] Horadam, A. F., Applications of modified Pell numbers to representations, *Ulam Quarterly*, 3(1) (1994), 34–53.
[4] Horadam, A. F., Jacobsthal representation numbers, *Fibonacci Quart.*, 34(1) (1996), 40–54.
[5] Daşdemir, A., On the Pell, Pell-Lucas and Modified Pell numbers by matrix method, *Applied Mathematical Sciences*, 5(64) (2011), 3173–3181.
[6] Daşdemir, A., On the Jacobsthal numbers by matrix method, *SDU Journal of Science Journal of Science*, 7(1) (2012), 69–76.
[7] Daşdemir, A., A study on the Jacobsthal and Jacobsthal-Lucas numbers by matrix method, *DUFED Journal of Sciences*, 3(1) (2014), 13–18.
[8] Arslan, S., Köken, F., The Jacobsthal and Jacobsthal-Lucas numbers via square roots of matrices, *Int. Math. Forum.*, 11(11) (2016), 513–520. http://doi.org/10.12988/imf.2016.6442
[9] Catarino, P., Campos, H., A note on Gaussian Modified Pell numbers, *Journal of Information and Optimization Sciences*, 39(6) (2018), 1363–1371. http://doi.org/10.1080/02522667.2018.1471267
[10] Radicic, B., On k-circulant matrices involving the Pell numbers, *Results in Mathematics*, 74(4) (2019), 200. https://doi.org/10.1007/s00025-019-1121-9
[11] Daşdemir, A., Mersene, Jacobsthal, and Jacobsthal-Lucas numbers with negative subscripts, *Acta Math. Univ. Comenian.*, 88(1) (2019), 142–156.
[12] Soykan, Y., Göcen, M., Properties of hyperbolic generalized Pell numbers, *Notes on Number Theory and Discrete Mathematics*, 26(4) (2020), 136–153. http://doi.org/10.7546/nntdm.2020.26.4.136-153
[13] Uygun, Ş., The relations between bi-periodic jacobsthal and bi-periodic jacobsthal lucas sequence, *Cumhuriyet Science Journal*, 42(2) (2021), 346–357. http://doi.org/10.17776/csj.770080
[14] Matveev, E. M., An explicit lower bound for a homogeneous rational linear form in the logarithms of algebraic numbers. II, *II. Izv. Math.*, 64(6) (2000), 1217–1269. http://doi.org/10.1070/IM2000v064n06ABEH000314
[15] Baker, A., Davenport, H., The equations $3x^2 - 2 = y^2$ and $8x^2 - 7 = z^2$, *Quart. J. Math. Oxford Ser.*, 20(1) (1969), 129–137.
[16] Dujella, A., Pethö, A., A generalization of a theorem of Baker and Davenport, *Quart. J. Math. Oxford Ser.*, 49(195) (1998), 291–306. https://doi.org/10.1093/qjmath/49.195.291