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Helicoidal Surfaces with Prescribed Curvatures in Some Conformally Flat Pseudo-Spaces of Dimensional Three

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Abstract

In this work, we consider the problem of finding explicit parametrizations for non-degenerate helicoidal surfaces with prescribed curvatures in some conformally flat pseudo-spaces with conformal pseudo-metrics whose conformal factors are related to three types of generic cylindrical functions. In the first two, we get a two-parameter family of these surfaces with prescribed extrinsic curvature or mean curvature given by smooth functions. In the last one, we discover a one-parameter family when both curvatures of these surfaces are zero; however, we find a two-parameter family when either one of those curvatures is zero. Also, we support them with examples.

1. Introduction

Mathematics is a branch of science that examines the properties of both abstract shapes and measurable quantities through equivalences. When the abstract shape is a smooth manifold, the term *diffeomorphism* refers to the idea that two manifolds are equivalent in terms of differentiability. *Isometry* is a concept that expresses metrical equivalence between two Riemannian manifolds. The angle and distance between two directions are both preserved in the concept of isometry. Furthermore, the term *conformal* is a more general concept than isometry, in which only the angle is preserved but not the distance. Let g and $\bar{g}$ denote the metrics of two Riemannian (or pseudo-Riemannian) manifolds, respectively. These Riemannian manifolds are said to be conformally equivalent if there exists a differentiable function λ, known as a conformal factor, that provides the equality

$$g = \frac{1}{\lambda^2}\bar{g}.$$

It is common knowledge that when the metric $\bar{g}$ is the Euclidean metric, the Euclidean space R^3 is conformally equivalent to the sphere S^3 (or the hyperbolic space H^3) through suitable conformal factors. The Minkowski space R^3_1 and the de Sitter S^3_1 (or the anti-de Sitter H^3_1) are conformally equivalent in accordance with the Minkowski metric $\bar{g}$. Conformally flat spaces with a bounded conformal factor have gained more attention in recent years. Keep in mind that selecting such a conformal factor results in the metric becoming complete, and as a result, the space in question is referred to as a complete Riemannian manifold.
Particular types of special surfaces, such as rotational and helicoidal surfaces, are surveyed in conformally flat spaces. The construction of a helicoidal surface, in contrast to that of a rotational surface, is performed using screw motions, which include a translation in addition to a rotation around an axis. The problem of finding the explicit parameterization of helicoidal

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surfaces are done in a wide variety of spaces. It should not be surprising that the properties of such surfaces remain unchanged under screw motions. So, it is important to determine the proper conformal factor if you want to conduct surveys of the above-mentioned surfaces in conformally flat spaces. A function f is said to be invariant under a transformation T of space into itself if the property $f(Tx) = f(x)$ holds for all x. If the conformal factor λ is a function satisfying this condition, then it makes sense to think of such surfaces in conformally flat spaces. An estimation for this kind of function can be found in the cartesian equation of standard geometric shapes like the sphere and the cylinder. Unlike the spherical function, which is only invariant under rotational symmetry, the cylindrical function is invariant under both rotational symmetry and translational symmetry. See [1, 2] for information on surveys carried out in the context of the spherical function. Refer to [3, 4, 5, 6, 7, 8] for the other one.
In a recent survey, Yerlikaya introduces the conformally flat pseudo-space of dimensional three and presents a non-degenerate surface's curvatures in this space. After investigating the presence of helicoidal surfaces in some conformally flat pseudo-spaces, the author works on the problem of determining the explicit parametrization of those surfaces. The author means by "some pseudo-spaces" that it is a conformally flat pseudo-space with a pseudo-metric that corresponds to the determined conformal factor, where the author preferred such a way that this pseudo-metric is a solution to the famous Einstein field equation. The process for determining such a conformal factor, which was just explained as being significant, is carried out in accordance with the causal character of the axis of helicoidal surfaces (see [8] for detail). In this study, we discuss the same problem for generic cylindrical conformal factors, taking the causal character of the axis into account once again.

2. Preliminaries

2.1. Basic Notations

Equipped the Minkowski space $\mathbb{R}^3_1$ with a conformally flat pseudo-metric specified by the angle-bracket notation

$$\langle w_1, w_2 \rangle_{g_\lambda} = \frac{1}{\lambda^2(p)} \langle w_1, w_2 \rangle_L, \quad \forall w_1, w_2 \in T_p\mathbb{R}^3_1,\ \forall p \in \mathbb{R}^3_1,$$

the resulting space is said to be the complete pseudo-Riemannian manifold if the conformal factor λ is bounded. From now on, unless otherwise stated, we shall refer to this pseudo-manifold as the conformally flat pseudo-space and represented by $\mathbb{F}^1_3$. Here, note that the pseudo-metric $\langle , \rangle_L$ is the Minkowski metric whose coefficients are

$$\langle e_1, e_1 \rangle_L = -1, \ \langle e_i, e_j \rangle_L = 1 \quad \text{for} \ 2 \le i = j \le 3,$$

$$\langle e_i, e_j \rangle_L = 0, \ \text{for} \ 1 \le i \ne j \le 3.$$

On the other hand, let M be a non-degenerate surface in $\mathbb{F}^1_3$. Given the Gaussian curvature K and the mean curvature H of the surface M in $\mathbb{R}^3_1$, both the extrinsic and mean curvatures of M are calculated as

$$\widetilde{K}_E = \varepsilon h^2 - 2H\lambda h + \lambda^2 K \tag{2.1}$$

and

$$\widetilde{H} = \lambda H - \varepsilon h, \quad \varepsilon = \langle N, N \rangle \tag{2.2}$$

respectively. For the unit normal vector field N of M, the function h holds $h = h(s,t) = \sum_{j=1}^{3} N^j \frac{\partial \lambda}{\partial x_j}$, where x_j for $1 \le j \le 3$ is an usual coordinate system of $\mathbb{R}^3_1$ [8].

2.2. The Process for Determining Proper Conformal Factors

Conformally flat spaces acquire special qualities by means of their conformal factors. If a space with a conformal factor λ has a transformation T that satisfies the equation $\lambda(Tx) = \lambda(x)$ for all x, then the space is invariant under the transformation T along an axis, as stated in the introduction. Thereby, it makes sense in this space to consider surfaces generated by the transformation T, because the properties of such surfaces remain unchanged under the one T. The infinitesimal isometry group, which consists of both the rotation and translation transformations of space, is represented by its Killing vector field. In [8], the author performs a calculation to determine the Killing vector field for conformal factors that correspond to the causal character of the axis.
Since we will be discussing helicoidal surfaces in this work, we consider that the transformation T must be both rotational and translational. The type of function that is invariant under both rotational and translational transformations is of the cylindrical type.

The spacelike axis: In this case, we take the conformal factor λ_h in

$$\lambda_h : \mathbb{R}^3_1 \to \mathbb{R}, \quad (x_1,x_2,x_3) \to \lambda_h(x_1,x_2,x_3) = \sqrt{x_2^2 - x_1^2},$$

which is directly related to a Lorentzian hyperbolic cylinder with x_3-axis in $\mathbb{R}^3_1$. In [8], taking the procedure of determining the Killing vector field into account, for the determined conformal factor above, we compute the Killing vector field V in $\left(\mathbb{F}^1_3\right)_{\lambda_h}$ as

$$V = c_1\left(x_2\frac{\partial}{\partial x_1} + x_1\frac{\partial}{\partial x_2}\right) + c_2\frac{\partial}{\partial x_3}$$

for some constants c_1 and c_2. This means that the corresponding isometry group consists of the translation along the x_3 axis given by

$$T(x_1,x_2,x_3) = (x_1,x_2,x_3+t) \tag{2.3}$$

and the rotation around the x_3 axis given by

$$R(x_1,x_2,x_3) = (x_1\cosh t + x_2\sinh t, x_1\sinh t + x_2\cosh t, x_3), \tag{2.4}$$

where $(x_1,x_2,x_3) \in \mathbb{R}^3_1$ and $t \in \mathbb{R}$.

Remark 2.1. *When considering the conformal factor λ_h, it is evident that Eqs.* (2.3) *and* (2.4) *both satisfy the function invariance mentioned in the introduction.*

So, we can maintain the following process.

When applying the profile curve $\gamma(s) = (0,s,n(s))$ to Eq. (2.4) together with Eq. (2.3), i.e.,

$$\begin{pmatrix} \cosh t & \sinh t & 0 \\ \sinh t & \cosh t & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ s \\ n(s) \end{pmatrix} + c\begin{pmatrix} 0 \\ 0 \\ t \end{pmatrix},$$

the resulting surface is said to be a helicoidal surface in $\left(\mathbb{F}^1_3\right)_{\lambda_h}$ with the spacelike axis of rotation, and so its parametrization is represented by

$$X : I \times \mathbb{R} \to \left(\mathbb{F}^1_3\right)_{\lambda_h}; (s,t) \to X(s,t) = \Big(s\sinh t, s\cosh t, n(s) + ct\Big), \tag{2.5}$$

where $n(s)$ is a smooth function and c is a constant.

The timelike axis: In this case, we determine the conformal factor λ_c as

$$\lambda_c : \mathbb{R}^3_1 \to \mathbb{R}, \quad (x_1,x_2,x_3) \to \lambda_h(x_1,x_2,x_3) = \sqrt{x_2^2 + x_3^2}.$$

This is related to a Lorentzian circular cylinder with x_1-axis in $\mathbb{R}^3_1$. Similarly, we compute the Killing vector field V in $\left(\mathbb{F}^1_3\right)_{\lambda_c}$ as

$$V = c_1\frac{\partial}{\partial x_1} + c_2\left(-x_3\frac{\partial}{\partial x_2} + x_2\frac{\partial}{\partial x_3}\right)$$

for some constants c_1 and c_2. This means that the corresponding isometry group consists of the translation along the x_1 axis given by

$$T(x_1,x_2,x_3) = (x_1+t,x_2,x_3) \tag{2.6}$$

and the rotation around the x_1 axis given by

$$R(x_1,x_2,x_3) = (x_1, x_2\cos t - x_3\sin t, x_2\sin t + x_3\cos t), \tag{2.7}$$

where $(x_1,x_2,x_3) \in \mathbb{R}^3_1$ and $t \in \mathbb{R}$.

Remark 2.2. *When considering the conformal factor λ_c, it is evident that Eqs.* (2.6) *and* (2.7) *both satisfy the function invariance mentioned in the introduction.*

In a manner that is analogous to the process that was just outlined, using Eq. (2.7) together with Eq. (2.6), we get a helicoidal surface in $\left(\mathbb{F}_3^1\right)_{\lambda_c}$ with the timelike axis of rotation, and so its parametrization is represented by

$$X : I \times \mathbb{R} \to \left(\mathbb{F}_3^1\right)_{\lambda_c}; (s,t) \to X(s,t) = \Big(n(s) + ct, s\cos t, s\sin t\Big). \tag{2.8}$$

The lightlike axis: In this case, we determine the conformal factor λ_p as

$$\lambda_p : \mathbb{R}_1^3 \to \mathbb{R}, \quad (x_1,x_2,x_3) \to \lambda_p(x_1,x_2,x_3) = \sqrt{x_2 - x_1}.$$

This is related to a Lorentzian parabolic cylinder with $(1,1,0)$-axis in $\mathbb{R}_1^3$. Thus, we compute the Killing vector field V in $\left(\mathbb{F}_3^1\right)_{\lambda_p}$ as

$$V = c_1\left(-x_3\frac{\partial}{\partial x_1} - x_3\frac{\partial}{\partial x_2} + (-x_1 + x_2)\frac{\partial}{\partial x_3}\right) + c_2\left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}\right).$$

for some constants c_1 and c_2. This means that the corresponding isometry group consists of the translation along the $(1,1,0)$ axis given by

$$T(x_1,x_2,x_3) = (x_1 + t, x_2 + t, x_3) \tag{2.9}$$

and the rotation around the $(1,1,0)$ axis given by

$$R(x_1,x_2,x_3) = \Big(\left(1 + t^2/2\right)x_1 - \left(t^2/2\right)x_2 + tx_3, \left(t^2/2\right)x_1 + \left(1 - t^2/2\right)x_2 + tx_3, tx_1 - tx_2 + x_3\Big), \tag{2.10}$$

where $(x_1,x_2,x_3) \in \mathbb{R}_1^3$ and $t \in \mathbb{R}$.

Remark 2.3. *When considering the conformal factor λ_p, it is evident that Eqs.* (2.9) *and* (2.10) *both satisfy the function invariance mentioned in the introduction.*

Similarly, from Eqs. (2.10) and (2.9), we have a helicoidal surface in $\left(\mathbb{F}_3^1\right)_{\lambda_p}$ with the lightlike axis of rotation generated by the profile curve $\gamma(s) = (s, n(s), 0)$. A parametrization of this surface is as follows

$$X : I \times \mathbb{R} \to \left(\mathbb{F}_3^1\right)_{\lambda_p}; (s,t) \to X(s,t) = \Big(\left(1 + t^2/2\right)s - \left(t^2/2\right)n(s) + ct, \left(t^2/2\right)s + \left(1 - t^2/2\right)n(s) + ct, (s - n(s))t\Big). \tag{2.11}$$

3. Results

Firstly, let's take a look at the helicoidal surface with the spacelike axis of rotation given by Eq. (2.5). For this surface, we have

$$K(s) = \frac{s^3 n' n'' + c^2}{\left[-c^2 + s^2\left(1 + n'^2\right)\right]\left|-c^2 + s^2\left(1 + n'^2\right)\right|} \tag{3.1}$$

and

$$H(s) = -\frac{s^2 n'^3 + n'\left(-2c^2 + s^2\right) + sn''\left(-c^2 + s^2\right)}{2\left[-c^2 + s^2\left(1 + n'^2\right)\right]\left|-c^2 + s^2\left(1 + n'^2\right)\right|^{\frac{1}{2}}}, \tag{3.2}$$

where the first one is the Gaussian curvature and the other one is the mean curvature. Assuming that $EG - F^2 = c^2 - s^2\left(1 + n'^2\right) < 0$, meanings that this surface is timelike, Eqs. (3.1) and (3.2) yield

$$K(s) = \frac{s^3 n' n'' + c^2}{\left[-c^2 + s^2\left(1 + n'^2\right)\right]^2} \tag{3.3}$$

and

$$H(s) = -\frac{s^2 n'^3 + n'\left(-2c^2 + s^2\right) + sn''\left(-c^2 + s^2\right)}{2\left[-c^2 + s^2\left(1 + n'^2\right)\right]^{\frac{3}{2}}}, \tag{3.4}$$

respectively. For the determined conformal factor λ_h, we have

$$\sum_{j=1}^{3} N^j \lambda_{,j} = \frac{sn'}{\sqrt{-c^2+s^2\left(1+n'^2\right)}}. \tag{3.5}$$

Inserting Eqs. (3.3), (3.4) and (3.5) into Eqs. (2.1) and (2.2), we get

$$\widetilde{K}_{E_{\lambda_h}} = \frac{s^2\left(sn'n''\left(-c^2+2s^2\right)+2s^2n'^4+n'^2\left(2s^2-3c^2\right)+c^2\right)}{\left[-c^2+s^2\left(1+n'^2\right)\right]^2} \tag{3.6}$$

and

$$\widetilde{H}_{\lambda_h} = -\frac{s\left(sn''\left(-c^2+s^2\right)+3s^2n'^3+n'\left(3s^2-4c^2\right)\right)}{2\left[-c^2+s^2\left(1+n'^2\right)\right]^{\frac{3}{2}}}. \tag{3.7}$$

In order to find a solution to Eq. (3.6), we turn Eq. (3.6) into

$$A'(s)+\left(\frac{5}{s}+\frac{4s}{c^2-2s^2}\right)A(s) = \frac{4}{c^2-2s^2}+\frac{2}{s^2}\widetilde{K}_{E_{\lambda_h}}(s), \tag{3.8}$$

where

$$A\left(s\right) = \frac{sn'^2-s}{-c^2+s^2\left(1+n'^2\right)}. \tag{3.9}$$

The general solution to Eq. (3.8) becomes

$$A(s) = \frac{c^2-2s^2}{s^5}\left[\int \frac{s^5}{c^2-2s^2}\left\{\frac{4}{c^2-2s^2}+\frac{2}{s^2}\widetilde{K}_{E_{\lambda_h}}(s)\right\}ds+c_1\right], \tag{3.10}$$

where c_1 is constant. When we compare Eq. (3.9) with Eq. (3.10), we obtain

$$\left[s^2\left(s^4-\psi(s)\right)\right]n'^2(s) = s^6+\left(s^2-c^2\right)\psi(s), \tag{3.11}$$

where

$$\psi(s) = \left(c^2-2s^2\right)\left[\int \frac{s^5}{c^2-2s^2}\left\{\frac{4}{c^2-2s^2}+\frac{2}{s^2}\widetilde{K}_{E_{\lambda_h}}(s)\right\}ds+c_1\right]. \tag{3.12}$$

The next theorem can now be established:

Theorem 3.1. *Let $\gamma(s) = (0,s,n(s))$ be a profile curve of the timelike helicoidal surface (2.5) in $\left(\mathbb{F}_3^1\right)_{\lambda_h}$. Assuming its extrinsic curvature at the point $(0,s,n(s))$ is represented by $\widetilde{K}_{E_{\lambda_h}}(s)$, there exists an open subinterval $\widetilde{I} \subset I$ concerning c_1 such that the function $n(s)$ is*

$$n\left(s\right) = \pm\int \frac{\sqrt{\left|s^6+\left(s^2-c^2\right)\psi(s)\right|}}{|s|\sqrt{\left|s^4-\psi(s)\right|}}ds+c_2, \tag{3.13}$$

where $\psi(s)$ is given by Eq.(3.12) and c_2 is a constant. Also, for the designated constant c_1 and some constants c and c_2, there exists the two-parameter family of curves such that

$$\gamma(s;\widetilde{K}_{E_{\lambda_h}}(s),c,c_1,c_2) = \left(0,s,\pm\int \frac{\sqrt{\left|s^6+\left(s^2-c^2\right)\psi(s)\right|}}{|s|\sqrt{s^4-\psi(s)}}ds+c_2\right), \quad s\in\widetilde{I}\cap\left(\mathbb{R}\backslash\left(-c/\sqrt{2},c/\sqrt{2}\right)\right) \tag{3.14}$$

Proof. For the known function $\widetilde{K}_{E_{\lambda_h}} : I \to \mathrm{R}$ referred to as the extrinsic curvature of the helicoidal surface (2.5), we have Eq. (3.11). For an arbitrary number c_1, we establish a function $\mathscr{F}(s, c_1) = s^4 - \psi$ on the product of sub-intervals containing the numbers s_0 and $\widetilde{c_1}$ which satisfy the equality

$$\widetilde{c_1} = -\left(\int \frac{s^5}{c^2 - 2s^2}\left\{\frac{4}{c^2 - 2s^2} + \frac{2}{s^2}\widetilde{K}_{E_{\lambda_h}}(s)\right\}ds\right)(s_0).$$

Based on the function $\mathscr{F}$ to be continuous, we find a product set $\widetilde{I} \times J$ where Eq. (3.11) turns into

$$n'^2(s) = \frac{s^6 + (s^2 - c^2)\,\psi(s)}{s^2\left(s^4 - \psi(s)\right)},$$

which is the equation whose integration gives Eq. (3.13). Combining Eqs. (3.9) and (3.10), we obtain Eq. (3.14). □

Theorem 3.2. *Let c and c_2 be arbitrary constants. Thus, for any c_1 and a smooth function $\widetilde{K}_{E_{\lambda_h}}$, setting an open subinterval $\widetilde{I}$ of I in which the function $n(s)$ given by Eq. (3.13) is defined, we can construct the two-parameter family of timelike helicoidal surfaces defined on $\widetilde{I} \times \mathbb{R} \subset \mathbb{R}^2$, with the extrinsic curvature $\widetilde{K}_{E_{\lambda_h}}(s)$, with the profile curve $\gamma(s; \widetilde{K}_{E_{\lambda_h}}(s), c, c_1, c_2)$, $s \in \widetilde{I}$.*

Proof. Eq. (3.13) is a solution to Eq. (3.8), which implies the requirement that concludes the proof. When considering the sub-intervals that are stated in the proof of Theorem (3.1), for two numbers $c_1 \in J$, $c_2 \in \mathbb{R}$, a function $\widetilde{K}_{E_{\lambda_h}}(s)$ and $c \in \mathbb{R}$, we obtain the desired family defined on $\widetilde{I} \times \mathbb{R} \subset \mathbb{R}^2$. □

Now considering Eq. (3.7), we constitute

$$B'(s) + \frac{4}{s}B(s) = -\frac{2}{s^2}\widetilde{H}_{\lambda_h}(s), \qquad s \neq 0 \tag{3.15}$$

where

$$B(s) = \frac{n'}{\sqrt{-c^2 + s^2\left(1 + n'^2\right)}}, \tag{3.16}$$

which implies that the general solution to Eq. (3.15) becomes

$$B(s) = -\frac{1}{s^4}\left[\int 2s^2\widetilde{H}_{\lambda_h}(s)\,ds + c_1\right], \tag{3.17}$$

where c_1 is constant. With Eqs. (3.16) and (3.17), we have

$$\left[s^2\left(s^6 - \left(\int 2s^2\widetilde{H}_{\lambda_h}(s)\,ds + c_1\right)^2\right)\right]n'^2(s) = \left(s^2 - c^2\right)\left(\int 2s^2\widetilde{H}_{\lambda_h}(s)\,ds + c_1\right)^2.$$

Theorem 3.3. *Let $\gamma(s) = (0, s, n(s))$ be a profile curve of the timelike helicoidal surface (2.5) in $\left(\mathbb{F}_3^1\right)_{\lambda_h}$. Assuming its mean curvature at the point $(0, s, n(s))$ is represented by $\widetilde{H}_{\lambda_h}(s)$, there exists an open subinterval $\widetilde{I} \subset I$ relating to c_1 such that the function $n(s)$ is*

$$n(s) = \pm\int \frac{\sqrt{|s^2 - c^2|}\left|\int 2s^2\widetilde{H}_{\lambda_h}(s)\,ds + c_1\right|}{|s|\left(\left|s^6 - \left(\int 2s^2\widetilde{H}_{\lambda_h}(s)\,ds + c_1\right)^2\right|\right)^{\frac{1}{2}}}ds + c_2, \tag{3.18}$$

where c_2 is a constant. Also, for the designated constant c_1 and some constants c and c_2, there exists the two-parameter family of curves such that

$$\gamma(s; \widetilde{H}_{\lambda_h}(s), c, c_1, c_2) = \left(0, s, \int \frac{\sqrt{s^2 - c^2}\left|\int 2s^2\widetilde{H}_{\lambda_h}(s)\,ds + c_1\right|}{|s|\left(s^6 - \left(\int 2s^2\widetilde{H}_{\lambda_h}(s)\,ds + c_1\right)^2\right)^{\frac{1}{2}}}ds + c_2\right), \; s \in \widetilde{I} \cap (\mathbb{R} \backslash (-c, c)). \tag{3.19}$$

Proof. Define the function $\mathscr{F}$ to be $(s,c_1) \to s^6 - \left(\int 2s^2 \widetilde{H}_{\lambda_h}(s)\, ds + c_1\right)^2$. So, the technique for proving that Eqs. (3.18) and (3.19) can be obtained for the known function $\widetilde{H}_{\lambda_h}$ referred to as the mean curvature of the helicoidal surface (2.5) is the same as that of Theorem (3.1). □

Example 3.4. *Let the mean curvature be $\widetilde{H}_{\lambda_h}(s) = -\frac{1}{s}$, which implies that the function $\mathscr{F}$ amounts $s^6 - (s^2+c_1)^2$. So, we get the inequality $s^2(1-s) < c_1 < s^2(1+s)$ such that $\mathscr{F}$ is positive. Establish two functions $f(s) := s^2(1+s)$ and $g(s) := s^2(1-s)$. For $c_1 = 0$, consider the interval $\left(11/10, \sqrt{2}\right)$. Thus, the number c_1 falls in the interval $\left(2-\sqrt{8}, 2+\sqrt{8}\right)$. In this way, we determine the sub-interval $\widetilde{I}$ of I to be $\left(11/10, \sqrt{2}\right)$ in accordance with the positivity of $\mathscr{F}$. Ultimately, for $c_2 = 0$ and $c = 1$, Eq. (3.19) lead into*

$$\gamma(s) = \left(0, s, \ln s\right), \qquad s \in \left(11/10, \sqrt{2}\right).$$

Replacing the last one into Eq. (2.5), we get

$$X(s,t) = \left(s\sinh t, s\cosh t, \ln s + t\right),$$

which is the parametrization of a timelike helicoidal surface.

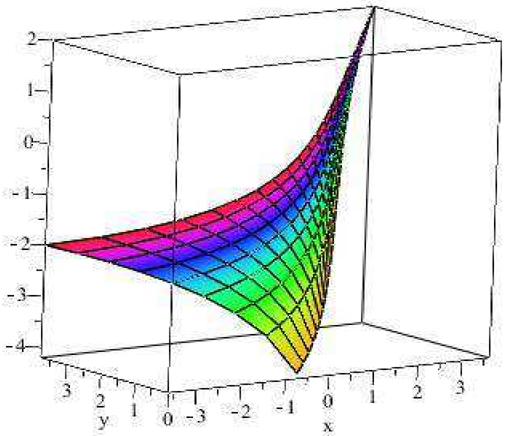

Figure 3.1: The graphic belongs to a timelike helicoidal surface of spacelike axis of rotation with $\widetilde{H}_{\lambda_h}(s) = -\frac{1}{s}$

Theorem 3.5. *Let c and c_2 be arbitrary constants. Thus, for any c_1 and a smooth function $\widetilde{H}_{\lambda_h}$, setting an open subinterval $\widetilde{I}$ of I in which the function $n(s)$ given by Eq. (3.18) is defined, we can construct the two-parameter family of timelike helicoidal surfaces defined on $\widetilde{I} \times \mathbb{R} \subset \mathbb{R}^2$, with the mean curvature $\widetilde{H}_{\lambda_h}(s)$, with the profile curve $\gamma(s; \widetilde{H}_{\lambda_h}(s), c, c_1, c_2)$, $s \in \widetilde{I}$.*

Proof. The process for proving that the desired family can be construct is the same as that of Theorem (3.2). □

Remark 3.6. *It is important to remember that analogous outcomes may be obtained if we pick $EG - F^2 = c^2 - s^2\left(1 + n'^2\right) > 0$, meanings that the helicoidal surface in $\left(\mathbb{F}_3^1\right)_{\lambda_h}$ is spacelike.*

Now, we consider the helicoidal surface given by Eq. (2.8). For this surface, taking $EG - F^2 = -c^2 + s^2\left(1 - n'^2\right) > 0$ into account, we have

$$K(s) = \frac{-s^3 n' n'' + c^2}{\left[-c^2 + s^2\left(1 - n'^2\right)\right]^2} \tag{3.20}$$

and

$$H(s)=\frac{s^2n'^3+n'\left(2c^2-s^2\right)-sn''\left(-c^2+s^2\right)}{2\left[-c^2+s^2\left(1-n'^2\right)\right]^{\frac{3}{2}}}. \tag{3.21}$$

Also, we get

$$\sum_{j=1}^{3}N^j\lambda_{,j}=-\frac{sn'}{\sqrt{-c^2+s^2\left(1-n'^2\right)}}. \tag{3.22}$$

Considering Eqs. (3.20), (3.21) and (3.22) together, from Eqs. (2.1) and (2.2), we obtain

$$\widetilde{K}_{E_{\lambda_h}}=\frac{s^2\left(sn'n''\left(c^2-2s^2\right)+2s^2n'^4+n'^2\left(3c^2-2s^2\right)+c^2\right)}{\left[-c^2+s^2\left(1-n'^2\right)\right]^2} \tag{3.23}$$

and

$$\widetilde{H}_{\lambda_h}=\frac{s\left(sn''\left(c^2-s^2\right)+3s^2n'^3+n'\left(4c^2-3s^2\right)\right)}{2\left[-c^2+s^2\left(1-n'^2\right)\right]^{\frac{3}{2}}}. \tag{3.24}$$

In a similar way, Eq. (3.23) turns into

$$A'(s)+\left(\frac{5}{s}-\frac{4s}{2s^2-c^2}\right)A(s)=\frac{4}{2s^2-c^2}-\frac{2}{s^2}\widetilde{K}_{E_{\lambda_c}}(s), \tag{3.25}$$

where

$$A(s)=\frac{sn'^2+s}{-c^2+s^2\left(1-n'^2\right)}. \tag{3.26}$$

The general solution to Eq. (3.25) becomes

$$A(s)=\frac{2s^2-c^2}{s^5}\left[\int\frac{s^5}{2s^2-c^2}\left\{\frac{4}{2s^2-c^2}-\frac{2}{s^2}\widetilde{K}_{E_{\lambda_c}}(s)\right\}ds+c_1\right], \tag{3.27}$$

where c_1 is constant. Comparing Eqs. (3.26) with (3.27), we have

$$\left[s^2\left(s^4+\psi(s)\right)\right]n'^2(s)=-s^6+\left(s^2-c^2\right)\psi(s), \tag{3.28}$$

where

$$\psi(s)=\left(2s^2-c^2\right)\left[\int\frac{s^5}{2s^2-c^2}\left\{\frac{4}{2s^2-c^2}-\frac{2}{s^2}\widetilde{K}_{E_{\lambda_c}}(s)\right\}ds+c_1\right].$$

In light of the assumed sign of $EG-F^2$, it follows that $s^4+\psi(s)=\frac{s^4\left(2s^2-c^2\right)}{-c^2+s^2-s^2n'^2}>0$. Thus, from (3.28), we can write

$$n(s)=\pm\int\frac{\sqrt{\left|\left(s^2-c^2\right)\psi(s)-s^6\right|}}{|s|\sqrt{s^4+\psi(s)}}ds+c_2 \tag{3.29}$$

Theorem 3.7. *Let $\gamma(s)=(n(s),s,0)$ be a profile curve of the spacelike helicoidal surface (2.8) in $\left(\mathbb{F}_3^1\right)_{\lambda_c}$. Assuming its extrinsic curvature at the point $(n(s),s,0)$ is represented by $\widetilde{K}_{E_{\lambda_c}}(s)$, for some constant c, c_1 and c_2, there exists the two-parameter family of the spacelike helicoidal surface constituted by curves*

$$\gamma(s;\widetilde{K}_{E_{\lambda_c}}(s),c,c_1,c_2)=\left(\pm\int\frac{\sqrt{\left|\left(s^2-c^2\right)\psi(s)-s^6\right|}}{|s|\sqrt{s^4+\psi(s)}}ds+c_2,s,0\right),\quad s\in\mathbb{R}\backslash(-c,c).$$

Inversely, let c and c_2 be arbitrary constants. Thus, for any c_1 and a smooth function $\widetilde{K}_{E_{\lambda_c}}(s)$, we can construct the two-parameter family of spacelike helicoidal surfaces defined on $\widetilde{I}\times\mathbb{R}\subset\mathbb{R}^2$, with the extrinsic curvature $\widetilde{K}_{E_{\lambda_c}}(s)$, with the profile curve $\gamma(s;\widetilde{K}_{E_{\lambda_c}}(s),c,c_1,c_2)$, $s\in\widetilde{I}$.

Proof. For the known function $\widetilde{K}_{E_{\lambda_c}}$, it is seen that the function $n(s)$ takes Eq. (3.29), which means that concludes the necessity. By defining the function $\mathscr{F}$ to be

$$(s,c_1)\to s^4+\left(2s^2-c^2\right)\left[\int\frac{s^5}{2s^2-c^2}\left\{\frac{4}{2s^2-c^2}-\frac{2}{s^2}\widetilde{K}_{E_{\lambda_c}}(s)\right\}ds+c_1\right],$$

it is possible to perform the inverse of the proof in a manner similar to Theorem (3.1). □

Example 3.8. *Let the extrinsic curvature be $\widetilde{K}_{E_{\lambda_c}}(s)=\frac{2s^2(6s^2+5)}{(2s^2+1)^2}$. For $c_1=0$, we find $\psi(s)=-\frac{s^2(3s^4+1)}{2s^2+1}$. For c_1, using Eq. (3.29), we get the profile curve to be $\gamma(s)=\left(\sqrt{3}s+\sqrt{2}+t,s,0\right)$. Thus, we write the parametrization of the corresponding helicoidal surface as*

$$X(s,t)=\left(\sqrt{3}s+\sqrt{2}+t,s\cos t,s\sin t\right).$$

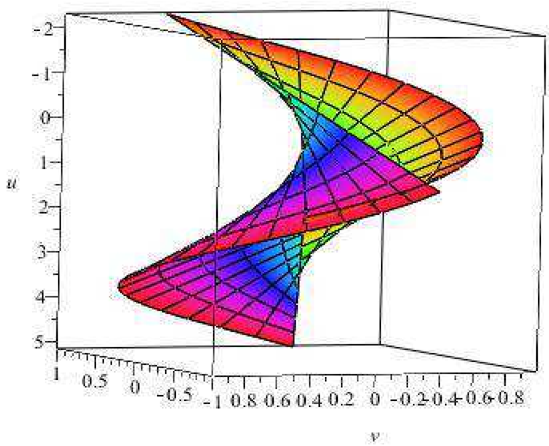

Figure 3.2: The graphic belongs to a spacelike helicoidal surface of timelike axis of rotation with $\widetilde{K}_{E_{\lambda_c}}(s)=\frac{2s^2(6s^2+5)}{(2s^2+1)^2}$

In a similar way, from Eq. (3.24), we write

$$B'(s)+\frac{4}{s}B(s)=-\frac{2}{s^2}\widetilde{H}_{\lambda_c}(s),\qquad s\neq 0 \tag{3.30}$$

where

$$B(s)=\frac{n'}{\sqrt{-c^2+s^2\left(1-n'^2\right)}}. \tag{3.31}$$

The general solution to Eq. (3.30) becomes

$$B(s)=-\frac{1}{s^4}\left[\int 2s^2\widetilde{H}_{\lambda_c}(s)\,ds+c_1\right], \tag{3.32}$$

where c_1 is constant. From Eq. (3.31) and Eq. (3.32), we obtain

$$n(s)=\pm\int\frac{\sqrt{s^2-c^2}\left|\int 2s^2\widetilde{H}_{\lambda_c}(s)\,ds+c_1\right|}{|s|\left(s^6+\left(\int 2s^2\widetilde{H}_{\lambda_c}(s)\,ds+c_1\right)^2\right)^{\frac{1}{2}}}ds+c_2,$$

Theorem 3.9. *Let $\gamma(s)=(n(s),s,0)$ be a profile curve of the spacelike helicoidal surface (2.8) in $\left(\mathbb{F}_3^1\right)_{\lambda_c}$. Assuming its mean curvature at the point $(n(s),s,0)$ is represented by $\widetilde{H}_{\lambda_c}(s)$, for some constant c, c_1 and c_2, there exists the two-parameter family of the spacelike helicoidal surface constituted by curves*

$$\gamma(s;\widetilde{H}_{\lambda_c}(s),c,c_1,c_2)=\left(\int\frac{\sqrt{s^2-c^2}\left|\int 2s^2\widetilde{H}_{\lambda_c}(s)\,ds+c_1\right|}{|s|\left(s^6+\left(\int 2s^2\widetilde{H}_{\lambda_c}(s)\,ds+c_1\right)^2\right)^{\frac{1}{2}}}ds+c_2,s,0\right),\quad s\in\mathbb{R}\backslash(-c,c)$$

Inversely, let c and c_2 be arbitrary constants. Thus, for any c_1 and a smooth function $\widetilde{H}_{\lambda_c}(s)$, we can construct the two-parameter family of spacelike helicoidal surfaces defined on $\widetilde{I}\times\mathbb{R}\subset\mathbb{R}^2$, with the mean curvature $\widetilde{H}_{\lambda_c}(s)$, with the profile curve $\gamma(s;\widetilde{H}_{\lambda_c}(s),c,c_1,c_2)$, $s\in\widetilde{I}$.

Proof. If the function $\widetilde{H}_{\lambda_c}(s)$ is known and by defining the function $\mathscr{F}$ to be

$$(s,c_1)\to s^6+\left(\int 2s^2\widetilde{H}_{\lambda_c}(s)\,ds+c_1\right)^2,$$

then the proof is reduced to nothing more than the proof of Theorem (3.7). □

Remark 3.10. *It is important to remember that analogous outcomes may be obtained if we pick $EG-F^2=-c^2+s^2\left(1-n'^2\right)<0$, meanings that the helicoidal surface in $\left(\mathbb{F}_3^1\right)_{\lambda_c}$ is timelike.*

Finally, for the helicoidal surface given by Eq. (2.11), we have

$$K(s)=\frac{n''(n-s)^3+c^2(1-n')^3}{(1-n')\left[(n-s)^2(n'+1)+c^2(1-n')\right]^2}\tag{3.33}$$

and

$$H(s)=-\frac{n''(n-s)^3+2c^2(1-n')^3+(n-s)^2(n'+1)(1-n')^2}{2\left[(1-n')\left((n-s)^2(n'+1)+c^2(1-n')\right)\right]^{\frac{3}{2}}}.\tag{3.34}$$

Assuming $EG-F^2<0$, observe that $1-n'>0$ and taking the conformal factor λ_p into account, we compute as

$$\sum_{j=1}^{3}N^j\lambda_{,j}=\frac{\sqrt{n-s}\,(1-n')}{2\sqrt{(1-n')\left((n-s)^2(n'+1)+c^2(1-n')\right)}}.\tag{3.35}$$

Inserting Eqs. (3.33), (3.34) and (3.35) into Eqs. (2.1) and (2.2), we get

$$\widetilde{K}_{E_{\lambda_p}}=\frac{3(n-s)\left(2n''(n-s)^3+(n-s)^2(n'+1)(1-n')^2+3c^2(1-n')^3\right)}{4(1-n')\left[(n-s)^2(n'+1)+c^2(1-n')\right]^2}\tag{3.36}$$

and

$$\widetilde{H}_{\lambda_p}=-\frac{\sqrt{n-s}\left(n''(n-s)^3+2(n-s)^2(n'+1)(1-n')^2+3c^2(1-n')^3\right)}{2\left[(1-n')\left((n-s)^2(n'+1)+c^2(1-n')\right)\right]^{\frac{3}{2}}}.\tag{3.37}$$

It is difficult to determine the general solution to Eqs. (3.36) and (3.37) unless in a few specific cases.

First, we look at the case where both $\widetilde{K}_{E_{\lambda_p}}$ and $\widetilde{H}_{\lambda_p}$ amount mutually to zero. Thus, by using $n(s)\neq s$, Eqs. (3.36) and (3.37) turn into

$$2n''(n-s)^3+(n-s)^2\left(n'+1\right)\left(1-n'\right)^2+3c^2\left(1-n'\right)^3=0 \tag{3.38}$$

$$n''(n-s)^3+2(n-s)^2\left(n'+1\right)\left(1-n'\right)^2+3c^2\left(1-n'\right)^3=0. \tag{3.39}$$

By Eqs. (3.38) and (3.39) , we obtain

$$(n-s)^2\left(1+n'\right)+c^2\left(1-n'\right)=0. \tag{3.40}$$

Assigned $n(s)-s=p(s)$, we turn Eq. (3.40) into

$$\left(p^2-c^2\right)p'+2p^2=0. \tag{3.41}$$

Then, the general solution to Eq. (3.41) is

$$p(s)=-(s+c_1)\pm\sqrt{(s+c_1)^2-c^2},$$

where c_1 is an integration constant. From $n(s)-s=p(s)$, we find

$$n(s)=c_1\pm\sqrt{(s+c_1)^2-c^2},\qquad c_1\in\mathbb{R}.$$

Hence, we construct a one-parameter family of curves

$$\gamma(s;n(s),c,c_1)=\left(s,c_1\pm\sqrt{(s+c_1)^2-c^2},0\right).$$

As a result, with Eq. (2.11), the helicoidal surface turns into

$$X(s,t)=\left(\left(1+\frac{t^2}{2}\right)s-\frac{t^2}{2}\left(c_1\pm\sqrt{(s+c_1)^2-c^2}\right)+ct,\frac{t^2}{2}s+\left(1-\frac{t^2}{2}\right)\left(c_1\pm\sqrt{(s+c_1)^2-c^2}\right)+ct,\left(\left(s-c_1\mp\sqrt{(s+c_1)^2-c^2}\right)\right)t\right).$$

We will talk about what Eq. (3.36) turns into when the extrinsic curvature $\widetilde{K}_{E_{\lambda_p}}$ is zero, which is the situation in which Eq. (3.38) only is valid. In that case, we turn Eq. (3.36) into

$$2p^3p''+\left(p^2-3c^2\right)p'^3+2p^2p'^2=0, \tag{3.42}$$

where $n(s)-s=p(s)$. Assigned $p'(s)$ to $\widetilde{p}(s)$, Eq. (3.42) turns into

$$\frac{1}{\widetilde{p}}\left(\frac{1}{\widetilde{p}}\right)'-\frac{1}{p}\frac{1}{\widetilde{p}}-\frac{p^2-3c^2}{2p^3}=0. \tag{3.43}$$

If we put $\frac{1}{\widetilde{p}}=w(p)$, Eq. (3.43) turns into

$$\frac{dw}{dp}-\frac{1}{p}w-\frac{p^2-3c^2}{2p^3}=0. \tag{3.44}$$

The general solution to Eq. (3.44) is

$$w(p)=\frac{2c_1p^3-p^2+c^2}{2p^2},$$

where c_1 is a constant. Thus, the function $n=n(s)$ supplies the equality

$$c_1n^3-(3c_1s+1)n^2+\left(3c_1s^2-2c_2\right)n-c_1s^3+2c_2s+s^2-c^2=0, \tag{3.45}$$

where c_2 is an integration constant.

Example 3.11. *For $c_1 = 0$, from Eq. (3.45), we find $n(s) = -c_2 \pm \sqrt{(s+c_2)^2 - c^2}, s \in (-\infty, -c-c_2) \cup (c-c_2, \infty)$. Using Eq. (2.11), we write*

$$X(s,t) = \left(\left(1+\frac{t^2}{2}\right) s - \frac{t^2}{2}\left(-c_2 \pm \sqrt{(s+c_2)^2 - c^2}\right) + ct, \frac{t^2}{2}s + \left(1-\frac{t^2}{2}\right)\left(-c_2 \pm \sqrt{(s+c_2)^2 - c^2}\right) + ct, \left(\left(s+c_2 \mp \sqrt{(s+c_2)^2 - c^2}\right)\right)t \right).$$

We now plot it putting for $c_2 = 0$ and $c = 3$.

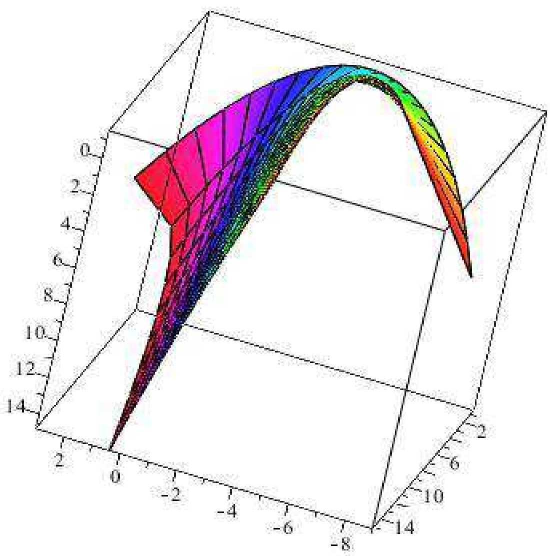

Figure 3.3: The graphic belongs to a minimal timelike helicoidal surface of lightlike axes of rotation with $\widetilde{K}_{E_{\lambda_p}} = 0$

Finally, we take an interest in Eq. (3.39), which requires a timelike helicoidal minimal surface. Similarly, the function $n(s)$ provides

$$\begin{aligned} &2c_1n^6 - 12c_1sn^5 + 30c_1s^2n^4 - 40c_1s^3n^3 + 5\left(6c_1s^4 - 1\right)n^2 \\ &-2\left(6c_1s^5 + 5c_2\right)n + 2c_1s^6 + 5s^2 + 10c_2s - 5c^2 = 0. \end{aligned} \tag{3.46}$$

Example 3.12. *For $c_1 = 0$, from Eq. (3.46), we get $n(s) = -c_2 \pm \sqrt{(s+c_2)^2 - 5c^2}, s \in \left(-\infty, -\sqrt{5}c - c_2\right) \cup \left(\sqrt{5}c - c_2, \infty\right)$, in which by Eq. (2.11), the parametric form of a timelike helicoidal minimal surface turns into*

$$\begin{aligned} X(s,t) = \Bigg(&\left(1+\frac{t^2}{2}\right) s - \frac{t^2}{2}\left(-c_2 \pm \sqrt{(s+c_2)^2 - 5c^2}\right) + ct, \\ &\frac{t^2}{2}s + \left(1-\frac{t^2}{2}\right)\left(-c_2 \pm \sqrt{(s+c_2)^2 - 5c^2}\right) + ct, \\ &\left(\left(s+c_2 \mp \sqrt{(s+c_2)^2 - 5c^2}\right)\right)t \Bigg). \end{aligned}$$

We now plot it putting for $c_2 = 0$ and $c = 3$.

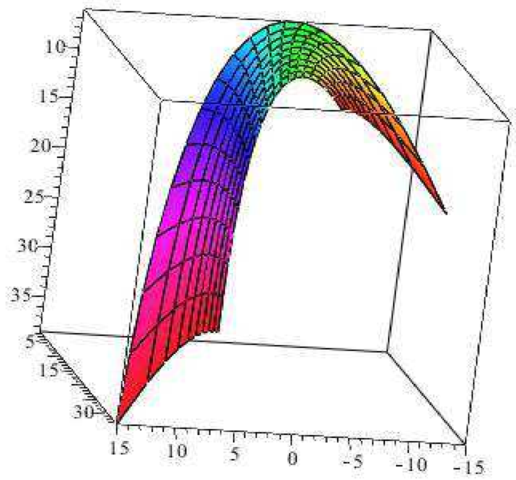

Figure 3.4: The graphic belongs to a minimal timelike helicoidal surface of lightlike axes of rotation with $\widetilde{K}_{E_{\lambda_p}} = 0$

Remark 3.13. *Observe that both helicoidal surfaces mentioned above, say Examples* (3.11) *and* (3.12) *satisfy both* $\widetilde{K}_{E_{\lambda_p}} = 0$ *and* $\widetilde{H}_{\lambda_p} = 0$.

Remark 3.14. *It is important to remember that analogous outcomes may be obtained if we pick*

$$EG - F^2 = (n' - 1)\left((n-s)^2 (n'+1) - c^2 (n'-1)\right) > 0.$$

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