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Research Article

# Extension of the Lotka-Volterra competition model

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## Abstract

In this paper, we introduce the $(p, q)$-Lotka-Volterra competition model which is extension of classical Lotka-Volterra competition model. The main purpose is to give some results on the existence and non-existence of positive solutions. Upper and lower solutions technique and comparison arguments plays a significant role in our main proof.

**Mathematics Subject Classification (2020).** 35J25, 35J55

**Keywords.** Lotka-Volterra competition model, $(p, q)$-Laplacian, upper and lower solutions

## 1. Introduction

As an important mathematical area, nonlinear partial diferential equations have been one of the most active research fields in the 21st century. Modelling and analyzing the dynamics of biological populations by means of differential equations is one of the primary concerns of many applied scientists. During the past few years, Lotka-Volterra models have been extensively studied in various disciplines of science and engineering fields for the description of different applied problems. A particular model which has been widely investigated is the following Lotka-Volterra system:

$$\begin{cases} \frac{du}{dt} = k_1\Delta u + u[a - bu - cv], & t > 0, \\ \frac{dv}{dt} = k_2\Delta v + v[d - eu - fv], & t > 0. \end{cases} \tag{1.1}$$

Here the equations are assumed to be satisfied in a cylinder $x \in \bar{\Omega}$, $t \in (0, \infty)$, where $\Omega$ is an open, bounded, smooth domain in $\mathbb{R}^n$. These equations are supplemented by linear boundary conditions on $\partial\Omega \times (0, \infty)$. The solutions to (1.1) represent population densities for the competing species.

The Lotka-Volterra system is the most fundamental model of the population dynamics of species in a competition/predator-prey/cooperating relationship. As such, it has been studied extensively, and much is known about its properties. Nonetheless, there is still much to be discovered. In particular, despite the apparent simplicity of the equations, it appears difficult to determine analytical expressions for their solution.

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In [2] by the method of upper and lower solutions and its associated monotone iterations, existence and non-existence results was derived for both the scalar problem and for systems with fractional diffusion. Also, in [7], the authors gave simple extensions of the Lotka-Volterra prey-Predator model.

In this paper, we are concerned with the existence and non-existence of positive solutions to the $(p,q)$-Lotka-Volterra model of the form:

$$\begin{cases} -\Delta_p u = a u^{p-1} - u^{r-1} - \alpha u^{p-1} v^{q-1}, & x \in \Omega, \\ -\Delta_q v = b v^{q-1} - v^{s-1} - \beta u^{p-1} v^{q-1}, & x \in \Omega, \\ u = 0 = v, & x \in \partial\Omega, \end{cases} \tag{1.2}$$

where $\Delta_p$ denotes the $p$-Laplacian operator defined by $\Delta_p z = div\,(|\nabla z|^{p-2}\nabla z)$, $p,q>1$, $r>p$, $s>q$, $a,b>0$, $\alpha,\beta\geq 0$ and $\Omega \subseteq \mathbb{R}^N$ is a bounded region with smooth boundary $\partial\Omega$ with $N \geq 1$. Here, $u$ and $v$ denote the densities of two species in $\Omega$ (the habitat), which is surrounded by inhospitable areas due to the homogeneous Dirichlet boundary conditions. In (1.2) we assume that the species diffuse following the $(p,q)$-Laplacian. In fact, the $(p,q)$-Laplacian operator acts as the diffusive mechanism describing the migration of $u$ and $v$ throughout $\Omega$.

If $p=q=2$ ( The Laplace operator) and $r=s=3$, then (1.2) becomes the classical Lotka-Volterra model which has been extensively studied by many scholars, the interested readers may refer to [9–11, 13, 16, 19–21, 25, 26] and the references therein. This paper is motivated, in part, by the mathematical difficulty posed by the $(p,q)$-Laplacian operator compared to the Laplacian operator ( $p=q=2$ ).

During the last few decades, there has been growing interest in the investigation of various composite type operators such as the $(p,q)$-Laplacian. This operator has a wide range of applications in physics and related sciences like chemical reaction design [3], biophysics [12], plasma physics [24] and models of elementary particles [6]. Furthermore, (1.2) arises in the theory of quasiregular and quasiconformal mappings or in the study of non-Newtonian fluids ( see [4]). In the non-Newtonian fluids theory, the pair $(p,q)$ is a characteristic quantity of the medium. Media with $(p,q)>(2,2)$ are called dilatant fluids and those with $(p,q)<(2,2)$ are called pseudoplastics. If $(p,q)=(2,2)$, they are Newtonian fluids. One can refer to [5, 14, 17, 22, 23] for some existence results of $(p,q)$-Laplacian systems.

Our first result deals with the existence of positive solution for (1.2) which has $(p,q)$-Laplacian operator. Let $\lambda_p, \lambda_q$ be the respective first eigenvalues of $-\Delta_p$, $-\Delta_q$ with Dirichlet boundary conditions and $\phi_p, \phi_q$ be the corresponding eigenfunctions with $\phi_p, \phi_q > 0$; $\Omega$ and $\|\phi_p\|_\infty = 1 = \|\phi_q\|_\infty$. Further, by Hopf's lemma $|\nabla\phi_p|, |\nabla\phi_q| > 0$ on $\partial\Omega$. We establish:

**Theorem 1.1** Let $c=\min\{a,b\}$ and $\gamma=\max\{\alpha,\beta\}$. If $c>\max\{(p/p-1)^{p-1}\lambda_p, (q/q-1)^{q-1}\lambda_q\}$, then there exists $\gamma^*$ such that for $\gamma<\gamma^*$, system (1.2) has a positive solution.

Next we will establish a nonexistence result for our model.

**Theorem 1.2** Let $d=\max\{a,b\}$. Then there exists $\lambda_0>0$ such that for $0<d<\lambda_0$, system (1.2) has no nontrivial nonnegative weak solution.

This article is organized as follows. In Section 2, we will recall some important results that are required for the development of this paper. Section 3 is dedicated to the proof of Theorem 1.1. Section 4 contains the proof of Theorem 1.2

## 2. Preliminaries

In this section, we recall some results concerning a lower and upper-solution method (see [1]) for $(p, q)$-Laplacian system.

**Definition 2.1.** A pair of functions $(u, v) \in W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$ is said to be a (weak) solution of (1.2), if for any $w \in C_0^\infty(\Omega)$, we have

$$\int_\Omega |\nabla u|^{p-2}\,\nabla u \cdot \nabla w\,dx = \int_\Omega \Big(au^{p-1} - u^{r-1} - \alpha u^{p-1}v^{q-1}\Big)w\,dx,$$

and

$$\int_\Omega |\nabla v|^{q-2}\,\nabla v \cdot \nabla w\,dx = \int_\Omega \Big(bv^{q-1} - v^{s-1} - \beta u^{p-1}v^{q-1}\Big)w\,dx.$$

We establish Theorem 1.1 by the method of lower and upper-solution.

**Definition 2.2.** We say that the pair $(\psi_1, \psi_2) \in W^{1,p}(\Omega) \cap C(\overline{\Omega}) \times W^{1,q}(\Omega) \cap C(\overline{\Omega})$ is a weak lower solution of (1.1) if $\psi_1, \psi_2 \le 0$ on $\in \partial\Omega$ and

$$\int_\Omega |\nabla \psi_1|^{p-2}\,\nabla\psi_1 \cdot \nabla w\,dx \le \int_\Omega (a\psi_1^{p-1} - \psi_1^{r-1} - \alpha\psi_1^{p-1}\psi_2^{q-1})w\,dx,$$
$$\int_\Omega |\nabla \psi_2|^{q-2}\,\nabla\psi_2 \cdot \nabla w\,dx \le \int_\Omega (b\psi_2^{q-1} - \psi_2^{s-1} - \beta\psi_1^{p-1}\psi_2^{q-1})w\,dx,$$

for all $w \in W = \{w \in C_0^\infty(\Omega) | w \ge 0, x \in \Omega\}$.

Similarly one defines a weak upper solution $(z_1, z_2)$ of system (1.1), by considering the reversed inequalities in the above definition.

**Notation.** If $u, v \in C(\Omega)$, with $u(x) \le v(x)$ for *a.e.* $x \in \Omega$, we denote by $[u, v]$ the set $\{w \in C(\Omega) : u(x) \le w(x) \le v(x),\ a.e.\ x \in \Omega\}$.

Then the following lower-upper solution result holds.

**Lemma 2.3.**( [8], p. 269). Let $(\psi_1, \psi_2)$ and $(z_1, z_2)$ be lower and upper solutions of (1.2) respectively such that $(\psi_1, \psi_2) \le (z_1, z_2)$, in $\Omega$. Then (1.2) has a solution $(u, v) \in W^{1,p}(\Omega) \cap C(\overline{\Omega}) \times W^{1,q}(\Omega) \cap C(\overline{\Omega})$ such that $(u, v) \in [(\psi_1, \psi_2), (z_1, z_2)]$.

## 3. Proof of Theorem 1.1

In this section, we use lower and upper solution method to prove Theorem 1.1. We adapt and extend the ideas used in [15, 18] to construct a crucial lower-solution.

**Construction of lower-solution**: Let $k_1, k_1 > 0$ be such that $k_1 \le \Big(\frac{1}{2}[a-(\frac{p}{p-1})^{p-1}\lambda_p]\Big)^{1/r-p}$ and $k_2 \le \Big(\frac{1}{2}[b-(\frac{q}{q-1})^{q-1}\lambda_q]\Big)^{1/s-q}$. Define $(\psi_1, \psi_2) = (k_1\phi_p^{\frac{p}{p-1}}, k_2\phi_q^{\frac{q}{q-1}})$. Let $w \in W$. Then, a calculation shows that

$$\begin{aligned}\int_\Omega |\nabla \psi_1|^{p-2}\,\nabla\psi_1\cdot\nabla w\,dx &= k_1^{p-1}(\frac{p}{p-1})^{p-1}\int_\Omega \phi_p\,|\nabla\phi_p|^{p-2}\,\nabla\phi_p\cdot\nabla w\,dx\\ &= k_1^{p-1}(\frac{p}{p-1})^{p-1}\int_\Omega \Big[|\nabla\phi_p|^{p-2}\,\nabla\phi_p\cdot\nabla(\phi_p w)-|\nabla\phi_p|^p\,w\Big]dx\\ &= k_1^{p-1}(\frac{p}{p-1})^{p-1}\int_\Omega(\lambda_p\,\phi_p^p-|\nabla\phi_p|^p)w\,dx.\end{aligned}$$

and

$$\begin{aligned}&\int_\Omega(a\psi_1^{p-1}-\psi_1^{r-1}-\alpha\psi_1^{p-1}\psi_2^{q-1})w\,dx\\ &=\int_\Omega(ak_1^{p-1}\phi_p^p-k_1^{r-1}\phi_p^{\frac{p(r-1)}{p-1}}-\alpha k_1^{p-1}k_2^{q-1}\phi_p^p\phi_q^q)w\,dx.\end{aligned}$$

Similarly

$$\int_\Omega |\nabla \psi_2|^{q-2}\,\nabla\psi_2\cdot\nabla w\,dx = k_2^{q-1}(\frac{q}{q-1})^{q-1}\int_\Omega(\lambda_q\,\phi_q^q-|\nabla\phi_q|^q)w\,dx,$$

and

$$\begin{aligned}&\int_\Omega(b\psi_2^{q-1}-\psi_2^{s-1}-\beta\psi_1^{p-1}\psi_2^{q-1})w\,dx\\ &=\int_\Omega(bk_2^{q-1}\phi_q^q-k_2^{s-1}\phi_q^{\frac{q(s-1)}{q-1}}-\beta k_1^{p-1}k_2^{q-1}\phi_p^p\phi_q^q)w\,dx.\end{aligned}$$

Since $|\nabla\phi_p|>0$ on $\partial\Omega$, and $\phi_p\in C^\infty(\bar\Omega)$, then, by continuity, there exist a $\delta$ neighborhood of $\bar\Omega$, say $\bar{\Omega_\delta}:=\{x\in\Omega\,:\,dist(x,\partial\Omega)\le\delta\}$ and $\eta>0$ such that $|\nabla\phi_p|\ge\eta$ in $\bar{\Omega_\delta}$ and similarly, $|\nabla\phi_q|\ge\eta$ in $\bar{\Omega_\delta}$. Let

$$\gamma^*=\min\Big\{\frac{(p/p-1)^{p-1}\eta^p}{k_2^{q-1}},\frac{(q/q-1)^{q-1}\eta^q}{k_1^{p-1}},\frac{\frac12[a-(\frac{p}{p-1})^{p-1}\lambda_p]}{k_2^{q-1}},\frac{\frac12[b-(\frac{q}{q-1})^{q-1}\lambda_q]}{k_1^{p-1}}\Big\}.$$

To prove $(\psi_1,\psi_2)$ is a lower-solution of (1.2), we need to establish:

$$\begin{aligned}&k_1^{p-1}(\frac{p}{p-1})^{p-1}\int_\Omega(\lambda_p\,\phi_p^p-|\nabla\phi_p|^p)wdx\\ &\le\int_\Omega(ak_1^{p-1}\phi_p^p-k_1^{r-1}\phi_p^{\frac{p(r-1)}{p-1}}-\alpha k_1^{p-1}k_2^{q-1}\phi_p^p\phi_q^q)wdx,\end{aligned}\tag{3.1}$$

and

$$\begin{aligned}&k_2^{q-1}(\frac{q}{q-1})^{q-1}\int_\Omega(\lambda_q\phi_q^q-|\nabla\phi_q|^q)wdx\\ &\le\int_\Omega(bk_2^{q-1}\phi_q^q-k_2^{s-1}\phi_q^{\frac{q(s-1)}{q-1}}-\beta k_1^{p-1}k_2^{q-1}\phi_p^p\phi_q^q)wdx,\end{aligned}\tag{3.2}$$

in $\Omega$ if $\gamma<\gamma^*$. To achieve (3.1) we split the term $k_1^{p-1}(\frac{p}{p-1})^{p-1}\lambda_p\phi_p^p$ into three parts, namely,

$$\begin{aligned}k_1^{p-1}(\frac{p}{p-1})^{p-1}\lambda_p\phi_p^p &= ak_1^{p-1}\phi_p^p-\frac12 k_1^{p-1}\phi_p^p\Big(a-(\frac{p}{p-1})^{p-1}\lambda_p\Big)\\ &- \frac12 k_1^{p-1}\phi_p^p\Big(a-(\frac{p}{p-1})^{p-1}\lambda_p\Big).\end{aligned}$$

Now to prove (3.1) holds in $\Omega$, it is enough to show the following three inequalities:

$$-\frac12 k_1^{p-1}\phi_p^p\Big(a-(\frac{p}{p-1})^{p-1}\lambda_p\Big)\le -k_1^{r-1}\phi_p^{\frac{p(r-1)}{p-1}},\ \text{in }\Omega,\tag{3.3}$$

$$-\frac{1}{2}k_1^{p-1}\phi_p^p\Big(a-(\frac{p}{p-1})^{p-1}\lambda_p\Big) \le -\alpha k_1^{p-1}k_2^{q-1}\phi_p^p\phi_q^q, \text{ in } \Omega\setminus\bar{\Omega_\delta}, \tag{3.4}$$

$$-k_1^{p-1}(\frac{p}{p-1})^{p-1}|\nabla\phi_p|^p \le -\alpha k_1^{p-1}k_2^{q-1}\phi_p^p\phi_q^q, \text{ in } \bar{\Omega_\delta}. \tag{3.5}$$

Since $\|\phi_p\|_\infty = 1$, from the choice of $k_1$, $\Big(a-(\frac{p}{p-1})^{p-1}\lambda_p\Big) \ge 2k_1^{r-p}\phi_p^{\frac{p(r-1)}{p-1}}$. Hence,

$$-\frac{1}{2}k_1^{p-1}\phi_p^p\Big(a-(\frac{p}{p-1})^{p-1}\lambda_p\Big) \le -k_1^{r-1}\phi_p^{\frac{p(r-1)}{p-1}}. \tag{3.6}$$

In $\Omega\setminus\bar{\Omega_\delta}$, since $\|\phi_p\|_\infty = 1 = \|\phi_q\|_\infty$, using $\gamma < \frac{\frac{1}{2}\Big(a-(\frac{p}{p-1})^{p-1}\lambda_p\Big)}{k_2^{q-1}}$, we have

$$-\frac{1}{2}k_1^{p-1}\phi_p^p\Big(a-(\frac{p}{p-1})^{p-1}\lambda_p\Big) \le -\alpha k_1^{p-1}k_2^{q-1}\phi_p^p\phi_q^q. \tag{3.7}$$

Next, we know that $\gamma < \frac{(\frac{p}{p-1})^{p-1}\eta^p}{k_2^{q-1}}$, and $|\nabla\phi_p| \ge \eta$, in $\overline{\Omega_\delta}$. Thus

$$\begin{aligned}
-k_1^{p-1}(\frac{p}{p-1})^{p-1}|\nabla\phi_p|^p &\le -k_1^{p-1}(\frac{p}{p-1})^{p-1}\eta^p \\
&\le -\gamma k_1^{p-1}k_2^{q-1} \\
&\le -\alpha k_1^{p-1}k_2^{q-1}\|\phi_p\|_\infty^p\|\phi_q\|_\infty^q \\
&\le -\alpha k_1^{p-1}k_2^{q-1}\phi_p^p\phi_q^q.
\end{aligned} \tag{3.8}$$

From (3.6),(3.7) and (3.8) we see that the inequalities (3.3), (3.4), (3.5) hold in $\Omega$, if $\gamma < \gamma^*$. Therefore,

$$\int_\Omega |\nabla\psi_1|^{p-2}\,\nabla\psi_1\cdot\nabla w\,dx \le \int_\Omega (a\psi_1^{p-1} - \psi_1^{r-1} - \alpha\psi_1^{p-1}\psi_2^{q-1})w\,dx.$$

Similarly

$$\int_\Omega |\nabla\psi_2|^{q-2}\,\nabla\psi_2\cdot\nabla w\,dx \le \int_\Omega (b\psi_2^{q-1} - \psi_2^{s-1} - \beta\psi_1^{p-1}\psi_2^{q-1})w\,dx.$$

Thus, $(\psi_1,\psi_2)$ is a lower solution to (1.2).

**Construction of upper-solution**: Let $e_p, e_q$ be the solution of $-\Delta_\zeta e_\zeta = 1$ in $\Omega$, $e_\zeta = 0$ on $\partial\Omega$ for $\zeta = p,q$. Let $G_1(y) = ay^{p-1} - y^{r-1}$ and $G_2(y) = by^{q-1} - y^{s-1}$. Since $G_1'(y) = y^{p-2}[a(p-1)-(r-1)y^{r-p}]$, $G_1(y) \le L_1 = G_1(y_1)$, where $y_1 = [a(p-1)/(r-1)]^{1/(r-p)}$. Similarly, $G_2(y) \le L_2 = G_2(y_2)$, where $y_2 = [b(q-1)/(s-1)]^{1/(s-q)}$. Let

$$(z_1,z_2) = \Big(L_1^{1/p-1}e_p(x), L_2^{1/q-1}e_q(x)\Big).$$

Then for $w \in W$,

$$\begin{aligned}
\int_\Omega |\nabla z_1|^{p-2}\,\nabla z_1\cdot\nabla w\,dx &= \int_\Omega L_1\,w\,dx \\
&\ge \int_\Omega (az_1^{p-1} - z_1^{r-1})\,w\,dx \\
&\ge \int_\Omega (az_1^{p-1} - z_1^{r-1} - \alpha z_1^{p-1}z_2^{q-1})\,w\,dx
\end{aligned}$$

Similarly

$$\int_\Omega |\nabla z_2|^{q-2}\,\nabla z_2\cdot\nabla w\,dx \ge \int_\Omega (bz_2^{q-1} - z_2^{s-1} - \beta z_1^{p-1}z_2^{q-1})\,w\,dx.$$

Clearly $z_1, z_2 \geq 0$ on $\partial\Omega$ and hence $(z_1, z_2)$ is a upper-solution of (1.2).

**Proof of Theorem 1.1.** Let $(\psi_1, \psi_2)$ is a lower solution to (1.2) for $\gamma < \gamma^*$ ( as constructed in the previous subsection). Then we can construct a upper solution $(z_1, z_2)$ of (1.2) ( as constructed in the previous subsection). Further, since $z_1, z_2 > 0$ in $\bar{\Omega}$, we can choose $L_1 \gg 1$ and $L_2 \gg 1$ such that $(z_1, z_2) \geq (\psi_1, \psi_2)$ in $\bar{\Omega}$. Hence by Lemma 2.3, (1.2) has a positive solution $(u, v) \in [(\psi_1, \psi_2), (z_1, z_2)]$ and Theorem 1.1 is proven.

## 4. Proof of Theorem 1.2

**Proof.** Let $(u, v)$ be a nontrivial nonnegative solution of (1.2). We prove Theorem 1.2 by arriving at a contradiction. Multiplying the equations of $u$ and $v$ by $u$ and $v$, respectively and then integrating over $\Omega$, we have

$$\int_\Omega |\nabla u|^p \, dx = \int_\Omega \left(au^{p-1} - u^{r-1} - \alpha u^{p-1} v^{q-1}\right) u \, dx,$$

and

$$\int_\Omega |\nabla v|^2 \, dx \quad = \textstyle\int_\Omega \left(bv^{q-1} - v^{s-1} - \beta u^{p-1} v^{q-1}\right) v \, dx.$$

Note that

$$\lambda_p = \inf_{z \in W_0^{1,p}} \frac{\int_\Omega |\nabla z|^p \, dx}{\int_\Omega z^p dx}, \quad \lambda_q = \inf_{z \in W_0^{1,q}} \frac{\int_\Omega |\nabla z|^q \, dx}{\int_\Omega z^q dx}.$$

Combining, we obtain

$$\lambda_p \int_\Omega u^p dx + \lambda_q \int_\Omega v^q dx \leq \int_\Omega (au^p - u^r - \alpha u^p v^{q-1}) \, dx + \int_\Omega (bv^q - v^s - \beta u^{p-1} v^q) \, dx.$$

Hence

$$\begin{aligned} &(\lambda_p - a) \int_\Omega u^p dx + (\lambda_q - b) \int_\Omega v^q dx \\ &\leq - \int_\Omega (u^r + \alpha u^p v^{q-1}) \, dx - \int_\Omega (v^s + \beta u^{p-1} v^q) \, dx \\ &\leq 0, \end{aligned}$$

which is a contradiction if $0 < d < \lambda_0 = \min\{\lambda_p, \lambda_q\}$, and we finish the proof of Theorem 1.2. **Acknowledgment.** This research is funded by Babol Noshirvani University Technology, research grant No P/M/1109.

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