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Reeb Flow Invariant Unit Tangent Sphere Bundles with the Kaluza-Klein Metric

Murat Altunbaş[1]

[1]*Department of Mathematics, Faculty of Science and Arts, Erzincan Binali Yıldırım University, Erzincan, Türkiye*

Abstract

Let (M,g) be an $n-$dimensional Riemannian manifold and T_1M its tangent sphere bundle with the contact metric structure $(\tilde{G},\eta,\phi,\xi)$, where $\tilde{G}$ is the Kaluza-Klein metric. Let $h=\frac{1}{2}\mathfrak{L}_\xi\phi$ be the structural operator and $l=\bar{R}(\cdot,\xi)\xi$ be the characteristic Jacobi operator on T_1M. In this paper, we find some conditions for the Reeb flow invariancy of the $(0,2)-$ type tensors L and H defined by $L(\tilde{X},\tilde{Y})=g(l\tilde{X},\tilde{Y})$ and $H(\tilde{X},\tilde{Y})=g(h\tilde{X},\tilde{Y})$ for all vector fields $\tilde{X}$ and $\tilde{Y}$ on T_1M.

Keywords: *Contact metric structure; Jacobi operator; Kaluza-Klein metric; tangent sphere bundle.*
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1. Introduction

Let (M,g) be a Riemannan manifold and T_1M its unit tangent sphere bundle. The standard contact structure on T_1M is given with respect to the Sasaki metric. A generalization of the Sasaki metric is the Kaluza-Klein metric. Certain geometric properties of unit tangent sphere bundles with respect to the Kaluza-Klein metric and compatible contact structures were studied by some authors (see for example [1],[7],[8]). An important problem on T_1M with a contact metric structure $(\eta,\phi,\xi,\bar{g})$ is to study the geodesic flow generated by the Reeb vector field ξ. This is called the Reeb flow and it leaves some structure tensors invariant. It is always true for ξ and η, since $\mathfrak{L}_\xi\xi=0$ and $\mathfrak{L}_\xi\eta=0$, where $\mathfrak{L}_\xi$ denotes the Lie differentiation in the direction ξ. On the other hand, Tashiro showed that $\mathfrak{L}_\xi g=0$ if and only if (M,g) has constant curvature $c=1$,[10]. However, there exist two fundamental tensors on T_1M, so called the structural operator $h=\frac{1}{2}\mathfrak{L}_\xi\phi$ and the characteristic Jacobi operator $l=\bar{R}(\cdot,\xi)\xi$. Boeckx *et al.* showed that T_1M fulfills $\mathfrak{L}_\xi h=0$ if and only if (M,g) is of constant curvature $c=1$ and T_1M fulfills $\mathfrak{L}_\xi l=0$ if and only if (M,g) is of constant curvature $c=0$ or $c=1$, [6]. In these papers, the unit tangent sphere bundles were considered with the standard contact metric structure.
In this paper, we deal with two $(0,2)-$type tensors L and H defined by $L(\tilde{X},\tilde{Y})=g(l\tilde{X},\tilde{Y})$ and $H(\tilde{X},\tilde{Y})=g(h\tilde{X},\tilde{Y})$ for all vector fields $\tilde{X}$ and $\tilde{Y}$ on T_1M and investigate the conditions under which these tensors are preserved by the geodesic flow. We endow unit the tangent sphere bundle T_1M with the Kaluza-Klein metric. Similar problems were considered in [9] with respect to the standard contact metric structure.

2. Preliminaries

In this section, we give some basic facts about contact metric manifolds. All manifolds are assumed to be smooth and connected. We may refer to [5] for more information about contact metric geometry.
Let $\bar{M}$ be a $(2n-1)$ dimensional differentiable manifold. If $\bar{M}$ admits a global 1-form η (a contact form) such that $\eta\wedge(d\eta)^n\neq 0$, then it is called a contact manifold. When η is given, there exists a unique vector field ξ (characteristic vector field) such that $\eta(\xi)=1$ and $d\eta(\xi,\bar{X})=0$ for all vector fields $\bar{X}$ on $\bar{M}$. Moreover, a Riemannian metric $\bar{g}$ is called an associated metric if there exists a $(1,1)-$tensor ϕ such that

$$\eta(\bar{X})=g(\bar{X},\xi),\ d\eta(\bar{X},\bar{Y})=\bar{g}(\bar{X},\phi\bar{Y}),\ \phi^2\bar{X}=-\bar{X}+\eta(\bar{X})\xi, \tag{2.1}$$

for all vector fields $\bar{X}$, $\bar{Y}$ on $\bar{M}$. It follows that

$$\phi\xi=0,\ \eta\circ\phi=0,\ \bar{g}(\phi\bar{X},\phi\bar{Y})=\bar{g}(\bar{X},\bar{Y})-\eta(\bar{X})\eta(\bar{Y}).$$

The quartet $(\eta,\bar{g},\phi,\xi)$ satisfying (2.1) is called a contact metric structure and the quintet $(\bar{M},\eta,\bar{g},\phi,\xi)$ a contact metric manifold.

Email addresses: **maltunbas@erzincan.edu.tr (Murat Altunbaş)**

Let $\bar{M}$ be a contact metric manifold. The structural operator h is defined by $h = \frac{1}{2}\mathfrak{L}_{\xi}\phi$, where $\mathfrak{L}$ is the Lie differentiation operator. The operator h is self-adjoint and it satisfies the following relations:

$$h\xi = 0,\ h\phi = -\phi h,\ \bar{\nabla}_{\bar{X}}\xi = -\phi\overline{X} - \phi h\overline{X},\ (\bar{\nabla}_{\xi}h)\phi = -\phi(\bar{\nabla}_{\xi}h),$$

where $\bar{\nabla}$ is the Levi-Civita connection of $\bar{g}$.
We denote by $\bar{R}$ the Riemannian curvature tensor defined by

$$\bar{R}(\bar{X},\bar{Y})\bar{Z} = \nabla_{\overline{X}}(\nabla_{\overline{Y}}\overline{Z}) - \nabla_{\overline{Y}}(\nabla_{\overline{X}}\overline{Z}) - \nabla_{[\overline{X},\overline{Y}]}\overline{Z}$$

for all vector fields $\bar{X},\ \bar{Y},\ \bar{Z}$ on $\bar{M}$.
The characteristic Jacobi operator is a $(1,1)-$tensor field and defined by $l = \bar{R}(\cdot,\xi)\xi$. We have

$$l = \phi l\phi - 2(h^2+\phi^2), \tag{2.2}$$
$$\overline{\nabla}_{\xi}h = \phi - \phi l - \phi h^2. \tag{2.3}$$

Let (M,g) be an $n-$Riemannian manifold with the Levi-Civita connection ∇. The tangent bundle TM of M is a $2n-$dimensional manifold with the projection map $\pi : TM \to M,\ \pi(p,u) = p$. The $g-$natural metric G on TM is defined by, [4]

$$\begin{cases} G(X^h,Y^h) = (\alpha_1+\alpha_3)(r^2)g(X,Y) + (\beta_1+\beta_3)(r^2)g(X,u)g(Y,u), \\ G(X^h,Y^v) = \alpha_2(r^2)g(X,Y) + \beta_2(r^2)g(X,u)g(Y,u), \\ G(X^v,Y^v) = \alpha_1(r^2)g(X,Y) + \beta_1(r^2)g(X,u)g(Y,u), \end{cases}$$

for all vector fields X,Y on M, where $\alpha_i,\beta_i : \mathbb{R}^+ \to \mathbb{R},\ i = 1,2,3$ are smooth functions, $r^2 = g(u,u)$ and X^h,X^v denote the horizontal lift and the vertical lift of X, respectively.
The unit tangent sphere bundle T_1M is a hypersurface of TM defined by $T_1M = \{(x,u) \in TM : g(u,u) = 1\}$. By definition, the $g-$natural metric on T_1M is the restriction of the $g-$natural metric of TM to its hypersurface T_1M. The $g-$natural metric on T_1M is defined by

$$\begin{cases} \bar{G}(X^h,Y^h) = (a+c)g(X,Y) + \beta(r^2)g(X,u)g(Y,u), \\ \bar{G}(X^h,Y^v) = bg(X,Y), \\ \bar{G}(X^v,Y^v) = ag(X,Y), \end{cases}$$

where $a,b,c \in \mathbb{R}$ and $\beta : [0,\infty) \to \mathbb{R}$. The vector field $N = \frac{1}{\sqrt{(a+c+d)\varphi}}[-bu^h + (a+c+d)u^v]$ is the unit normal vector field, where $d = \beta(1)$ and $\varphi = a(a+c+d) - b^2$. The tangential lift X^t is given by $X^t = X^v - \sqrt{\frac{\varphi}{a+c+d}}g(X,u)N$. Since the tangent space $(T_1M)_{(x,u)}$ of T_1M at (x,u) is spanned by vectors of the form X^h and Y^t, the Riemannian metric g on T_1M, induced from G, is completely determined by

$$\begin{cases} \bar{G}(X^h,Y^h) = (a+c)g(X,Y) + dg(X,u)g(Y,u), \\ \bar{G}(X^h,Y^t) = bg(X,Y), \\ \bar{G}(X^t,Y^t) = ag(X,Y) - \frac{\varphi}{a+c+d}g(X,u)g(Y,u), \end{cases} \tag{2.4}$$

for all vector fields X,Y on M. The particular cases of the metric G are listed below:
i) The Sasaki metric if $a = 1,\ b = c = d = 0$,
ii) The Cheeger-Gromoll type metric if $b = d = 0,\ a = 1/2^m,\ c = 1-a$,
iii) The Kaluza-Klein type metric if $b = 0$ (and $a,\ a+c > 0,\ a+c+d > 0$),
iv) The Kaluza-Klein metric if $b = d = 0$ (and $a,\ a+c > 0$).
In this paper, we deal with the Kaluza-Klein metric. From (2.4), the Kaluza-Klein metric $\tilde{G}$ is defined by

$$\begin{cases} \tilde{G}(X^h,Y^h) = (a+c)g(X,Y), \\ \tilde{G}(X^h,Y^t) = 0, \\ \tilde{G}(X^t,Y^t) = a(g(X,Y) - g(X,u)g(Y,u)). \end{cases} \tag{2.5}$$

The Levi-Civita connection $\tilde{\nabla}$ of $\tilde{G}$ is given by, [7]

$$\begin{aligned} \tilde{\nabla}_{X^h}Y^h &= (\nabla_XY)^h - \frac{1}{2}(R(X,Y)u)^v, \\ \tilde{\nabla}_{X^h}Y^t &= (\nabla_XY)^t + \frac{a}{2(a+c)}(R(u,Y)X)^h, \\ \tilde{\nabla}_{X^t}Y^h &= \frac{a}{2(a+c)}(R(u,X)Y)^h, \\ \tilde{\nabla}_{X^t}Y^t &= -g(Y,u)X^t. \end{aligned} \tag{2.6}$$

where ∇ and R are the Levi-Civita connection and the Riemannian curvature tensor of g, respectively. For the characteristic Jacobi operator l, we need only two components of the Riemannian curvature tensor $\tilde{R}$ which are given by

$$\widetilde{R}(X^t,Y^h)Z^h = -\frac{1}{2}\{R(Y,Z)(X-g(X,u)u)\}^t + \frac{a}{4(a+c)}\{R(Y,R(u,X)Z)u\}^t - \frac{a}{2(a+c)}\{(\nabla_YR)(u,X)Z\}^h, \tag{2.7}$$

$$\widetilde{R}(X^t,Y^h)Z^h = (R(X,Y)Z)^h + \frac{a}{2(a+c)}\{R(u,R(X,Y)u)Z\}^h - \frac{a}{4(a+c)}\{R(u,R(Y,Z)u)X - R(u,R(X,Z)u)Y\}^h + \frac{1}{2}\{(\nabla_ZR)(X,Y)u\}^t$$

for all vector fields X,Y,Z on M, [3].

From [2], we have a contact metric structure $(\tilde{G},\eta,\phi,\xi)$ on T_1M satisfying

$$\xi = 2u^h,\ \phi X^t = \frac{1}{4(a+c)}(-X^h + \frac{1}{2}g(X,u)\xi),\ \phi X^h = \frac{1}{4a}X^t, \eta(X^h) = \frac{1}{2}g(X,u), \eta(X^t) = 0. \tag{2.8}$$

We also have

$$\begin{aligned} hX^t &= \frac{1}{4a}X^t - \frac{1}{4(a+c)}(R_uX)^t, \\ hX^h &= -\frac{1}{4a}X^h + \frac{1}{8a}g(X,u)\xi + \frac{1}{4(a+c)}(R_uX)^h, \end{aligned} \tag{2.9}$$

where $R_u = R(\cdot,u)u$ is the Jacobi operator associated with the unit vector u.
Using formulae (2.7), we obtain

$$\begin{aligned} lX^t &= \frac{a}{a+c}(R_u^2X)^t + \frac{2a}{a+c}(R_u'X)^t, \\ lX^h &= 4(R_uX)^h - \frac{3a}{a+c}(R_u^2X)^h + 2(R_u'X)^t, \end{aligned} \tag{2.10}$$

where $R_u^2X = R(R(\cdot,u)u,u)u$ and $R_u' = (\nabla_uR)(\cdot,u)u$. Using (2.3), (2.8) and (2.9) we get

$$\begin{aligned} (\tilde{\nabla}_\xi h)X^t &= \frac{-1}{32a(a+c)^2}(R_uX)^h + \frac{16a(a+c)+1}{64(a+c)^3}(R_u^2X)^h - \frac{1}{2(a+c)}(R_u'X)^t + (\frac{1-16a^2}{64a^2(a+c)})(X^h - g(X,u)u^h), \\ (\tilde{\nabla}_\xi h)X^h &= \frac{-64a(a+c)+2}{64a^2(a+c)}(R_uX)^t + \frac{48a(a+c)-1}{64a(a+c)^2}(R_u^2X)^t + \frac{1}{2(a+c)}(R_u'X)^h + \frac{16a^2-1}{64a^3}X^t. \end{aligned} \tag{2.11}$$

From (2.6) and (2.10), we get

$$\begin{aligned} (\tilde{\nabla}_\xi l)X^t &= \frac{4a}{a+c}(R_u'R_uX + R_uR_u'X)^t + \frac{4a}{a+c}(R_u''X + R_u^2X - \frac{a}{a+c}R_u^3X)^h, \\ (\tilde{\nabla}_\xi l)X^h &= 8(R_u'X - \frac{a}{a+c}(R_u'R_uX + R_uR_u'X))^h + 4(R_u''X + R_u^2X - \frac{a}{a+c}R_u^3X)^t. \end{aligned} \tag{2.12}$$

3. Lie differentiations of the tensor fields L and H

Theorem 3.1. *Let (M,g) be a space with constant curvature k and T_1M be its unit tangent sphere bundle with the contact metric structure defined by (2.8). Then T_1 M fulfills $\mathfrak{L}_\xi L = 0$ if the following system of equations is satisfied*

$$\begin{cases} \dfrac{32a^2(a+c)-a}{8(a+c)^3}k^3 + \dfrac{8a+3-32a(a+c)}{8(a+c)^2}k^2 - \dfrac{4a+1}{a+c}k = 0, \\ \dfrac{4a(8a+c)-1}{8(a+c)^2}k^3 + \dfrac{8a+3-32a(a+c)}{8(a+c)}k^2 - \frac{4a+1}{4a^2}k = 0. \end{cases} \tag{3.1}$$

Proof. From [9], we know that $\mathfrak{L}_\xi L = 0$ is equivalent to

$$\tilde{\nabla}_\xi l = l\phi - \phi l + l\phi h - h\phi l. \tag{3.2}$$

If (M,g) is a space with constant curvature k, then equations (2.9)-(2.12) reduce to

$$hX^t = (\frac{1}{4a} - \frac{k}{4(a+c)})X^t,\ hX^h = (\frac{-1}{4a} + \frac{k}{4(a+c)})(X^h - g(X,u)u^h),$$

$$lX^t = \frac{a}{a+c}k^2X^t,\ lX^h = (4k - \frac{3a}{a+c}k^2)(X^h - g(X,u)u^h), \tag{3.3}$$

$$(\tilde{\nabla}_\xi h)X^t = (\frac{-1}{32a(a+c)^2}k + \frac{16a(a+c)+1}{64(a+c)^3}k^2 + \frac{1-16a^2}{64a^2(a+c)})(X^h - g(X,u)u^h),$$

$$(\tilde{\nabla}_\xi h)X^h = (\frac{-64a(a+c)+2}{64a^2(a+c)}k + \frac{48a(a+c)-1}{64a(a+c)^2}k^2 + \frac{16a^2-1}{64a^3})X^t,$$

$$(\tilde{\nabla}_\xi l)X^t = \frac{4a}{a+c}(k^2 - \frac{a}{a+c}k^3)(X^h - g(X,u)u^h),$$

$$(\tilde{\nabla}_\xi l)X^h = 4(k^2 - \frac{a}{a+c}k^3)X^t.$$

Using equations (3.3) in (3.2), we obtain equation (3.1). $\square$

Example 3.2. *It is clear that if* (M,g) *is a space with constant curvature* 0, *then* T_1M *with the contact metric structure defined by* (2.8) *satisfies* $\mathfrak{L}_\xi L = 0$.

Theorem 3.3. *Let* (M,g) *be a space with constant curvature* k *and* T_1M *be its unit tangent sphere bundle with the contact metric structure defined by* (2.8). *Then* T_1 M *fulfills* $\mathfrak{L}_\xi H = 0$ *if the following system of equations is satisfied*

$$\begin{cases} \dfrac{16a(a+c)+1}{64(a+c)^3}k^2 + \dfrac{4a-1}{32a(a+c)^2}k + \dfrac{-16a^2-8a+1}{64a^2(a+c)} = 0, \\ \\ \dfrac{48a(a+c)-1}{64a(a+c)^2}k^2 + \dfrac{8a+2-64a(a+c)}{64a(a+c)^2}k + \dfrac{16a^2-8a-1}{64a^3} = 0. \end{cases} \tag{3.4}$$

Proof. From [9], we know that $\mathfrak{L}_\xi H = 0$ is equivalent to

$$\tilde{\nabla}_\xi h = 2h\phi. \tag{3.5}$$

Using equations (3.3) in (3.5), we occur equation (3.4). □

Corollary 3.4. *[9] Let* (M,g) *be a space with constant curvature* -1 *or* 1. *Then* T_1M *with the standard contact metric structure fulfills* $\mathfrak{L}_\xi H = 0$.

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