http://communications.science.ankara.edu.tr

Commun.Fac.Sci.Univ.Ank.Ser. A1 Math. Stat.
Volume 73, Number 1, Pages 259–273 (2024)
DOI:10.31801/cfsuasmas.1316623
ISSN 1303-5991 E-ISSN 2618-6470

Research Article; Received:June 19, 2023; Accepted: October 6, 2023

FRACTIONAL APPROACH FOR DIRAC OPERATOR INVOLVING M-TRUNCATED DERIVATIVE

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Abstract. In this study, we examine the basic spectral information for systems governed by the Dirac equation with distinct boundary conditions, utilizing a modified form of local derivatives known as M-truncated derivative (MTD). The spectral information discussed includes the representation of solutions in the form of integral equations, the asymptotics vector-valued eigenfunctions and eigenvalues, and their normalized forms, all within the context of the MTD method that incorporates truncated Mittag-Leffler functions. This type of MTD provides the features of integer-order operator theory. Also, by virtue of the parameters α and γ , we analyze and compare the solutions with graphs in terms of different potentials, different eigenvalues and different orders. Thus, the aim of this article is to consider spectral structure of Dirac system in frame of M-truncated derivative by proping with visual analysis.

1. Introduction

Studies related to several types of differential equations are always attracted by scientists. Because the differential equations have the speciality to model more complex natural systems. Also, the main advantage of fractional derivatives is that it allows us to achieve better results in modeling. Many fractional integral and derivatives like Liouville-Caputo, Riemann-Liouville, Hilfer, Atangana-Baleanu, Caputo-Fabrizio, etc. has been introduced and studied by scientists in [4, 12, 13, 28]. In recently, Khalil et al. has described the local derivative, which is also referred to as the conformable derivative depending on the basic limit definitions of the derivative firstly in [20]. The conformable derivative is very useful in applied mathematics because it shows parallel features to the ordinary derivative like quotient of two functions and the derivative of the product. Also it enables changing of order between $0 < \alpha \leq 1$. Because of this reason, many scientists have applied this

2020 *Mathematics Subject Classification.* 34A08, 34A45, 34L30.
Keywords. M-truncated derivative, Dirac operator, spectral data, visual results.
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derivative to their studies like [1–3, 5, 6, 8, 9, 16–18, 20]. Proportional $\alpha-$ derivative has similar features with conformable derivative but it differs in its limit definition which presented by Katugampola [19] . It was studied in [6, 7]. In recently, M-fractional derivative containing a Mittag-Leffler function with one parameter has been introduced by Sousa and Oliveira in [25–27]. Benefiting from the definition of these four local derivatives as mentioned above, M-truncated derivative is introduced by Sousa and Oliveira in [25] and it represents a generalization of the other four local derivatives because of the additional parameter inside definition. All other definitions of local derivative such as Katugampola, M-truncated derivative are adaptations of conformable derivative. In these derivatives, basic formulas such as quotient of two function, derivative of the product, chain rule, Leibniz rule etc. shares similarities with conformable derivatives. Spectral analysis of M-truncated derivative for Sturm-Liouville problem and some applications containing truncated Mittag- Leffler function are studied in [24, 29–31].

Dirac equation has a big importance in the modern field of atomic physics. The deepest meaning of the Dirac equation was that any relative definition of a particle necessarily includes not only the wave function of a single particle, but also multiple wave functions representing the potential of other particles. Dirac equation systems have applications in many branches of science like electrical engineering, mathematics and physics. New applications of conclusions and opinions from this topic shed light on future problems such as inverse problems of spectral theory. A first-order matrix linear differential equation whose solution is a 4-component wave function (a spinor) is so important in physics and mathematics [10, 14, 15].

Let L be a matrix operator defined by

$$\begin{pmatrix} V(x)+m & 0 \\ 0 & V(x)-m \end{pmatrix}$$

where $V(x)$ is a potential function, m is the mass of a particle and $y(x)$ denote a two component vector function $y(x)=\begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix}$. Then let's consider the equation

$$\left(B\frac{d}{dx}+L-\lambda I\right)y=0$$

where λ is a parameter and

$$B=\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\ I=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

is equivalent to a system

$$\begin{aligned} \frac{dy_2(x,\lambda)}{dx}+(V(x)+m)\,y_1(x,\lambda) &= \lambda y_1(x,\lambda), \\ -\frac{dy_1(x,\lambda)}{dx}+(V(x)-m)\,y_2(x,\lambda) &= \lambda y_2(x,\lambda). \end{aligned}$$

The basic analysis of the spectral structure for the Dirac operator means that finding asymptotic behaviors of the eigenvalues, the vector-valued eigenfunctions and the norming constants and showing the reality of the eigenvalues and the ortogonality of the eigen-vector-functions, etc. This type of analysis is called a direct problem. In this article, the reality of the eigenvalues, and the orthogonality of the eigenfunctions have been shown and the asymptotic formulas for the eigenvalues, eigen-vector-functions, the normalized eigen-vector-functions and the norming constants have been obtained in terms of M-truncated derivative for Dirac system having separated boundary conditions. The studies on the direct and inverse eigenvalue problems can be viewed from [11, 21–23]. Basic spectral features of linear differential operators including conformable derivatives, which inspired our work, were studied by [2,3,5,6,25]. Authors have established an existence and uniqueness theorem for a conformable fractional Dirac system in study [2]. Also they have addresses the existence of a spectral function for a singular conformable Dirac system in [3]. The M-truncated derivative can be employed in studies related to eigenvalue problems and spectral analysis. It is particularly beneficial in such analyses related to Dirac operators based on fractional derivatives. Our primary reason for selecting this particular local derivative is its inclusivity of other local derivatives, owing to the presence of an additional parameter associated with the Mittag-Leffler function. Differing from the literature, our results are more comprehensive compared to other local derivatives due to the presence of the parameter associated with the Mittag-Leffler function.

The layout of this research is presented in the following way: in section 2, we present to definitions and fundamental properties of MTD. In section 3, the spectral structure of Dirac system is studied. This main part of our study involves the reality of the eigenvalues, the orthogonality of the eigenvector-functions, and asymptotic formulas for the vector-valued eigen-functions, the eigenvalues, the norming constants and their normalized forms. Section 4 prerents detailed discussion about simulation anaysis by supporting with the graphs for different values of α, γ and λ. In part 5, the remarks of main results close the paper.

2. Preliminaries

In this part, we assign some necessary definitions, theorems and lemmas related to MTD.

Definition 1. *[25] The concept of the truncated Mittag-Leffler function with a single parameter is introduced through,*

$$ {}_iE_\gamma\left(z\right)=\sum_{k=0}^{i}\frac{z^k}{\Gamma\left(\gamma k+1\right)}. \tag{1} $$

Definition 2. *[25] Let $f:[0,\infty)\to\mathbb{R}$ be a function for $t>0$, then MTD of f with order $0<\alpha\leq 1$ is defined by*

$$ {}_iT_M^{\alpha,\gamma} f(t) = \lim_{\varepsilon\to 0} \frac{f\left(t_i E_\gamma\left(\varepsilon t^{-\alpha}\right)\right) - f(t)}{\varepsilon} \tag{2} $$

where ${}_iE_\gamma(.)$ is the truncated Mittag-Leffler function defined in (1) for $\gamma>0$.

Definition 3. *[25] The M- integral is defined as follows*

$$ \left({}_MI_a^{\alpha,\gamma} f\right)(\tau) = \int_a^\tau f(t)\, d_{\alpha,\gamma} t = \Gamma(\gamma+1) \int_a^\tau \frac{f(t)}{t^{1-\alpha}} dt $$

where $\gamma>0$, $\alpha\in(0,1]$, and f is defined in $(a,\tau]$.

Lemma 1. *[25] Let $\alpha\in(0,1]$, $\gamma>0$ and f,g be $\alpha-$differentiable at a point $t>0$. Then,*

1. ${}_iT_M^{\alpha,\gamma}(af+bg) = a_iT_M^{\alpha,\gamma} f + b_iT_M^{\alpha,\gamma}(g)$ *for* $a,b\in\mathbb{R}$;
2. ${}_iT_M^{\alpha,\gamma}(t^n) = nt^{n-\alpha}$ *for all* $n\in\mathbb{R}$;
3. ${}_iT_M^{\alpha,\gamma}(fg) = f_iT_M^{\alpha,\gamma}(g) + g_iT_M^{\alpha,\gamma}(f)$;
4. ${}_iT_M^{\alpha,\gamma}\left(\dfrac{f}{g}\right) = \dfrac{g_iT_M^{\alpha,\gamma}(f) - f_iT_M^{\alpha,\gamma}(g)}{g^2}$;
5. ${}_iT_M^{\alpha,\gamma}(c) = 0$, *$c$ is a constant*;
6. ${}_iT_M^{\alpha,\gamma}(fog)(t) = f'(g(t))_i T_M^{\alpha,\gamma} g(t)$, *for f is differentiable at $g(t)$;*
7. *If f is differentiable, thus* ${}_iT_M^{\alpha,\gamma}(f)(t) = \frac{t^{1-\alpha}}{\Gamma(\gamma+1)} \frac{df(t)}{dt}$.

Theorem 1. *Assume that f,$g:[a,b]\to\mathbb{R}$ and fg is differentiable. Then, we have*

$$ \Gamma(\gamma+1)\int_a^b s^{\alpha-1} f(s)_i\, T_M^{\alpha,\gamma} g(s)\, ds = f(t)\, g(t)|_a^b - \Gamma(\gamma+1) \int_a^b s^{\alpha-1} g(s)_i\, T_M^{\alpha,\gamma} f(s)\, ds. $$

The $L^2_{\alpha,\gamma}(0,\pi)$ is a Hilbert space with inner product

$$ (y,z) = \int_0^\pi y^T(x,\lambda_1)\, z(x,\lambda_2)\, d_{\alpha,\gamma} x, $$

where $y^T = (y_1,y_2)$ and $d_{\alpha,\gamma}x = \Gamma(\gamma+1)\, x^{\alpha-1} dx$.

In the next section, we will analyze the Dirac systems in terms of the MTD and we are able to obtain general representations of solutions that involve parameters α and γ. Additionally, using the MTD approach, we can also present asymptotic formulas for eigen-vector-functions and eigenvalues. The general results which found in main results correspond to classical Dirac systems when $\alpha=1$ and $\gamma=1$.

3. Main Results

Let us consider Dirac system containing M-tuncated derivative as follows:

$$\begin{array}{l} {}_iT_M^{\alpha,\gamma} y_2(x) + p(x) y_1(x) = \lambda y_1(x), 0 < \alpha \leq 1, x \in [0,\pi] \\ -\ {}_iT_M^{\alpha,\gamma} y_1(x) + r(x) y_2(x) = \lambda y_2(x), \end{array} \tag{3}$$

where ${}_iT_M^{\alpha,\gamma}$ is MTD operator, $p(x)$ and $r(x)$ are continuous and real-valued functions on $[0,\pi]$, $y(x)$ is $2\alpha-$continuously differentiable on $[0,\pi]$, ${}_iT_M^{\alpha,\gamma} y(x)$ is continuous on $[0,\pi]$. Deal with the system (3) subject to boundary conditions

$$y_1(0)\sin a + y_2(0)\cos a = 0, \tag{4}$$

$$y_1(\pi)\sin b + y_2(\pi)\cos b = 0, \tag{5}$$

where a and b are real constants.

Let symbolize the solution of (3) by $\varphi(x,\lambda) = \begin{pmatrix} \varphi_1(x,\lambda) \\ \varphi_2(x,\lambda) \end{pmatrix}$ satisfying the following initial conditions

$$\varphi_1(0,\lambda) = \cos a,\ \ \varphi_2(0,\lambda) = -\sin a. \tag{6}$$

Theorem 2. *Let λ_1 and λ_2 be two distinct eigenvalues of the problem $(3)-(5)$. Then the corresponding eigen-vector-functions $y(x,\lambda_1)$ and $z(x,\lambda_2)$ are orthogonal on $L^2_{\alpha,\gamma}(0,\pi)$ Hilbert space, that is,*

$$\int_0^\pi y^T(x,\lambda_1) z(x,\lambda_2) d_{\alpha,\gamma}x = 0, \qquad \lambda_1 \neq \lambda_2. \tag{7}$$

Proof. Since the $y(x,\lambda_1)$ and $z(x,\lambda_2)$ satisfy the system (3), we have

$$\begin{array}{rcl} {}_iT_M^{\alpha,\gamma} y_2(x,\lambda_1) + p(x) y_1(x,\lambda_1) & = & \lambda_1 y_1(x,\lambda_1), \\ -{}_iT_M^{\alpha,\gamma} y_1(x,\lambda_1) + r(x) y_2(x,\lambda_1) & = & \lambda_1 y_2(x,\lambda_1), \\ \\ {}_iT_M^{\alpha,\gamma} z_2(x,\lambda_2) + p(x) z_1(x,\lambda_2) & = & \lambda_2 z_1(x,\lambda_2), \\ -{}_iT_M^{\alpha,\gamma} z_1(x,\lambda_2) + r(x) z_2(x,\lambda_2) & = & \lambda_2 z_2(x,\lambda_2). \end{array}$$

If we multiply these equations by $z_1(x,\lambda_2)$, $z_2(x,\lambda_2)$, $-y_1(x,\lambda_1)$ and $-y_2(x,\lambda_1)$, respectively, and sum together, we get

$$\begin{array}{l} (\lambda_1-\lambda_2)(z_1(x,\lambda_2) y_1(x,\lambda_1) + z_2(x,\lambda_2) y_2(x,\lambda_1)) \\ \qquad = {}_iT_M^{\alpha,\gamma} \{z_1(x,\lambda_2) y_2(x,\lambda_1) - z_2(x,\lambda_2) y_1(x,\lambda_1)\}. \end{array}$$

Applying the integral ${}_MI_0^{\alpha,\gamma}$ from 0 to π on both side of the last equality, one can find

$$(\lambda_1-\lambda_2)\int_0^\pi y^T(x,\lambda_1) z(x,\lambda_2) d_{\alpha,\gamma}x = (z_1(x,\lambda_2) y_2(x,\lambda_1) - z_2(x,\lambda_2) y_1(x,\lambda_1))|_0^\pi.$$

By virtue of boundary conditions (4) and (5) , one can obtain

$$(\lambda_1 - \lambda_2) \int_0^{\pi} y_{\lambda_1}^T(x)\, z_{\lambda_2}(x)\, d_{\alpha,\gamma}x = 0.$$

□

Theorem 3. *All eigenvalues of the problem defined by* $(3)-(5)$ *are real.*

Proof. Let $\lambda_1 = a + ib$ be an eigenvalue with eigenfunction $y(x, \lambda_1)$. Since $p(x)$ and $r(x)$ real-valued functions, $\lambda_2 = \bar{\lambda}_1 = a - ib$ is also an eigenvalue with the eigenfunctions $\bar{y}(x, \lambda_2)$. By considering Theorem 2, we have

$$\begin{aligned} \left(\lambda - \bar{\lambda}\right) \int_0^{\pi} y^T(x, \lambda_1)\, \bar{y}(x, \lambda_2)\, d_{\alpha,\gamma}x &= 0, \\ \left(\lambda - \bar{\lambda}\right) \int_0^{\pi} \left\{ y_1^2(x, \lambda_1) + y_2^2(x, \lambda_1) \right\} d_{\alpha,\gamma}x &= 0, \end{aligned}$$

and since $y(x) \neq 0$, we have $\lambda = \bar{\lambda}$. □

Theorem 4. *The solution of the system* (3) *satisfying the initial conditions* (6) *provides the following integral equation system,*

$$\begin{aligned} \varphi_1(x, \lambda) = {} & \cos\left(\frac{\lambda\Gamma(\gamma+1)\, x^{\alpha}}{\alpha} - a\right) - \int_0^x \sin\left(\lambda\Gamma(\gamma+1)\left(\frac{t^{\alpha} - x^{\alpha}}{\alpha}\right)\right) p(t)\, \varphi_1(t, \lambda)\, d_{\alpha,\gamma}t \\ & + \int_0^x \cos\left(\lambda\Gamma(\gamma+1)\left(\frac{t^{\alpha} - x^{\alpha}}{\alpha}\right)\right) r(t)\, \varphi_2(t, \lambda)\, d_{\alpha,\gamma}t, \end{aligned} \tag{8}$$

$$\begin{aligned} \varphi_2(x, \lambda) = {} & \sin\left(\frac{\lambda\Gamma(\gamma+1)\, x^{\alpha}}{\alpha} - a\right) - \int_0^x \cos\left(\lambda\Gamma(\gamma+1)\left(\frac{t^{\alpha} - x^{\alpha}}{\alpha}\right)\right) p(t)\, \varphi_1(t, \lambda)\, d_{\alpha,\gamma}t \\ & - \int_0^x \sin\left(\lambda\Gamma(\gamma+1)\left(\frac{t^{\alpha} - x^{\alpha}}{\alpha}\right)\right) r(t)\, \varphi_2(t, \lambda)\, d_{\alpha,\gamma}t. \end{aligned} \tag{9}$$

Proof. By using the variation of parameters method given in [17], we express the representation of the solutions as follow:

$$\begin{aligned} \varphi_1(x, \lambda) &= -c_1(x)\, y_1(x) + c_2(x)\, y_2(x) \\ \varphi_2(x, \lambda) &= c_1(x)\, y_2(x) + c_2(x)\, y_1(x) \end{aligned}$$

where

$$\begin{aligned}
c_1(x) &= -\int_0^x \left(p(t)\, y_1(t)\, \varphi_1(t,\lambda) - r(t)\, y_2(t)\, \varphi_2(t,\lambda)\right) d_{\alpha,\gamma}t + c_1, \\
c_2(x) &= \int_0^x \left(p(t)\, y_1(t)\, \varphi_2(t,\lambda) + r(t)\, y_2(t)\, \varphi_1(t,\lambda)\right) d_{\alpha,\gamma}t + c_2, \\
y_1(x) &= \sin \frac{\lambda\Gamma(\gamma+1)\, x^\alpha}{\alpha} \text{ and } y_2(x) = \cos \frac{\lambda\Gamma(\gamma+1)\, x^\alpha}{\alpha}.
\end{aligned}$$

If we benefit from the initial conditions (6), it can be easily seen (8) and (9). □

Theorem 5. *As* $|\lambda| \to \infty$, *the estimates are provided as follows:*

$$\varphi_1(x,\lambda) = \cos(\xi(x,\lambda) - a) + O\left(\frac{1}{\lambda}\right), \tag{10}$$

$$\varphi_2(x,\lambda) = \sin(\xi(x,\lambda) - a) + O\left(\frac{1}{\lambda}\right), \tag{11}$$

$$\frac{\partial \varphi_1(x,\lambda)}{\partial \lambda} = -\Gamma(\gamma+1)\frac{x^\alpha}{\alpha}\sin(\xi(x,\lambda) - a) + O(1), \tag{12}$$

$$\frac{\partial \varphi_2(x,\lambda)}{\partial \lambda} = \Gamma(\gamma+1)\frac{x^\alpha}{\alpha}\cos(\xi(x,\lambda) - a) + O(1), \tag{13}$$

for $0 \le x \le \pi$ *where*

$$\xi(x,\lambda) = \frac{\lambda\Gamma(\gamma+1)}{\alpha} x^\alpha + \frac{1}{2}\int_0^x (p(t) + r(t))\, d_{\alpha,\gamma}t. \tag{14}$$

Proof. Let us introduce by $\varphi(x,\lambda)$ the solution of the system (3) satisfying the initial conditions (6). If the problem (3), (6) is considered for $p(x) = r(x) \equiv 0$, the solution of this problem stand for $\psi(x,\lambda)$. Therby, one can easily obtain that

$$\psi_1(x,\lambda) = \cos\left(\frac{\lambda\Gamma(\gamma+1)}{\alpha} x^\alpha - a\right), \tag{15}$$

$$\psi_2(x,\lambda) = \sin\left(\frac{\lambda\Gamma(\gamma+1)}{\alpha} x^\alpha - a\right). \tag{16}$$

If the solution of the problem (3), (6) is applied to the transformation matrix operator, we have [22]

$$\varphi(x,\lambda) = R(x)\, \psi(x,\lambda) + \int_0^x K(x,s)\, \psi(s,\lambda)\, d_{a,\gamma}s \tag{17}$$

in here $R(x)$ and $K(x,s)$ are matrices of second-order that can be continuously differentiated twice,

$$R(x)=\begin{pmatrix}\gamma(x) & \beta(x)\\ -\beta(x) & \gamma(x)\end{pmatrix} \tag{18}$$

and $\gamma(x)$ and $\beta(x)$ can be computed as below:

$$\begin{aligned}\gamma(x) &= \cos\left(\frac{1}{2}\int\limits_0^x (p(t)+r(t))\,d_{\alpha,\gamma}t\right),\\ \beta(x) &= -\sin\left(\frac{1}{2}\int\limits_0^x (p(t)+r(t))\,d_{\alpha,\gamma}t\right),\end{aligned}$$

for $\kappa=1$. Thereby considering by (17) and (18), we find the formulas

$$\begin{aligned}\varphi_1(x,\lambda) &= \cos(\xi(x,\lambda)-a)+\int\limits_0^x K_{11}(x,s)\cos\left(\tfrac{\lambda\Gamma(\gamma+1)}{\alpha}s^{\alpha}-a\right)d_{\alpha,\gamma}s\\ &\quad+\int\limits_0^x K_{12}(x,s)\sin\left(\tfrac{\lambda\Gamma(\gamma+1)}{\alpha}s^{\alpha}-a\right)d_{\alpha,\gamma}s\\ \varphi_2(x,\lambda) &= \sin(\xi(x,\lambda)-a)+\int\limits_0^x K_{11}(x,s)\cos\left(\tfrac{\lambda\Gamma(\gamma+1)}{\alpha}s^{\alpha}-a\right)d_{\alpha,\gamma}s\\ &\quad+\int\limits_0^x K_{12}(x,s)\sin\left(\tfrac{\lambda\Gamma(\gamma+1)}{\alpha}s^{\alpha}-a\right)d_{\alpha,\gamma}s\end{aligned} \tag{19}$$

where $K_{ij}(x,s)$ are the components of the matrix $K(x,s)$ for $i,j=1,2$ from (17). To gain the asymptotics in (10) and (11), it is enough to integrate by parts the integrals including in (19), because of the differentiability of the functions $K_{ij}(x,s)$. In a similar manner, if we differentiate (19) in terms of λ, we obtain the asymptotics in (12) and (13).

Additionally, we demonstrate the asymptotic behaviors of the eigenvalues using the MTD approach, enabling us to observe how the formulas change as α and γ vary.

□

Theorem 6. *The eigenvalues of the problem outlined by equations (3) to (5) in their asymptotic forms are given as follows:*

$$\lambda_{\pm n}=\frac{\alpha}{\Gamma(\gamma+1)\pi^{\alpha}}(\pm n\pi+c)+O\left(\frac{1}{n}\right), \tag{20}$$

where

$$c=a-b-\frac{1}{2}\int\limits_0^x (p(t)+r(t))\,d_{\alpha,\gamma}t.$$

Proof. The eigenvalues of the given problem overlap with the roots of the characteristic function

$$\triangle(\lambda) = \varphi_1(\pi,\lambda)\sin b + \varphi_2(\pi,\lambda)\cos b.$$

If we put asymptotics of the eigen-vector-functions $\varphi_1(\pi,\lambda)$ and $\varphi_2(\pi,\lambda)$ from the estimates (11) into $\triangle(\lambda)$, we obtain

$$\cos(\xi(x,\lambda)-a)\sin b + \sin(\xi(x,\lambda)-a)\cos b + O\left(\lambda^{-1}\right) = 0.$$

After some calculation with the aid of trigonometric functions, we reach

$$\sin\left(\frac{\lambda\Gamma(\gamma+1)\pi^{\alpha}}{\alpha} + c\right) + O\left(\lambda^{-1}\right) = 0. \tag{21}$$

It is clearly seen that the equation (21), for large $|\lambda|$, has solutions in the form

$$\frac{\lambda\Gamma(\gamma+1)\pi^{\alpha}}{\alpha} + c = n\pi + \delta_n,$$

it is obvious that $sin\delta_n = O\left(n^{-1}\right)$, i.e. $\delta_n = O\left(n^{-1}\right)$. Therefore the asymptotic formula for eigenvalues is obtained in (20). □

Theorem 7. *The asymptotic formula for the norming constants is given by*

$$\rho_n = \sqrt{\frac{\pi^{\alpha}\Gamma(\gamma+1)}{\alpha}} + O\left(\frac{1}{n}\right).$$

Proof. By utilizing the asymptotic formula for eigenvalues given in (20), we can reobtain the asymptotics for eigen-vector-functions as follows:

$$\varphi_1(x,\lambda_n) = \cos(\xi_n - a) + O\left(n^{-1}\right) \tag{22}$$

$$\varphi_2(x,\lambda_n) = \sin(\xi_n - a) + O\left(n^{-1}\right) \tag{23}$$

where $\xi(x,\lambda_n) = \xi_n = \frac{\lambda_n\Gamma(\gamma+1)}{\alpha}x^{\alpha} + \frac{1}{2}\int\limits_0^x (p(t)+r(t))\,d_{\alpha,\gamma}t.$

To reach at the asymptotic expression for the norming constants, take in consideration the following integral

$$\begin{aligned}
\rho_n^2 &= \int\limits_0^{\pi} \left\{\varphi_1^2(x,\lambda_n) + \varphi_2^2(x,\lambda_n)\right\} d_{\alpha,\gamma}x, \\
&= \int\limits_0^{\pi} \left\{\cos^2(\xi_n - a)\,x + \sin^2(\xi_n - a)\right\} d_{\alpha,\gamma}x + O\left(\frac{1}{n}\right), \\
&= \frac{\pi^{\alpha}\Gamma(\gamma+1)}{\alpha} + O\left(\frac{1}{n}\right).
\end{aligned}$$

Hence, the proof is completed. □

Theorem 8. *Asymptotic expression of the normalized vector-valued eigenfunctions is given in the form,*

$$\widetilde{\varphi}(x,\lambda_n) = \begin{pmatrix} \sqrt{\dfrac{\alpha}{\pi^{\alpha}\Gamma(\gamma+1)}}\cos(\xi_n - a) + O\left(n^{-1}\right) \\ \sqrt{\dfrac{\alpha}{\pi^{\alpha}\Gamma(\gamma+1)}}\sin(\xi_n - a) + O\left(n^{-1}\right) \end{pmatrix}.$$

Proof. The proof can be easily seen with the help of Theorem 7. □

4. Illustrative Results

In the current section, the representation of the solutions $y_1(x)$ and $y_2(x)$ for Dirac equation is offered by means of MTD under different orders of α, different potentials and different values of λ. If the values of α increases while the value of γ is constant, Figure 1 (a) and (b) have showed a right-sided shift for the solution curves. If the values of γ increases while the value of α is constant, Figure 2 (a) and (b) have showed a smaller right-sided shift in the solutions than Figure 1. Figure 3 demonstrates the acting for the solutions when $q = 0, 1, 2, 3$. Also, the roots of the characteristic function are computed detailed under different values of α in Table 1. If one pays attention to Figure 4 (a), (b) and (c), it can be easily seen that the value of α increases which is equal to 0.1, 0.3, 0.5, respectively, the frequency of the oscillation interval decreases. That is as the value of α decreases the number of eigenvalues of considered problem increases. Thereby α is changed the mobility of the solutions curves increases and it provides important advantage in applications of spectral analysis. Lastly, in Figure 5 (a) and (b), the graphs of eigenfunctions corresponding to different eigenvalues were plotted according to the changing the value of α and γ, respectively. Also Figure 5 (b) shows that eigenfunctions overlap for different values of γ. The main purpose in drawing graphs with different values is that one can observe the behavior of representations of solutions curves for Dirac equation in light of MTD. Also the approximate eigenvalues are given for different orders of α and γ in Table 1. Assume that $a = 1$, $b = \frac{\pi}{4}$ for all figures.

Table 1. The roots of $\triangle(\lambda)$ for $x = \pi$

α	λ_1	λ_2	γ	λ_1	λ_2
0.1	−0.2945	0.0215	0.1	−0.8679	0.0636
0.3	−0.7028	0.0515	0.3	−0.9200	0.0674
0.5	−0.9316	0.0683	0.5	−0.9316	0.0683
0.7	−1.0374	0.0760	0.7	−0.9087	0.0666
0.9	−1.0609	0.0777	0.9	−0.8585	0.0629
0.99	−1.0527	0.0771	0.99	−0.8291	0.0607

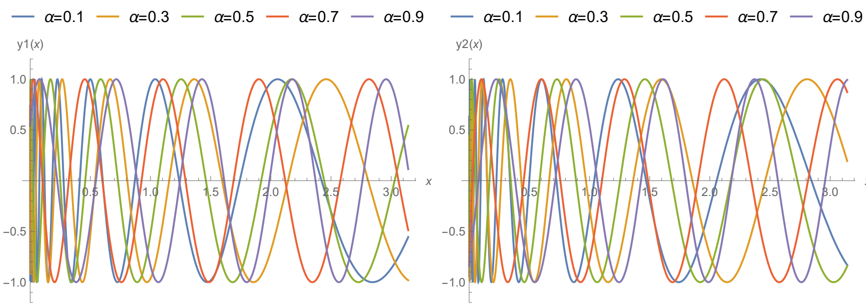

FIGURE 1. Comparative analysis for different orders of $\alpha, \lambda = 10, \quad p(x) = r(x) = 0, \gamma = 0.5$

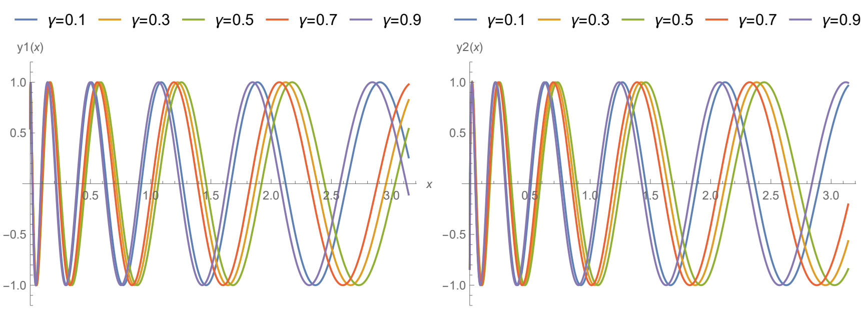

FIGURE 2. Comparative analysis for different orders of $\gamma, \ \lambda = 10, \quad p(x) = r(x) = 0, \alpha = 0.5.$

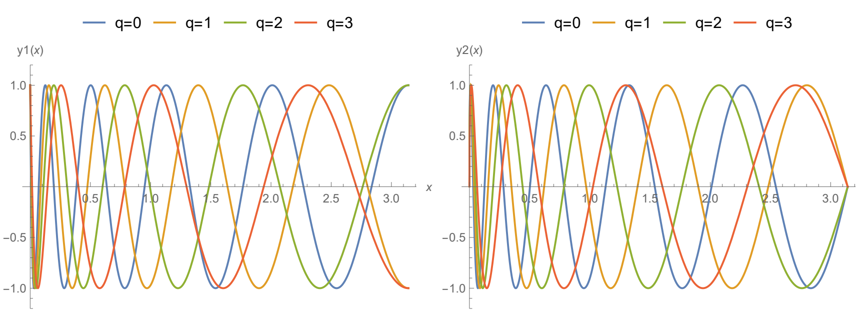

FIGURE 3. Comparative analysis for different values of the potentials, $\lambda = 10, \quad p(x) = r(x) = q, \gamma = 0.5, \alpha = 0.5$

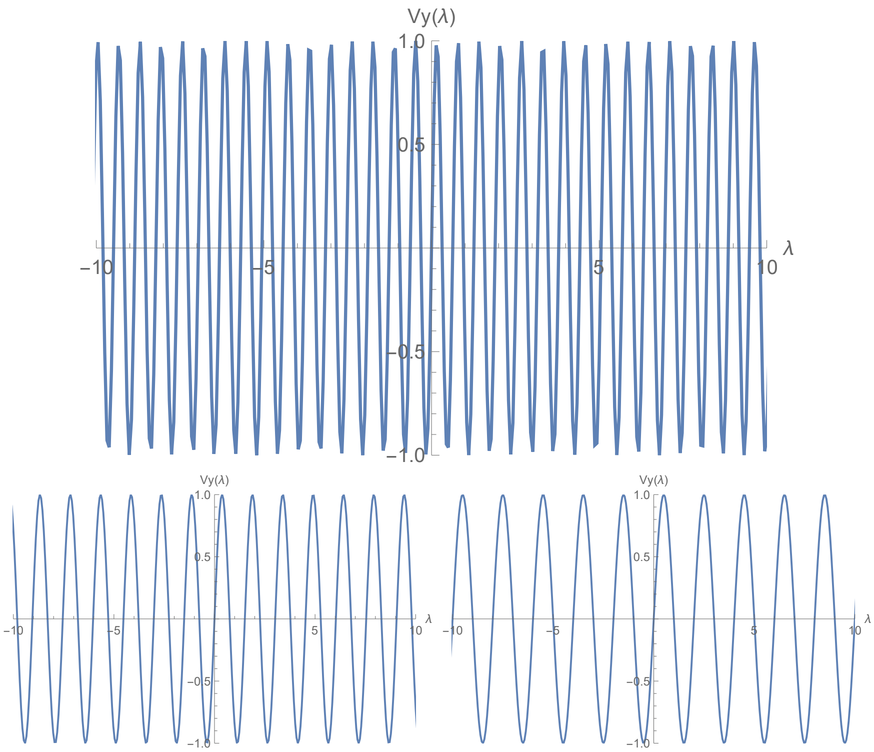

FIGURE 4. Comparisons of the roots of the characteristic function under different orders, $\lambda = 10, \quad p(x) = r(x) = 0, \gamma = 0.5$

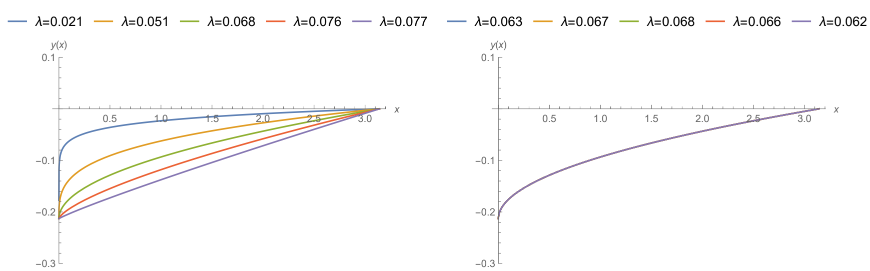

FIGURE 5. Comparisons of the eigenfunctions benefit from Table 1 under different order of α and γ

5. Conclusion

In here, we analyzed spectral structure of Dirac systems which has been studied by Levitan and Sargsjan [22] for integer order case in light of MTD. For one-dimensional Dirac operator in sense of MTD, its fundamental spectral theory is given systematically and behaviours of eigen-vector-functions are observed with graphics under different orders, potentials and eigenvalues. We obtain the representations of the solutions and asymptotics for the norming constants, the eigenvalues, eigen-vector-functions, and the normalized eigen-vector-functions. To gain these important results, certain calculations like variation of parameters method, Leibniz rule, and so forth are made in sense of MTD. The most important advantage of MTD is that this definition offers the features of the integer-order calculus. MTD give us the change to examine derivatives of infinite order . Also, we give comparative analysis of the solutions by graphs with different orders α and γ, different eigenvalues and different potentials. Thereby, we observe the behaviours of the mobility of the solutions. Thus, we have supplied a large amount of spectral theory for the considered problem in terms of MTD.

Author Contribution Statements As the sole author, the work is entirely the author's own.

Declaration of Competing Interests The author declares that there is no competing interest regarding the publication of this paper.

Acknowledgements The author would like to thank the editor and the reviewers for their valuable comments and suggestions which resulted in substantial improvement in the presentation of the paper.

References

[1] Abdeljawad, T., On conformable fractional calculus, *J. Comput. Appl. Math.*, 279 (2015), 57-66. https://doi.org/10.1016/j.cam.2014.10.016

[2] Allahverdiev, B. P., Tuna, H. One-dimensional conformable fractional Dirac system, *Bol. Soc. Mat. Mex.*, 26(1) (2020), 121-146. https://doi.org/10.1007/s40590-019-00235-5

[3] Allahverdiev, B. P., Tuna, H., Spectral expansion for singular conformable fractional Dirac systems, *Rend. Circ. Mat. Palermo*, 69(3) (2020), 1359–1372. https://doi.org/10.1007/s12215-019-00476-3

[4] Allahverdiev, B. P., Tuna, H., Regular fractional Dirac type systems, *Facta Univ. Ser. Math. Inform.*, 36(3) (2021), 489-499. https://doi.org/10.22190/FUMI200318036A

[5] Al-Refai, M., Abdeljawad, T., Fundamental results of conformable Sturm-Liouville eigenvalue problems, *Complexity*, (2017), 1-7. https://doi.org/10.1155/2017/3720471

[6] Anderson, D. R., Ulness, D. J., Newly defined conformable derivatives, *Adv. Dyn. Syst. Appl.*, 10(2) (2015), 109-137.

[7] Anderson, D. R., Ulness, D. J., Properties of the Katugampola fractional derivative with potential application in quantum mechanics, *J. Math. Phys.*, 56(6) (2015), 063502. https://doi.org/10.1063/1.4922018.

[8] Atangana, A., Baleanu, D., Alsaedi, A., New properties of conformable derivative, *Open Math.*, 13(1) (2015), 889-898. https://doi.org/10.1515/math-2015-0081

[9] Baleanu, D., Jarad, F., Uğurlu, E., Singular conformable sequential differential equations with distributional potentials. *Quaest. Math.*, 42(3) (2019), 277-287. https://doi.org/10.2989/16073606.2018.1445134

[10] Bjorken, J. D., Drell, S. D., Relativistic Quantum Mechanics, McGraw-Hill, New York, 1964.

[11] Ercan, A., Panakhov, E. S., Stability of the spectral problem for Dirac operators, *Aip Conf. Proc.*, 1738 (2016), 290010.

[12] Ercan, A., On the fractional Dirac systems with non-singular operators, *Thermal Sci.*, 23(6) (2019), 2159-2168. https://doi.org/10.2298/TSCI190810405E

[13] Ercan, A., Bas, E., Regular spectral problem for conformable Dirac system with simulation analysis, *J. Interdiscip. Math.*, 24(6) (2021), 1497-1514. https://doi.org/10.1080/09720502.2020.1827507

[14] Greiner, W., Miller, B., Rafelski, J., Quantum Electrodynamics of Strong Fields, Springer, Berlin, 1985.

[15] Greiner, W., Relativistic Quantum Mechanics: Wave Equations, Springer, Berlin, 1994.

[16] Hammad, M. A., Khalil, R., Abel's formula and wronskian for conformable fractional differential equations, *Int. J. Differ. Equ. Appl.*, 13(3) (2014), 177-183. http://dx.doi.org/10.12732/ijdea.v13i3.1753

[17] Horani, M. A., Hammad, M. A., Khalil, R., Variation of parameters for local fractional nonhomogenous linear-differential equations, *J. Math. Comput. Sci.*, 16 (2016), 147-153.

[18] Jarad, F., Uğurlu, E., Abdeljawad, T., Baleanu, D., On a new class of fractional operators, *Adv. Difference Equ.*, 247 (2017), 16. https://doi.org/10.1186/s13662-017-1306-z

[19] Katugampola, U. N., A new fractional derivative with classical properties, *arXiv preprint*, (2014), arXiv:1410.6535v2.

[20] Khalil, R., Horani, M. A., Yousef, A., Sababheh, M., A new definition of fractional derivative, *J. Comput. Appl. Math.*, 264 (2014), 65-70. https://doi.org/10.1016/j.cam.2014.01.002

[21] Levitan, B. M., Sargsjan, I. S., Introduction to Spectral Theory: Selfadjoint Ordinary Differential Operators, American Mathematical Society, Providence, R.I., 1975.

[22] Levitan, B. M., Sargsjan, I. S., Sturm-Liouville and Dirac Operators, Kluwer Academic Publishers Group, Dordrecht, 1991.

[23] Mamedov, K. R., Akcay, O., Inverse problem for a class of Dirac operators by the Weyl function, *Dynam. Systems Appl.*, 26(1) (2017), 183-195.

[24] Ozarslan, R., Bas, E., Baleanu, D., Acay, B., Fractional physical problems including wind-influenced projectile motion with Mittag-Leffler kernel, *AIMS Math.*, 5(1) (2019), 467-481. https://doi.org/10.3934/math.2020031

[25] Vanterler da C. Sousa J., Capelas de Oliveira, E., A new truncated M-fractional derivative type unifying some fractional derivative types with classical properties, *Int. J. Anal. Appl.*, 16(1) (2018), 83-96.

[26] Vanterler da C. Sousa J., Capelas de Oliveira, E., Leibniz type rule: ψ-Hilfer fractional operator, *Nonlinear Sci. Numer. Simul.*, 77 (2019), 305–311. https://doi.org/10.1016/j.cnsns.2019.05.003

[27] Vanterler da C. Sousa J., Capelas de Oliveira, E., Mittag–Leffler functions and the truncated V-fractional derivative, *Mediterr. J. Math.*, 14(6) (2017), 244. https://doi.org/10.1007/s00009-017-1046-z

[28] Yalçınkaya, Y., Some fractional Dirac systems, *Turkish J. Math.*, 47(1) (2023), 110-122. https://doi.org/10.55730/1300-0098.3349

[29] Yusuf, A., Inc, M., Aliyu, A. I., Two-strain epidemic model involving fractional derivative with Mittag-Leffler kernel, *Chaos*, 28(12) (2018), 123121, 11. https://doi.org/10.1063/1.5074084

[30] Yusuf, A., Sulaiman, T. A., Mirzazadeh, M., Hosseini, K., M-truncated optical solitons to a nonlinear Schrödinger equation describing the pulse propagation through a two-mode optical fiber, *Opt. Quant. Electron*, 53(10) (2021), 558. https://doi.org/10.1007/s11082-021-03221-2

[31] Yusuf, A., Sulaiman, T. A., Inc, M., Abdel-Khalek, S., Mahmoud, K. H., *M*-truncated optical soliton and their characteristics to a nonlinear equation governing the certain instabilities of modulated wave trains, *AIMS Math.*, 6(9) (2021), 9207–9221. https://doi.org/10.3934/math.2021535