Bilecik Şeyh Edebali Üniversitesi Fen Bilimleri Dergisi
Bilecik Seyh Edebali University Journal of Science
11(2), 270-275, 2024
DOİ: 10.35193/bseufbd.1315932
e-ISSN: 2458-7575 (https://dergipark.org.tr/tr/pub/bseufbd)

Araştırma Makalesi - Research Article

A New Method for Solving Fisher-Type Equations

Fisher Tipi Denklemleri Çözmek İçin Yeni Bir Metod

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Geliş / Received: 17/06/2023 ***Revize / Revised: 22/11/2023*** ***Kabul / Accepted: 16/12/2023***

ABSTRACT

In this article, we implemented the Fourier Adomian Decomposition Method (FADM) which depends on the Fourier transform method and the Adomian decomposition method to solve Fisher-type equations. Besides, two examples are represented to show the accuracy and validity of the proposed method.

Keywords- Fourier Transform Method, Fisher Equation, Adomian Decomposition Method

ÖZ

Bu makalede, Fisher tipi denklemleri çözmek için Fourier dönüşüm metodu ve Adomian Ayrıştırma Metoduna bağlı Fourier Adomian ayrıştırma metodunu (FADM) uyguladık. Ayrıca önerilen metodun dogrulugunu göstermek için iki örnek verildi.

Anahtar Kelimeler- Fourier Dönüşüm Metodu, Fisher Denklem, Adomian Ayrıştırma Metodu

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I. INTRODUCTION

The Fisher-type equations are a class of partial differential equations commonly used to model the spatial dynamics of population growth and propagation. Named after biologist and statistician R.A. Fisher, these equations play a crucial role in understanding the spread and distribution of biological species, epidemics, and other phenomena with spatial components [1,2]. Several methods for solving Fisher-type equations have been developed by many researchers. Humaira and Shah [3] have employed Laplace Adomian decomposition method to solve the general Fisher's equation. In [4], Yıldırım and Bayram, implemented the reduced differential transform method (RDTM) to construct explicit /exact solutions of Fisher-type equations. Ağırseven and Öziş [5] used the homotopy perturbation method for solving Fisher type equations. In addition, Bhalekar and Patade [6] used the decomposition method to give an analytical solution to Fisher's equation. Other methods for the Fishers equation are the Exp-function method [7], Q-function method [8], Adomian decomposition method [9], Laplace transform, and new homotopy perturbation methods [10]. In this article, we propose a method, namely, The Fourier Adomian Decomposition Method (FADM) to solve Fisher-type equations given by

$$u_t = u_{xx} + \alpha u(1-u). \tag{1}$$

The paper is organized as follows: In Section 2, basic definitions and theorems related to the Fourier transform and Adomian decomposition method. In Section 3, some examples have been given for the solution of the fisher equation by using FADM. Finally, we have given a conclusion.

In this article, the Fisher-type equations have been solved using the proposed method for several reasons: Firstly, FADM is known for its versatility, it can handle nonlinear problems like Fisher-type equations with relative ease. Its ability to break down complex problems into more manageable parts makes it a valuable tool in the researcher's toolbox. Secondly, FADM has a solid theoretical foundation. Leveraging the Fourier transform and the Adomian polynomials, it provides a systematic and efficient way to approximate solutions. This method allows you to express the solution as a series, making it easier to work with and potentially yielding more accurate results. Finally, FADM often proves computationally efficient, especially when dealing with problems that might be challenging for other numerical methods. This can save valuable time and resources in the research process. The choice to use FADM boils down to its adaptability, theoretical robustness, and computational efficiency, which are essential when tackling complex Fisher-type equations.

II. PRELIMINARIES AND THEOREMS

Definition 2.1. : [12] The Fourier transform of $f(t)$ is given by

$$\mathcal{F}[f(t)] = F(w) = \int_{-\infty}^{\infty} f(t).e^{-iwt}dt \tag{2}$$

Definition 2.2. : [12] The inverse Fourier transform of $F(w)$ is given by

$$f(t) = \mathcal{F}^{-1}[F(w)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(w).e^{iwt}dt \tag{3}$$

Theorem 2.1.[11-12] (Linearity of Fourier Transform) If $\mathcal{F}[f_1(t)] = F_1(w), \mathcal{F}[f_2(t)] = F_2(w)$, then

$$\mathcal{F}[r_1.f_1(t) + r_2.f_2(t)] = r_1.F_1(w) + r_2.F_2(w),$$

where r_1, r_2 are arbitrary constants.

Theorem 2.2.[11-12] Let $f(t)$ be continuous or partially continuous in $(-\infty,\infty)$, and $f(t), f'(t), f''(t), \ldots, f^{(n-1)}(t) \to 0$ for $|t| \to \infty$. Also, if $f(t), f'(t), f''(t), \ldots, f^{(n-1)}(t)$ are absolutely integrable in $(-\infty,\infty)$, then

$$\mathcal{F}\left[f^{(n)}(t)\right] = (iw)^n \mathcal{F}[f(t)] \tag{4}$$

Definition 2.3. The Dirac delta function is given by

$$\delta(t) = \begin{cases} 0, & t \neq 0 \\ \infty, & t = 0 \end{cases}$$

Some properties of the Dirac Delta distribution are as follows [12]:

i. $\int_{-\infty}^{\infty} \delta(t)dt = 1$

ii. $\int_{-\infty}^{\infty} f(t).\delta(t-t_0)dt = f(t_0)$ (5)

iii. $\int_{-\infty}^{\infty} f(t).\delta^{(n)}(t-t_0)dt = (-1)^n.f^{(n)}(t_0)$ (6)

iv. $(t-t_0)^n\delta^{(n)}(t_0) = (-1)^n n!\,\delta(t-t_0)$ (7)

Where $\delta(t-t_0)$ is given by

$$\delta(t-t_0) = \begin{cases} 0, & t \neq t_0 \\ \infty, & t = t_0 \end{cases}$$

Theorem 2.3. [11-12] The Fourier transform of $\delta(t)$ is 1.

Theorem 2.4.[11-13] The Fourier transforms for some functions are following

$i)\mathcal{F}[1] = 2\pi.\delta(w)$

$ii)\ \mathcal{F}[t^n] = 2\pi.i^n.\delta^{(n)}(w)$

$iii)\mathcal{F}[e^{iw_0t}] = 2\pi\delta(w-w_0)$

$iv)\mathcal{F}[e^{at}] = 2\pi\delta(w+ia)$

Lemma 1. The Fourier Transform of Partial derivative functions are following:

$$\mathcal{F}\left[\frac{\partial f}{\partial x}\right] = iwF(w,y) \quad (8)$$

$$\mathcal{F}\left[\frac{\partial f}{\partial y}\right] = \frac{\partial F(w,y)}{\partial y} \quad (9)$$

$$\mathcal{F}\left[\frac{\partial^2 f}{\partial x^2}\right] = -w^2F(w,y) \quad (10)$$

$$\mathcal{F}\left[\frac{\partial^2 f}{\partial y^2}\right] = \frac{\partial^2 F(w,y)}{\partial y^2} \quad (11)$$

III. FADM FOR FISHER'S EQUATION

In this section, we will show the reliability of the proposed method and its compatibility with some real physical processes.

Consider the following general form of the nonlinear diffusion equation with the specified initial condition:

$$u_t = u_{xx} + F(u) \quad (12)$$

$$u(x,0) = f(x) \quad (13)$$

where $F(u)$ is a continuous nonlinear function that satisfies the conditions

$$F(0) = F(1) = 0$$

$$F'(0) > 0 > F'(1)$$

$$F(u) > 0, 0 < u < 1$$

The methodology consists of applying Fourier transform first on both sides of Eq. (12)

$$\mathcal{F}(u_t) = \mathcal{F}(u_{xx}) + \mathcal{F}(F(u))$$

Using the differentiation property of Fourier transform, we get

$$\frac{\partial U(w,t)}{\partial t} = (iw)^2U(w,t) + \mathcal{F}(F(u)) \quad (14)$$

The second step in the Fourier Adomian decomposition method is that we represent solution as an infinite series given by

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + \cdots$$

Where the component $u_n(x,t), n \geq 0$ will be determined in a recursive manner.

Also, by applying inverse Fourier transform to Eq (14), our required recursive relation is given below

$$\mathcal{F}^{-1}\left[\frac{\partial U(w,t)}{\partial t}\right] = -\mathcal{F}^{-1}\big(w^2U(w,t)\big)+\mathcal{F}^{-1}\Big(\mathcal{F}\big(F(u)\big)\Big)$$

$$\frac{\partial u_{n+1}}{\partial t} = -\mathcal{F}^{-1}\big(w^2U_n(w,t)\big) + \mathcal{F}^{-1}\big(\mathcal{F}(A_n)\big), n \geq 0$$

Where $u_0 = u(x,0) = f(x)$ from the initial condition, and $A_n{}'s$ are the Adomian polynomials.

Example 1: [3,5,6,9,10] Consider the following Fisher's equation

$$u_t = u_{xx} + \alpha u(1-u) \tag{15}$$

with initial condition $u(x,0) = \lambda$ for $\alpha = 1$

The exact solution of Eq (15) is

$$u(x,t) = \frac{\lambda e^t}{1-\lambda+\lambda e^t}$$

By using Fourier transform of Eq (15), we obtain

$$\mathcal{F}(u_t) = \mathcal{F}(u_{xx}) + \mathcal{F}(u) - \mathcal{F}(u^2)$$

$$\frac{\partial U(w,t)}{\partial t} = (iw)^2 U(w,t) + \mathcal{F}(u) - \mathcal{F}(u^2)$$

Now, using the inverse Fourier transform of the previous equation, we obtain

$$\mathcal{F}^{-1}\left[\frac{\partial U(w,t)}{\partial t}\right] = -\mathcal{F}^{-1}\big(w^2 U(w,t)\big) + \mathcal{F}^{-1}\big(\mathcal{F}(u)\big) - \mathcal{F}^{-1}\big(\mathcal{F}(u^2)\big)$$

$$\frac{\partial u_{n+1}}{\partial t} = -\mathcal{F}^{-1}\big(w^2 U_n(w,t)\big) + u_n - \mathcal{F}^{-1}\big(\mathcal{F}(A_n)\big)$$

The first few components of $u_n(x,t)$ are given by

$$u_0 = \lambda,\ \ A_0 = u_0^2,\quad U_0(w,t) = \mathcal{F}(u_0) = \mathcal{F}(\lambda) = 2\pi\lambda.\delta(w)$$

$$\frac{\partial u_1}{\partial t} = -\mathcal{F}^{-1}\big(w^2 U_0(w,t)\big) + u_0 - \mathcal{F}^{-1}\big(\mathcal{F}(A_0)\big) = u_0 - u_0^2 = \lambda - \lambda^2$$

$$u_1 = \lambda t - \lambda^2 t,\ A_1 = 2u_0.u_1 = 2\lambda(\lambda t - \lambda^2 t),\ U_1(w,t) = \mathcal{F}(u_1) = 2\pi\delta.(\lambda t - \lambda^2 t)$$

$$\frac{\partial u_2}{\partial t} = -\mathcal{F}^{-1}\big(w^2 U_1(w,t)\big) + u_1 - \mathcal{F}^{-1}\big(\mathcal{F}(A_1)\big) == -\mathcal{F}^{-1}\big(w^2 2\pi\delta(\lambda t - \lambda^2 t)\big) + (\lambda t - \lambda^2 t) - 2\lambda(\lambda t - \lambda^2 t)$$

$$= \lambda t - \lambda^2 t - 2\lambda^2 t + 2\lambda^3 t$$

$$u_2 = \frac{\lambda t^2}{2} - \frac{3\lambda^2 t^2}{2} + \lambda^3 t^2,$$

$$A_2 = 2u_0.u_2 + u_1^2 = (2\lambda^2 - 5\lambda^3 + 3\lambda^4)t^2$$

$$U_2(w,t) = \mathcal{F}(u_2) = 2\pi\delta\left(\frac{\lambda t^2}{2} - \frac{3\lambda^2 t^2}{2} + \lambda^3 t^2\right)$$

$$\frac{\partial u_3}{\partial t} = -\mathcal{F}^{-1}\big(w^2 U_2(w,t)\big) + u_2 - \mathcal{F}^{-1}\big(\mathcal{F}(A_2)\big)$$

$$= -\mathcal{F}^{-1}\left(w^2 2\pi\delta\left(\frac{\lambda t^2}{2} - \frac{3\lambda^2 t^2}{2} + \lambda^3 t^2\right)\right) + \frac{\lambda t^2}{2} - \frac{3\lambda^2 t^2}{2} + \lambda^3 t^2 - (2\lambda^2 - 5\lambda^3 + 3\lambda^4)t^2$$

$$= \frac{\lambda t^2}{2} - \frac{3\lambda^2 t^2}{2} + \lambda^3 t^2 - (2\lambda^2 - 5\lambda^3 + 3\lambda^4)t^2$$

$$u_3 = \left(\frac{\lambda}{2} - \frac{7\lambda^2}{2} + 6\lambda^3 - 3\lambda^4\right)\frac{t^3}{3}$$

$$\vdots$$

and so on. Therefore, on taking the sum of the above iterations, we get

$$u(x,t)=\sum_{n=0}^{\infty}u_n(x,t)=\lambda+(\lambda-\lambda^2)t+\left(\frac{\lambda}{2}-\frac{3\lambda^2}{2}+\lambda^3\right)t^2+\left(\frac{\lambda}{6}-\frac{7\lambda^2}{6}+2\lambda^3-\lambda^4\right)t^3+\cdots$$

Which leads to the exact solution of equation (15)

$$u(x,t)=\frac{\lambda e^t}{1-\lambda+\lambda e^t}$$

Example 2[**5**, **6**] Consider the following Fisher's equation

$$u_t=u_{xx}+6u(1-u) \tag{16}$$

with initial condition $u(x,0)=\frac{1}{(1+e^x)^2}$

By using Fourier transform of Eq (16), we obtain

$$\mathcal{F}(u_t)=\mathcal{F}(u_{xx})+6\mathcal{F}(u)-6\mathcal{F}(u^2)$$

$$\frac{\partial U(w,t)}{\partial t}=(iw)^2U(w,t)+6\mathcal{F}(u)-6\mathcal{F}(u^2)$$

Now, using the inverse Fourier transform of the previous equation, we obtain

$$\frac{\partial u_{n+1}}{\partial t}=-\mathcal{F}^{-1}\big(w^2U_n(w,t)\big)+\mathcal{F}^{-1}\big(\mathcal{F}(A_n)\big)$$

The first few components of $u_n(x,t)$ are given by

$$u_0=\frac{1}{(1+e^x)^2},\ A_0=6u_0-6u_0^2$$

$$\frac{\partial u_1}{\partial t}=-\mathcal{F}^{-1}\big(w^2U_0(w,t)\big)+A_0=\frac{\partial^2u_0}{\partial x^2}+6u_0-6u_0^2=\frac{4e^{2x}-2e^x}{(1+e^x)^4}+\frac{6}{(1+e^x)^2}-\frac{6}{(1+e^x)^4}=\frac{10e^x}{(1+e^x)^3}$$

$$u_1(x,t)=\frac{10e^xt}{(1+e^x)^3}$$

$$A_1=6u_1-12u_0u_1$$

$$\frac{\partial u_2}{\partial t}=-\mathcal{F}^{-1}\big(w^2U_1(w,t)\big)+A_1=\frac{\partial^2u_1}{\partial x^2}+A_1=\frac{50e^x(e^{2x}-1)}{(1+e^x)^4}t$$

$$u_2(x,t)=\frac{25e^x(e^{2x}-1)}{(1+e^x)^4}t^2$$

$$A_2=6u_2-12u_0u_2-6u_1^2$$

$$\frac{\partial u_3}{\partial t}=-\mathcal{F}^{-1}\big(w^2U_2(w,t)\big)+A_2=\frac{\partial^2u_2}{\partial x^2}+A_2=-125\frac{e^x(-1+7e^x-4e^{2x})}{(1+e^x)^5}t^2$$

$$u_3(x,t)=-\frac{125}{3}\frac{e^x(-1+7e^x-4e^{2x})}{(1+e^x)^5}t^3$$

$$\vdots$$

and so on. Therefore, on taking the sum of the above iterations, we get

$$u(x,t)=\frac{1}{(1+e^x)^2}+\frac{10e^xt}{(1+e^x)^3}+\frac{25e^x(e^{2x}-1)}{(1+e^x)^4}t^2-\frac{125}{3}\frac{e^x(-1+7e^x-4e^{2x})}{(1+e^x)^5}t^3+\cdots$$

Which matches highly accurately with the literature

IV. CONCLUSION

In this study, employing the Fourier Adomian Decomposition Method (FADM) to solve Fisher-type equations has proven to be a good choice. Through its versatile approach, FADM effectively handles the inherent nonlinearity of Fisher equations, providing a systematic framework for obtaining accurate approximations. The theoretical underpinnings of FADM, utilizing Fourier transform and Adomian polynomials, contribute to the method's reliability and efficiency. The ability to decompose complex problems into more manageable components facilitates a clearer understanding of the solution process. Furthermore, FADM showcases its prowess, demonstrating efficiency in tackling challenges that may pose difficulties for other numerical methods. This computational advantage translates to saved time and resources in the research endeavour. Finally, the application of FADM in solving Fisher-type equations emerges as a robust and efficient methodology, offering a promising avenue for researchers seeking accurate solutions to complex mathematical problems in various scientific domains.

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