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Parallel Curves Based on Normal Vector

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Research Article

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Abstract
In this paper, the definition of parallel curves based on normal vector is given and the curvature, torsion, Frenet frame of this curve are determined. Furthermore, special cases in curves such as circle and helix are also be exemplified.

Keywords: Curve, Frenet frame, curvature, torsion

Normal Vektöre Dayalı Paralel Eğriler

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Öz
Bu çalışmada, normal vektöre dayalı paralel eğrilerin tanımı verilmiş ve bu eğrinin eğriliği, burulması, Frenet çatısı belirlenmiştir. Ayrıca, çember ve helis gibi eğrilerdeki özel durumlar da örneklendirilmiştir.

Anahtar Kelimeler: Eğri, Frenet çatısı, eğrilik, burulma

Introduction

Curves are one of the main tools of differential geometry. Different kinds of curves are determined for giving descriptions and explaining the theory about curves in differential geometry. In the literature, there are many articles on this subject. Arslan and Hacısalihoğlu [1] investigate the harmonic curvatures of a Frenet curve, Ergin [2] studies generalized Darboux curves, Liu, and Wang [3] examine Mannheim partner curves, İlarslan and Nesovic [4] analyze the rectifying curves, Has and Yılmaz [5] work on quaternionic Bertrand curves. Sağıroğlu [6] studies global differential invariants of affine curves in two dimensional space. These results are carried out in the Euclidean space. Also, curves in different space from Euclidean space are investigated. Divjak [7] investigates special curves on ruled surfaces in Galilean and Pseudo-Galilean spaces, Bükcü and Karacan [8] work on Bishop frame of the spacelike curves in Minkowski 3-Space. Şenyurt et al. [9] define tangent, principal, normal and binormal wise associated curves such that each of these vectors of any given curve lies on the osculating, normal and rectifying plane of its partner, respectively. Güler [10] defines the quasi parallel curve with the help of the quasi frame of a given curve. In addition, curves are the most important examples of their role in Pattern Recognition, Computer Graphics and Computer Vision because of being suitable for modeling real life problems. [11] and [12] are their applications to these subjects. The classical differential geometry of curves is investigated with respect to conformable fractional derivative and fractional integral in Gözütok, Çoban and Sağıroğlu [13]. Curvature and torsion of a conformable curve are defined and the geometric interpretation of these two functions is done. In [14], Aldossary and Gazwani give the similar definition of the unit speed curves using binormal vector. These curves are called parallel curves. The main goal of this paper is to investigate parallel curves using normal vector and to study the associated geometry of these curves. We give a similar definition in [14] using the normal vector. The aim of this study is contribution to the literature on the theory of curves in three-dimensional space. In addition, the studies discussed here will later be expanded to surfaces and their geometric properties will be examined. Also, the instrinsic geometric formulas will be derived from the curvatures. We refer to [15] for the definitions of curvature, torsion and Frenet frame of a curve.

Parallel Curves Based on Normal Vector

Definition1. Let α be a unit speed curve in R^3 and $\boldsymbol{t}(s), \boldsymbol{n}(s), \boldsymbol{b}(s)$ be its Franet frame in a point $\alpha(s)$. Including r to represent a real number constant, parallel curve based on normal vector of the curve α is defined as

$$\boldsymbol{\alpha_n}(s) = \boldsymbol{\alpha}(s) + r\boldsymbol{n}(s).$$

The reason why the curve $\alpha(s)$ taken as unit speed is the convenience in calculations. Fisrtly, the parametrization of the curve according to its arc lenght is obtained and calculations are made for this situation, and then calculations are made when it is arbitrary speed. The reason for this is that when we calculate a parallel curve based on a curve with unit speed, the resulting curve will generally not have unit speed.

Frenet Frame, Curvature and Torsion Functions of the Parallel Curves Based on Normal Vector

Now we consider the parallel curve based on normal vector of α;

$$\boldsymbol{\alpha_n}(s) = \boldsymbol{\alpha}(s) + r\boldsymbol{n}(s)$$

Let the arc length function of this curve be s_2. In this case, the unit tangent vector of the curve $\alpha_n(s)$ is obtained as

$$\boldsymbol{t_n}(s_2) = \frac{d\boldsymbol{\alpha_n}}{ds}.\frac{ds}{ds_2} = (\boldsymbol{t} + r\boldsymbol{n}')\frac{ds}{ds_2} = (\boldsymbol{t} + r(-\kappa\boldsymbol{t} + \tau\boldsymbol{b})\frac{ds}{ds_2}=((1 - r\kappa)\boldsymbol{t} + r\tau\boldsymbol{b})\frac{ds}{ds_2}$$

Here, $\boldsymbol{t}, \boldsymbol{n}, \boldsymbol{b}$ and κ, τ are Frenet frame vectors and the corresponding curvatures of the curve $\alpha_n(s)$, respectively. If we take the dot product with $\boldsymbol{t}$ of both sides;

$$\langle \boldsymbol{t_n}, \boldsymbol{t} \rangle = (1 - r\kappa)\frac{ds}{ds_2}$$

is obtained. If we also multiply both sides of this equation by the vector $\boldsymbol{b}$, we get

$$\langle \boldsymbol{t_n}, \boldsymbol{b} \rangle = r\tau\frac{ds}{ds_2}.$$

Multiplying both sides by the vector $\boldsymbol{n}$, the equation

$$\langle \boldsymbol{t_n}, \boldsymbol{n} \rangle = 0$$

is obtained, which shows that the vectors $\boldsymbol{t_n}$ and $\boldsymbol{n}$ are orthogonal. If the norm of both sides of the above equation is taken, then we obtain that

$$\|\boldsymbol{t_n}(s_2)\| = \left\|((1 - r\kappa)\boldsymbol{t} + r\tau\boldsymbol{b})\frac{ds}{ds_2}\right\| = \|(1 - r\kappa)\boldsymbol{t} + r\tau\boldsymbol{b}\|\frac{ds}{ds_2}$$

$$= \frac{ds}{ds_2}[((1 - r\kappa)\boldsymbol{t} + r\tau\boldsymbol{b}).((1 - r\kappa)\boldsymbol{t} + r\tau\boldsymbol{b})]^{1/2}$$

$$= \frac{ds}{ds_2}[(1 - r\kappa)^2 + r^2\tau^2]^{1/2}$$

By using $\|\boldsymbol{t_n}(s_2)\| = 1$, it follows that

$$\frac{ds_2}{ds} = \sqrt{(1 - r\kappa)^2 + r^2\tau^2}$$

From here, we get

$$\frac{ds}{ds_2} = \frac{1}{\sqrt{(1 - r\kappa)^2 + r^2\tau^2}}$$

Note that this equation also shows the relationship between the arc lenght parameters of these two curves.

Theorem 1. The expression of the Frenet frame of the parallel curve $\boldsymbol{\alpha_n}(s)$ in terms of the Frenet frame of the curve $\boldsymbol{\alpha}(s)$ is in the form;

$$\boldsymbol{t_n}(s)=\frac{(1-r\kappa)\boldsymbol{t}+r\tau\boldsymbol{b}}{\sqrt{(1-r\kappa)^2+r^2\tau^2}}$$

$$\boldsymbol{b_n}(s)=\frac{\begin{array}{c}(r\tau-r^2\kappa\tau-r^2\tau^3)\boldsymbol{t}+\begin{pmatrix}-r\tau'+r^2\kappa\tau'\\-r^2\kappa'\tau\end{pmatrix}\boldsymbol{n}+\\(\kappa-2r\kappa^2-r\tau^2+r^2\kappa^3+r^2\kappa\tau^2)\boldsymbol{b}\end{array}}{F(s)}$$

$$\begin{aligned}\boldsymbol{n_n}(s)=\frac{1}{F(s).A(s)}&\{(-r^2\tau\tau'+r^3\kappa\tau\tau'-r^3\kappa'\tau^2)\boldsymbol{t}\\&+(-r^2\tau^2+r^3\kappa\tau^2+r^3\tau^4+\kappa-3r\kappa^2-r\tau^2+3r^2\kappa^3+2r^2\kappa\tau^2-r^3\kappa^4\\&-r^3\kappa^2\tau^2)\boldsymbol{n}+(r\tau'-2r^2\kappa\tau'+r^2\kappa'\tau+r^3\kappa^2\tau'-r^3\kappa\kappa'\tau)\boldsymbol{b}\}\end{aligned}$$

where

$$A(s)=\sqrt{(1-r\kappa)^2+r^2\tau^2}$$

and

$$F(s)=\sqrt{\begin{array}{c}(-r\tau\kappa+r^2\kappa^2\tau+r\tau^3)^2+(-r\tau'+r^2\kappa\tau'-r^2\kappa'\tau)^2+\\(\kappa-2r\kappa^2-r\tau^2+r^2\kappa^3+r^2\kappa\tau^2)^2\end{array}}.$$

Proof. Let us determine the Frenet frame of a curve $\boldsymbol{\alpha_n}(s)$ in terms of the Frenet frame of $\boldsymbol{\alpha}(s)$ in the general case. It is known that

$$\frac{d\boldsymbol{\alpha_n}(s)}{ds}=\frac{d\boldsymbol{\alpha}(s)}{ds}+r\frac{d\boldsymbol{n}(s)}{ds}=\boldsymbol{t}+r(-\kappa\boldsymbol{t}+\tau\boldsymbol{b})=(1-r\kappa)\boldsymbol{t}+r\tau\boldsymbol{b}.$$

If we take the norm of this expression, then it holds that

$$\left\|\frac{d\boldsymbol{\alpha_n}(s)}{ds}\right\|=\sqrt{(1-r\kappa)^2+r^2\tau^2}=A(s)$$

Hence, the unit tangent vector $\boldsymbol{t_n}(s)$ is obtained as;

$$\boldsymbol{t_n}(s)=\frac{\frac{d\boldsymbol{\alpha_n}(s)}{ds}}{\left\|\frac{d\boldsymbol{\alpha_n}(s)}{ds}\right\|}=\frac{(1-r\kappa)\boldsymbol{t}+r\tau\boldsymbol{b}}{\sqrt{(1-r\kappa)^2+r^2\tau^2}}$$

Since

$$\frac{d^2\boldsymbol{\alpha_n}(s)}{ds^2} = -r\kappa'\boldsymbol{t} + (1-r\kappa)\frac{d\boldsymbol{t}}{ds} + r\tau'\boldsymbol{b} + r\tau\frac{d\boldsymbol{b}}{ds}$$

$$= -r\kappa'\boldsymbol{t} + (\kappa - r\kappa^2 - r\tau^2)\boldsymbol{n} + r\tau'\boldsymbol{b}$$

and

$$\frac{d\boldsymbol{\alpha_n}(s)}{ds} \times \frac{d^2\boldsymbol{\alpha_n}(s)}{ds^2} = [(1-r\kappa)\boldsymbol{t} + r\tau\boldsymbol{b}] \times [-r\kappa'\boldsymbol{t} + (\kappa - r\kappa^2 - r\tau^2)\boldsymbol{n} + r\tau'\boldsymbol{b}]$$

$$= (\kappa - r\kappa^2 - r\tau^2 - r\kappa^2 + r^2\kappa^3 + r^2\kappa\tau^2)\boldsymbol{t}\times\boldsymbol{n} + (r\tau' - r^2\kappa\tau')\boldsymbol{t}\times\boldsymbol{b} - r^2\kappa'\tau\boldsymbol{b}\times\boldsymbol{t} + (r\tau\kappa - r^2\kappa^2\tau - r^2\tau^3)\boldsymbol{b}\times\boldsymbol{n}$$

$$= (-r\tau\kappa + r^2\kappa^2\tau + r^2\tau^3)\boldsymbol{t} + (-r\tau' + r^2\kappa\tau' - r^2\kappa'\tau)\boldsymbol{n} + (\kappa - 2r\kappa^2 - r\tau^2 + r^2\kappa^3 + r^2\kappa\tau^2)\boldsymbol{b}$$

and by taking the norm at second term, it is obtained

$$\left\|\frac{d\boldsymbol{\alpha_n}(s)}{ds} \times \frac{d^2\boldsymbol{\alpha_n}(s)}{ds^2}\right\| = \sqrt{\begin{matrix}(-r\tau\kappa + r^2\kappa^2\tau + r^2\tau^3)^2 + (-r\tau' + r^2\kappa\tau' - r^2\kappa'\tau)^2 + \\ (\kappa - 2r\kappa^2 - r\tau^2 + r^2\kappa^3 + r^2\kappa\tau^2)^2\end{matrix}} = F(s)$$

Also the third derivation is obtained as follows;

$$\frac{d^3\boldsymbol{\alpha_n}(s)}{ds^3} = -r\kappa''\,\boldsymbol{t} - r\kappa'\boldsymbol{t}' + (\kappa' - 2r\kappa\kappa' - 2r\tau\tau')\boldsymbol{n} + (\kappa - r\kappa^2 - r\tau^2)\boldsymbol{n}' + r\tau''\boldsymbol{b} + r\tau'\boldsymbol{b}'$$

$$= -r\kappa''\,\boldsymbol{t} - r\kappa\kappa'\boldsymbol{n} + (\kappa' - 2r\kappa\kappa' - 2r\tau\tau')\boldsymbol{n} + (-\kappa^2 + r\kappa^3 + r\kappa\tau^2)\boldsymbol{t} + (\kappa\tau - r\kappa^2\tau - r\tau^3)\boldsymbol{b} + r\tau''\boldsymbol{b} - r\tau\tau'\boldsymbol{n}$$

$$= (-r\kappa'' - \kappa^2 + r\kappa^3 + r\kappa\tau^2)\boldsymbol{t} + (-3r\kappa\kappa' + \kappa' - 3r\tau\tau')\boldsymbol{n} + (\kappa\tau - r\kappa^2\tau - r\tau^3 + r\tau'')\boldsymbol{b}$$

The product required for the torsion calculation is given by the following;

$$\left\langle\left(\frac{d\boldsymbol{\alpha_n}(s)}{ds} \times \frac{d^2\boldsymbol{\alpha_n}(s)}{ds^2}\right), \frac{d^3\boldsymbol{\alpha_n}(s)}{ds^3}\right\rangle$$
$$= (-r\tau\kappa + r^2\kappa^2\tau + r^2\tau^3)(-r\kappa'' - \kappa^2 + r\kappa^3 + r\kappa\tau^2)$$
$$+ (-r\tau' + r^2\kappa\tau' - r^2\kappa'\tau)(-3r\kappa\kappa' + \kappa' - 3r\tau\tau')$$
$$+ (\kappa - 2r\kappa^2 - r\tau^2 + r^2\kappa^3 + r^2\kappa\tau^2)(\kappa\tau - r\kappa^2\tau - r\tau^3 + r\tau'') = G(s)$$

We have that;

$$\boldsymbol{t_n}(s) = \frac{\frac{d\boldsymbol{\alpha_n}(s)}{ds}}{\left\|\frac{d\boldsymbol{\alpha_n}(s)}{ds}\right\|} = \frac{(1-r\kappa)\boldsymbol{t} + r\tau\boldsymbol{b}}{\sqrt{(1-r\kappa)^2 + r^2\tau^2}}$$

Now, let us calculate that the vectors $\boldsymbol{b_n}(s)$ and $\boldsymbol{n_n}(s)$.

$$\boldsymbol{b_n}(s) = \frac{\boldsymbol{\alpha}_n'(s) \times \boldsymbol{\alpha}_n''(s)}{\|\boldsymbol{\alpha}_n'(s) \times \boldsymbol{\alpha}_n''(s)\|}$$

$$= \frac{\begin{array}{c}(-r\tau\kappa + r^2\kappa^2\tau + r^2\tau^3)\boldsymbol{t} + (-r\tau' + r^2\kappa\tau' - r^2\kappa'\tau)\boldsymbol{n} + \\ (\kappa - 2r\kappa^2 - r\tau^2 + r^2\kappa^3 + r^2\kappa\tau^2)\boldsymbol{b}\end{array}}{F(s)}$$

and

$$\boldsymbol{n_n}(s) = \boldsymbol{b_n}(s) \times \boldsymbol{t_n}(s)$$

$$= \frac{1}{F(s)}\left[\begin{array}{c}(-r\tau\kappa + r^2\kappa^2\tau + r^2\tau^3)\boldsymbol{t} + (-r\tau' + r^2\kappa\tau' - r^2\kappa'\tau)\boldsymbol{n} \\ +(\kappa - 2r\kappa^2 - r\tau^2 + r^2\kappa^3 + r^2\kappa\tau^2)\boldsymbol{b}\end{array}\right] \times \frac{1}{A(s)}[(1 - r\kappa)\boldsymbol{t} + r\tau\boldsymbol{b}]$$

$$\begin{aligned}= \frac{1}{F(s).A(s)}\{&(-r^2\tau^2\kappa + r^3\kappa^2\tau^2 + r^3\tau^4)\boldsymbol{t} \times \boldsymbol{b} \\ &+ (-r\tau' + r^2\kappa\tau' - r^2\kappa'\tau + r^2\kappa\tau' - r^3\kappa^2\tau' + r^3\kappa\kappa'\tau)\boldsymbol{n} \times \boldsymbol{t} \\ &+ (-r^2\tau\tau' + r^3\kappa\tau\tau' - r^3\kappa'\tau^2)\boldsymbol{n} \times \boldsymbol{b} \\ &+ (\kappa - 2r\kappa^2 - r\tau^2 + r^2\kappa^3 + r^2\kappa\tau^2 - r\kappa^2 + 2r^2\kappa^3 + r^2\kappa\tau^2 - r^3\kappa^4 - r^3\kappa^2\tau^2)\boldsymbol{b} \\ &\times \boldsymbol{t}\}\end{aligned}$$

$$\begin{aligned}= \frac{1}{F(s).A(s)}\{&(-r^2\tau\tau' + r^3\kappa\tau\tau' - r^3\kappa'\tau^2)\boldsymbol{t} \\ &+ (-r^2\tau^2 + r^3\kappa\tau^2 + r^3\tau^4 + \kappa - 3r\kappa^2 - r\tau^2 + 3r^2\kappa^3 + 2r^2\kappa\tau^2 - r^3\kappa^4 \\ &- r^3\kappa^2\tau^2)\boldsymbol{n} + (r\tau' - 2r^2\kappa\tau' + r^2\kappa'\tau + r^3\kappa^2\tau' - r^3\kappa\kappa'\tau)\boldsymbol{b}\}.\end{aligned}$$

Theorem 2. The expression of the curvature and torsion functions of the parallel curve $\boldsymbol{\alpha_n}(s)$ in terms of the curvature and torsion functions of the $\boldsymbol{\alpha}(s)$ are as follows:

$$\kappa_n(s) = \frac{\sqrt{\begin{array}{c}(-r\tau\kappa + r^2\kappa^2\tau + r^2\tau^3)^2 + (-r\tau' + r^2\kappa\tau' - r^2\kappa'\tau)^2 + \\ (\kappa - 2r\kappa^2 - r\tau^2 + r^2\kappa^3 + r^2\kappa\tau^2)^2\end{array}}}{((1 - r\kappa)^2 + r^2\tau^2)^{3/2}}$$

$$\tau_n(s) = \frac{G(s)}{F^2(s)}$$

where

$$G(s) = (-r\tau\kappa + r^2\kappa^2\tau + r^2\tau^3)(-r\kappa'' - \kappa^2 + r\kappa^3 + r\kappa\tau^2) + (-r\tau' + r^2\kappa\tau' - r^2\kappa'\tau)(-3r\kappa\kappa' + \kappa' - 3r\tau\tau') + (1 - 2r\kappa - r\tau^2 + r^2\kappa^2 + r^2\kappa\tau^2)(\kappa\tau - r\kappa^2\tau - r\tau^3 + r\tau'')$$

and

$$F(s) = \sqrt{\begin{array}{c}(-r\tau\kappa + r^2\kappa^2\tau + r^2\tau^3)^2 + (-r\tau' + r^2\kappa\tau' - r^2\kappa'\tau)^2 + \\ (\kappa - 2r\kappa^2 - r\tau^2 + r^2\kappa^3 + r^2\kappa\tau^2)^2\end{array}}.$$

Proof. The curvature and torsion functions of the $\boldsymbol{\alpha_n}(s)$ in terms of the curvature and torsion functions of $\boldsymbol{\alpha}(s)$ are as follows:

$$\kappa_n(s) = \frac{\|\boldsymbol{\alpha}_n'(s) \times \boldsymbol{\alpha}_n''(s)\|}{\|\boldsymbol{\alpha}_n'(s)\|^3} = \frac{\sqrt{\begin{matrix}(-r\tau\kappa + r^2\kappa^2\tau + r^2\tau^3)^2 + (-r\tau' + r^2\kappa\tau' - r^2\kappa'\tau)^2 + \\ (\kappa - 2r\kappa^2 - r\tau^2 + r^2\kappa^3 + r^2\kappa\tau^2)^2\end{matrix}}}{((1-r\kappa)^2 + r^2\tau^2)^{3/2}}$$

$$\tau_n(s) = \frac{\langle(\boldsymbol{\alpha}_n'(s) \times \boldsymbol{\alpha}_n''(s)), \boldsymbol{\alpha}_n'''(s)\rangle}{\|\boldsymbol{\alpha}_n'(s) \times \boldsymbol{\alpha}_n''(s)\|^2} = \frac{G(s)}{F^2(s)}$$

where

$$G(s) = (-r\tau\kappa + r^2\kappa^2\tau + r^2\tau^3)(-r\kappa'' - \kappa^2 + r\kappa^3 + r\kappa\tau^2) + (-r\tau' + r^2\kappa\tau' - r^2\kappa'\tau)(-3r\kappa\kappa' + \kappa' - 3r\tau\tau') + (1 - 2r\kappa - r\tau^2 + r^2\kappa^2 + r^2\kappa\tau^2)(\kappa\tau - r\kappa^2\tau - r\tau^3 + r\tau'')$$

and

$$F(s) = \sqrt{\begin{matrix}(-r\tau\kappa + r^2\kappa^2\tau + r^2\tau^3)^2 + (-r\tau' + r^2\kappa\tau' - r^2\kappa'\tau)^2 + \\ (\kappa - 2r\kappa^2 - r\tau^2 + r^2\kappa^3 + r^2\kappa\tau^2)^2\end{matrix}}$$

as the proof of the first theorem.

Example 1. Consider the curve $\boldsymbol{\alpha}(s) = \left(acos\frac{s}{a}, asin\frac{s}{a}, 0\right)$, $a > 0$ and calculate the Frenet apparatus of parallel curve.

The definition of the parallel curve based on normal vector of the curve $\boldsymbol{\alpha}(s)$ is $\boldsymbol{\alpha_n}(s) = \boldsymbol{\alpha}(s) + r\boldsymbol{n}(s)$. Let us calculate the normal vector $\boldsymbol{n}(s)$ of the curve $\boldsymbol{\alpha}(s)$. Due the fact that the curve $\boldsymbol{\alpha}(s)$ is a unit speed curve, we obtain that

$$\boldsymbol{t}'(s) = \boldsymbol{\alpha}''(s) = \left(-\frac{1}{a}cos\frac{s}{a}, -\frac{1}{a}sin\frac{s}{a}, 0\right)$$

and

$$\|\boldsymbol{t}'(s)\| = \sqrt{\frac{1}{a^2}cos^2\frac{s}{a} + \frac{1}{a^2}sin^2\frac{s}{a}} = \frac{1}{a}.$$

So the normal vector is obtained as

$$\boldsymbol{n}(s) = \left(-cos\frac{s}{a}, -sin\frac{s}{a}, 0\right).$$

From here, the equation of the curve $\boldsymbol{\alpha_n}(s)$ is,

$$\boldsymbol{\alpha_n}(s) = \left(acos\frac{s}{a}, asin\frac{s}{a}, 0\right) + r\left(-cos\frac{s}{a}, -sin\frac{s}{a}, 0\right) = \left(acos\frac{s}{a} - rcos\frac{s}{a}, asin\frac{s}{a} - rsin\frac{s}{a}, 0\right).$$

Then we get that

$$\boldsymbol{\alpha_n}'(s) = \left(-sin\frac{s}{a} + \frac{r}{a}sin\frac{s}{a}, cos\frac{s}{a} - \frac{r}{a}cos\frac{s}{a}, 0\right)$$

and

$$\|\boldsymbol{\alpha_n}'(s)\| = \left(\left(-sin\frac{s}{a} + \frac{r}{a}sin\frac{s}{a}\right)^2 + \left(cos\frac{s}{a} - \frac{r}{a}cos\frac{s}{a}\right)^2\right)^{1/2}$$

$$= \sqrt{1 - \frac{2r}{a} + \frac{r^2}{a^2}} = \sqrt{\frac{a^2 - 2ar + r^2}{a^2}} = \sqrt{\frac{(a-r)^2}{a^2}} = \frac{|a-r|}{a}$$

So our curve is not unit speed. If we make the necessary calculations on arbitrary speed curves, we get that

$$\boldsymbol{\alpha_n}''(s) = \left(-\frac{1}{a}cos\frac{s}{a} + \frac{r}{a^2}cos\frac{s}{a}, -\frac{1}{a}sin\frac{s}{a} + \frac{r}{a^2}sin\frac{s}{a}, 0\right)$$

$$\boldsymbol{\alpha_n}'''(s) = \left(\frac{1}{a^2}sin\frac{s}{a} - \frac{r}{a^3}sin\frac{s}{a}, -\frac{1}{a^2}cos\frac{s}{a} + \frac{r}{a^3}cos\frac{s}{a}, 0\right)$$

$$\boldsymbol{\alpha_n}'(s) \times \boldsymbol{\alpha_n}''(s) = \begin{vmatrix} U_1 & U_2 & U_3 \\ -sin\frac{s}{a} + \frac{r}{a}sin\frac{s}{a} & cos\frac{s}{a} - \frac{r}{a}cos\frac{s}{a} & 0 \\ -\frac{1}{a}cos\frac{s}{a} + \frac{r}{a^2}cos\frac{s}{a} & -\frac{1}{a}sin\frac{s}{a} + \frac{r}{a^2}sin\frac{s}{a} & 0 \end{vmatrix}$$

$$= \left(0,0,\frac{a^2 - 2ak + r^2}{a^3}\right) = \left(0,0,\frac{(a-r)^2}{a^3}\right)$$

and

$$\|\boldsymbol{\alpha_n}'(s) \times \boldsymbol{\alpha_n}''(s)\| = \sqrt{0^2 + 0^2 + \left(\frac{(a-r)^2}{a^3}\right)^2} = \frac{(a-r)^2}{a^3}$$

$$\langle \boldsymbol{\alpha_n'}(s) \times \boldsymbol{\alpha_n''}(s), \boldsymbol{\alpha_n}'''(s)\rangle = 0$$

So the Frenet apparatus of the curve are;

$$\boldsymbol{t_n}(s) = \frac{\boldsymbol{\alpha_n}'(s)}{\|\boldsymbol{\alpha_n}'(s)\|} = \frac{a}{|a-r|}\left(-sin\frac{s}{a} + \frac{r}{a}sin\frac{s}{a}, cos\frac{s}{a} - \frac{r}{a}cos\frac{s}{a}, 0\right)$$

$$\boldsymbol{b_n}(s) = \frac{\boldsymbol{\alpha_n}'(s) \times \boldsymbol{\alpha_n}''(s)}{\|\boldsymbol{\alpha_n}'(s) \times \boldsymbol{\alpha_n}''(s)\|} = \frac{a^3}{(a-r)^2}\left(0,0,\frac{(a-r)^2}{a^3}\right) = (0,0,1)$$

$$\boldsymbol{n_n}(s) = \boldsymbol{b_n}(s) \times \boldsymbol{t_n}(s) = \frac{a}{|a-r|}\begin{vmatrix} U_1 & U_2 & U_3 \\ 0 & 0 & 1 \\ -sin\frac{s}{a} + \frac{r}{a}sin\frac{s}{a} & cos\frac{s}{a} - \frac{r}{a}cos\frac{s}{a} & 0 \end{vmatrix}$$

$$= \frac{a}{|a-r|}\left(-cos\frac{s}{a} + \frac{r}{a}cos\frac{s}{a}, -sin\frac{s}{a} + \frac{r}{a}sin\frac{s}{a}, 0\right).$$

The curvature and torsion functions of the curve are;

$$\kappa_n(s) = \frac{\|\boldsymbol{\alpha_n}'(s) \times \boldsymbol{\alpha_n}''(s)\|}{\|\boldsymbol{\alpha_n}'(s)\|^3} = \frac{\frac{(a-r)^2}{a^3}}{\left(\frac{|a-r|}{a}\right)^3} = \frac{1}{|a-r|}$$

$$\tau_n(s) = \frac{\langle \boldsymbol{\alpha}'_{\boldsymbol{n}}(s) \times \boldsymbol{\alpha}''_{\boldsymbol{n}}(s), \boldsymbol{\alpha_n}'''(s)\rangle}{\|\boldsymbol{\alpha_n}'(s) \times \boldsymbol{\alpha_n}''(s)\|^2} = \frac{0}{\left(\frac{(a-r)^2}{a^3}\right)^2} = 0.$$

This shows that parallel curve is also a circle (Figure 1).

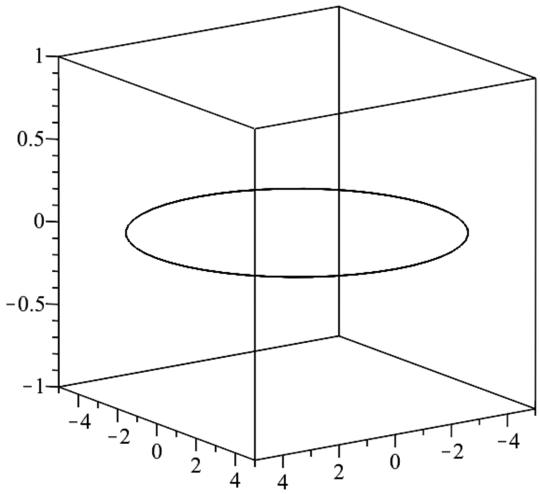

Figure 1. *Parallel curve $\alpha_n(s)$ for $a = 2$ and $r = -3$*

Remark: It is pointed out that parallel curves give a family of circles centered on the origin for $r \neq a$.

Example 2. Consider the helix $\boldsymbol{\alpha}(s) = \left(cos\frac{s}{\sqrt{2}}, sin\frac{s}{\sqrt{2}}, \frac{s}{\sqrt{2}}\right)$. Then let us calculate the Frenet apparatus of the parallel curve.

The definition of the parallel curve is $\boldsymbol{\alpha_n}(s) = \boldsymbol{\alpha}(s) + r\boldsymbol{n}(s)$. Let us calculate the normal vector $\boldsymbol{n}(s)$ of the curve $\boldsymbol{\alpha}(s)$. Using the fact that the curve $\boldsymbol{\alpha}(s)$ is a unit speed curve, we have that

$$\boldsymbol{t}'(s) = \boldsymbol{\alpha}''(s) = \left(-\frac{1}{2}cos\frac{s}{\sqrt{2}}, -\frac{1}{2}sin\frac{s}{\sqrt{2}}, 0\right)$$

and

$$\|\boldsymbol{t}'(s)\| = \sqrt{\frac{1}{4}cos^2\frac{s}{\sqrt{2}} + \frac{1}{4}sin^2\frac{s}{\sqrt{2}}} = \frac{1}{2}.$$

It holds that

$$\boldsymbol{n}(s) = \left(-cos\frac{s}{\sqrt{2}}, -sin\frac{s}{\sqrt{2}}, 0\right)$$

Then the equation of the curve $\boldsymbol{\alpha_n}(s)$ is,

$$\begin{aligned}\boldsymbol{\alpha_n}(s) &= \left(cos\frac{s}{\sqrt{2}}, sin\frac{s}{\sqrt{2}}, \frac{s}{\sqrt{2}}\right) + r\left(-cos\frac{s}{\sqrt{2}}, -sin\frac{s}{\sqrt{2}}, 0\right)\\ &= \left(cos\frac{s}{\sqrt{2}} - rcos\frac{s}{\sqrt{2}}, sin\frac{s}{\sqrt{2}} - rsin\frac{s}{\sqrt{2}}, \frac{s}{\sqrt{2}}\right)\\ &= \left((1-r)cos\frac{s}{\sqrt{2}}, (1-r)sin\frac{s}{\sqrt{2}}, \frac{s}{\sqrt{2}}\right).\end{aligned}$$

So it can be easily obtained that

$$\boldsymbol{\alpha_n}'(s) = \left(-\frac{(1-r)}{\sqrt{2}}sin\frac{s}{\sqrt{2}}, \frac{(1-r)}{\sqrt{2}}cos\frac{s}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

and

$$\|\boldsymbol{\alpha_n}'(s)\| = \left(\left(-\frac{(1-r)}{\sqrt{2}}sin\frac{s}{\sqrt{2}}\right)^2 + \left(\frac{(1-r)}{\sqrt{2}}cos\frac{s}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2\right)^{1/2} = \sqrt{\frac{r^2-2r+2}{2}}.$$

Hence our curve is not unit speed curve. If we make the necessary calculations on arbitrary speed curves, then the following equalities hold:

$$\boldsymbol{\alpha_n}''(s) = \left(-\frac{(1-r)}{2}cos\frac{s}{\sqrt{2}}, -\frac{(1-r)}{2}sin\frac{s}{\sqrt{2}}, 0\right)$$

$$\boldsymbol{\alpha_n}'''(s) = \left(\frac{(1-r)}{2\sqrt{2}}sin\frac{s}{\sqrt{2}}, -\frac{(1-r)}{2\sqrt{2}}cos\frac{s}{\sqrt{2}}, 0\right)$$

$$\boldsymbol{\alpha_n}'(s) \times \boldsymbol{\alpha_n}''(s) = \begin{vmatrix} U_1 & U_2 & U_3 \\ -\frac{(1-r)}{\sqrt{2}}sin\frac{s}{\sqrt{2}} & \frac{(1-r)}{\sqrt{2}}cos\frac{s}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{(1-r)}{2}cos\frac{s}{\sqrt{2}} & -\frac{(1-r)}{2}sin\frac{s}{\sqrt{2}} & 0 \end{vmatrix}$$

$$= \left(\frac{(1-r)}{2\sqrt{2}}sin\frac{s}{\sqrt{2}}, -\frac{(1-r)}{2\sqrt{2}}cos\frac{s}{\sqrt{2}}, \frac{(1-r)^2}{2\sqrt{2}}\right)$$

and

$$\|\boldsymbol{\alpha_n}'(s)\times\boldsymbol{\alpha_n}''(s)\| = \sqrt{\left(\frac{1-r}{2\sqrt{2}}\right)^2+\left(\frac{(1-r)^2}{2\sqrt{2}}\right)^2} = \frac{|1-r|\sqrt{2-2r+r^2}}{2\sqrt{2}}$$

$$\langle\boldsymbol{\alpha}_{\boldsymbol{n}}'(s)\times\boldsymbol{\alpha}_{\boldsymbol{n}}''(s),\boldsymbol{\alpha}_{\boldsymbol{n}}^{'''(s)}\rangle = \left(\frac{(1-r)}{2\sqrt{2}}\right)^2 sin^2\frac{s}{\sqrt{2}}+\left(\frac{(1-r)}{2\sqrt{2}}\right)^2 cos^2\frac{s}{\sqrt{2}} = \left(\frac{(1-r)}{2\sqrt{2}}\right)^2$$

Then Frenet frame of the curve is;

$$\boldsymbol{t_n}(s) = \frac{\boldsymbol{\alpha_n}'(s)}{\|\boldsymbol{\alpha_n}'(s)\|} = \frac{\sqrt{2}}{\sqrt{r^2-2r+2}}\left(-\frac{(1-r)}{\sqrt{2}}sin\frac{s}{\sqrt{2}},\frac{(1-r)}{\sqrt{2}}cos\frac{s}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)$$

$$\boldsymbol{b_n}(s) = \frac{\boldsymbol{\alpha}_{\boldsymbol{n}}'(s)\times\boldsymbol{\alpha}_{\boldsymbol{n}}''(s)}{\|\boldsymbol{\alpha}_{\boldsymbol{n}}'(s)\times\boldsymbol{\alpha}_{\boldsymbol{n}}''(s)\|} = \frac{2\sqrt{2}}{|1-r|\sqrt{2-2r+r^2}}\left(\frac{(1-r)}{2\sqrt{2}}sin\frac{s}{\sqrt{2}},-\frac{(1-r)}{2\sqrt{2}}cos\frac{s}{\sqrt{2}},\frac{(1-r)^2}{2\sqrt{2}}\right)$$

$$= \pm\frac{1}{\sqrt{2-2r+r^2}}\left(sin\frac{s}{\sqrt{2}},-cos\frac{s}{\sqrt{2}},1-r\right)$$

$$\boldsymbol{n_n}(s) = \boldsymbol{b_n}(s)\times\boldsymbol{t_n}(s) = \pm\frac{\sqrt{2}}{2-2r+r^2}\begin{vmatrix} U_1 & U_2 & U_3 \\ sin\frac{s}{\sqrt{2}} & -cos\frac{s}{\sqrt{2}} & 1-r \\ -\frac{(1-r)}{\sqrt{2}}sin\frac{s}{\sqrt{2}} & \frac{(1-r)}{\sqrt{2}}cos\frac{s}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{vmatrix}$$

$$= \pm\frac{\sqrt{2}}{2-2r+r^2}\left(\left(-\frac{1}{\sqrt{2}}-\frac{(1-r)^2}{\sqrt{2}}\right)cos\frac{s}{\sqrt{2}},\left(-\frac{(1-r)^2}{\sqrt{2}}-\frac{1}{\sqrt{2}}\right)sin\frac{s}{\sqrt{2}},0\right)$$

$$= \pm\left(-cos\frac{s}{\sqrt{2}},-sin\frac{s}{\sqrt{2}},0\right).$$

The curvature and torsion functions of the curve are given as follows:

$$\kappa_n(s) = \frac{\|\boldsymbol{\alpha_n}'(s)\times\boldsymbol{\alpha_n}''(s)\|}{\|\boldsymbol{\alpha_n}'(s)\|^3} = \frac{\frac{|1-r|\sqrt{2-2r+r^2}}{2\sqrt{2}}}{\left(\sqrt{\frac{r^2-2r+2}{2}}\right)^3} = \frac{|1-r|}{2(r^2-2r+2)}$$

$$\tau_n(s) = \frac{\langle\boldsymbol{\alpha}_{\boldsymbol{n}}'(s)\times\boldsymbol{\alpha}_{\boldsymbol{n}}''(s),\boldsymbol{\alpha_n}'''(s)\rangle}{\|\boldsymbol{\alpha_n}'(s)\times\boldsymbol{\alpha_n}''(s)\|^2} = \frac{\left(\frac{(1-r)}{2\sqrt{2}}\right)^2}{\left(\frac{|1-r|\sqrt{2-2r+r^2}}{2\sqrt{2}}\right)^2} = \pm\frac{1-r}{\sqrt{2-2r+r^2}}.$$

This shows that parallel curve is also a helix for $r \neq 1$ (Figure 2).

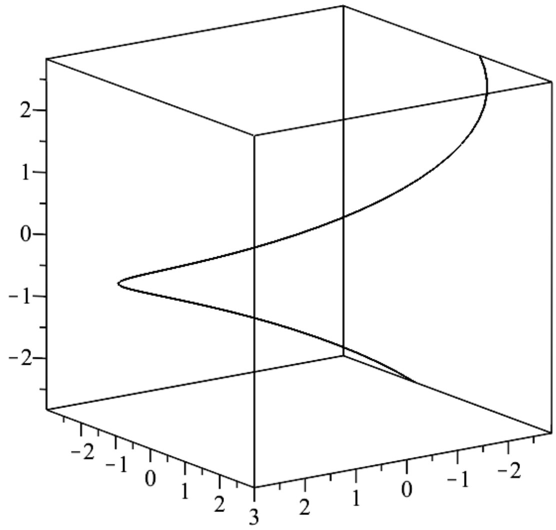

Figure 2. *Parallel curve* $\alpha_n(s)$ *for* $r = -2$

Remark: For $r \neq 1$, parallel curves give a family of helices whose axis is the z axis.

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References

[1] Arslan, K., & Hacısalihoğlu, H. H. (2000). On harmonic curvatures of a frenet curve. *Communications De La Faculte des Sciences de L'Universite D'Ankara*", *49*, 015-023. https://doi.org/10.1501/Commua1_0000000374

[2] Ergin, A. A. (1992). On the generalized darboux curves, *Communications De La Faculte des Sciences de L'Universite D'Ankara*", *41*, 073-077. https://doi.org/10.1501/Commua1_0000000499

[3] Liu, H., & Wang, F. (2008). Mannheim partner curves in 3-space. *Journal of Geometry*, *88*(1-2), 120-126. https://doi.org/10.1007/s00022-007-1949-0

[4] İlarslan, K., & Nesovic, E. (2008). Some characterization of rectifying curves in the Euclidean Space E^4, *Turkish Journal of Mathematics*, *32*, 21-30.

[5] Has, A., & Yılmaz, B. (2022). On quaternionic bertrand curves in euclidean 3-space. *Turkish Journal of. Mathematics and Computer Science*, *14*(2), 355-365. https://doi.org/10.47000/tjmcs.1021801

[6] Sağıroğlu, Y. (2015). Global differential invariants of affine curves in R2. *Far East Journal of Mathematical Sciences*, *96*(4), 497-515.

[7] Divjak, B. (2003). Special curves on ruled surface in galilean and pseudo-galilean space. *Acta Mathematica Hungarica, 98*(3), 203-215. https://doi.org/10.1023/A:1022821824927

[8] Bükcü, B., & Karacan, M. K. (2008). Bishop frame of the spacelike curve with a spacelike principal normal in minkowski 3-space. *Communications De La Faculte des Sciences de L'Universite D'Ankara*", *57*(1), 13-22. https://doi.org/10.1501/Commua1_0000000185

[9] Şenyurt, S., Canlı, D., & Ayvacı, K. H. (2022). Associated curves from a different point of view in E^3. *Communications Faculty of Sciences University of Ankara Series A1 Mathematics and Statistics, 71*(3), 826-845. https://doi.org/10.31801/cfsuasmas.1026359

[10] Güler, F. (2022). The quasi parallel curve of a space curve. *Celal Bayar University Journal of Science, 18*(2), 203-206. https://doi.org/10.18466/cbayarfbe.955974

[11] Srivastava, A., Klassen, E., Shantanu, H. J., & Jermyn, I. H. (2011). Shape analysis of elastic curves in euclidean spaces. *IEEE Transactions on Pattern Analysis and Machine Intelligence, 33*(7), 1415-1428. https://doi.10.1109/TPAMI.2010.184

[12] S´anchez-Reyes, J. (2015). Detecting symmetries in polynomial B´ezier curves. *Journal of Computational and Applied Mathematics, 288*, 274-283. https://doi.org/10.1016/j.cam.2015.04.025

[13] Gözütok, U., Çoban, H. A., & Sağıroğlu, Y. (2019). Frenet frame with respect to conformable derivative. *Filomat, 33*(6), 1541-1550. https://doi.org/10.2298/FIL1906541G

[14] Aldossary, M., T., & Gazwani, M. A. (2020). Motion of parallel curves and surfaces in euclidean 3-space R^3. *Global Journal of Advanced Research on Classical and Modern Geometries, 9*(1), 43-56.

[15] O'Neill, B. (2006). *Elementary Differential Geometry, Revised Second Edition,* Elsevier, USA.