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Cotton Solitons on Three Dimensional Almost α-paracosymplectic Manifolds

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ABSTRACT

In this paper, we study Cotton solitons on three-dimensional almost α-paracosymplectic manifolds. We especially focus on three-dimensional almost α-paracosymplectic manifolds with harmonic vector field ξ and characterize them for all possible types of operator h. Finally, we construct an example which satisfies our results.

Keywords: *Cotton soliton, almost para-Kenmotsu manifold, almost paracosymplectic manifold.*

AMS Subject Classification (2020): *Primary: 53B30 ; Secondary: 53C15; 53C25; 53D10.*

1. Introduction

The study of almost paracontact geometry was introduced by Kaneyuki and Williams in [18]. A systematic study of almost paracontact metric manifolds was carried out in paper of Zamkovoy [28]. The curvature identities for different classes of almost paracontact metric manifolds were obtained e.g. in [9, 26, 28].

Recently, a long awaited survey article, [5], concerning almost cosymplectic manifolds as Blair's monograph [2] about contact metric manifolds appeared.

Almost paracontact metric structure is given by a pair (η, Φ), where η is a 1-form and Φ is a 2-form and $\eta \wedge \Phi^n$ is a volume element. There exists a unique vector field ξ, called the characteristic or Reeb vector field, such that $i_\xi \eta = 1, i_\xi \Phi = 0$. The Riemannian or pseudo-Riemannian geometry appears if we try to introduce a *compatible* structure which is a metric or pseudo-metric g and ϕ $(1,1)$-tensor field, such that

$$\Phi(X,Y) = g(\phi X, Y), \qquad \phi^2 = \epsilon(I - \eta \otimes \xi). \tag{1.1}$$

We have almost paracontact metric structure for $\epsilon = +1$ and almost contact metric for $\epsilon = -1$. Then, the triple (ϕ, ξ, η) is called almost paracontact structure or almost contact structure, resp.

Combining the assumption concerning the forms η and Φ, we obtain many different types of almost (para)contact manifolds, e.g. (para)contact if η is contact form and $d\eta = \Phi$, almost (para)cosymplectic if $d\eta = 0$ and $d\Phi = 0$, almost (para)Kenmotsu if $d\eta = 0$ and $d\Phi = 2\eta \wedge \Phi$.

Recently, geometers discovered many similarities between almost contact metric and almost paracontact metric manifolds. However, the situation is more delicate: there are examples of almost paracontact metric manifolds without Riemannian counterparts e.g. [10, 21, 24].

The study of geometric evolution equations is one of the principal research subjects motivated by either physical or mathematical questions. Several years ago, the notion of the Yamabe flow was introduced by Richard Hamilton at the same time as the Ricci flow (see [16, 17]), as a tool for constructing metrics of constant scalar curvature in a given conformal class of Riemannian metrics on (M^n, g) $(n \geq 3)$. On a smooth semi-Riemannian manifold, the Yamabe flow can be defined as the evolution of the semi-Riemannian metric g_0 in time t to $g = g(t)$ by the equation

$$\frac{\partial}{\partial t} g = -rg, \quad g(0) = g_0,$$

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where r denotes the scalar curvature which corresponds to g.

The significance of the Yamabe flow lies in the fact that it is a natural geometric deformation to metrics of constant scalar curvature. One notes that Yamabe flow corresponds to the fast diffusion case of the porous medium equation (the plasma equation) in mathematical physics. In dimension $n = 2$ the Yamabe flow is equivalent to the Ricci flow (defined by $\frac{\partial}{\partial t}g = -2S(t)$, where S stands for Ricci tensor). However in dimension $n > 2$ the Yamabe and Ricci flows do not agree, since the first one preserves the conformal class of the metric but the Ricci flow does not in general. Just as a Ricci soliton is a special solution of the Ricci flow, a Yamabe soliton is a special solution of the Yamabe flow that moves by one parameter family of diffeomorphisms ϕ_t generated by a fixed (time-independent) vector field V on M, and homotheties, i.e. $g(.,t) = \sigma(t)\phi_t{}^*g_0$. Küpeli Erken and De published a lot of paper about Yamabe and Ricci solitons in paracontact geomerty [22, 11, 12].

In 1918, Hermann Weyl [27] first appeared the Weyl tensor is of enormous consequence in theoretical physics and the general theory of relativity. The Weyl tensor is similar to Riemannian curvature tensor and behaves like a force that a body feels when moving on a geodesic. This force is called tidal force which changes the shape of the body and like Riemann curvature tensor, it does not pass on the information that how the volume of the body changes. Weyl tensor is a significant tool in the study of manifold geometry. However, geometers need to find a distinct way in three-dimension. In general, Cotton tensor C, is a non-vanishing conformal invariant on a three-dimensional paracontact metric manifold contrary to Weyl tensor. The $(0, 2)$- Cotton tensor C is defined by

$$C_{ij} = \frac{1}{2\sqrt{g}} C_{nmi}\epsilon^{nml} g_{lj}, \tag{1.2}$$

where ϵ^{ijk} denotes the Levi-Civita permutation symbol ($\epsilon^{123} = 1$) and $g = |det(g_{ij})|$. It is trace-free and divergence-free tensor. The Cotton tensor is involved in many physics subjects such as Chern-Simons theory or topological massive gravity [1, 8, 13, 15, 23]. The relation between Cotton tensor and the energy momentum in Einstein's theory investigated in [14]. In particular, the field equation in topologically massive gravity implies a proportionality between the Einstein and Cotton tensors. The fact that the Einstein tensor consists of second-order derivatives on the metric whereas the Cotton tensor is of order 3 implies that an exact solution to this field equation is in general difficult to find. Indeed, most of the solutions for the field equation in the topological massive gravity are constructed on homogeneous spaces.

In [19], a new geometric flow based on the conformally invariant Cotton tensor was introduced. A Cotton flow is a one-parameter family $g(t)$ of three-dimensional metrics satisfying

$$\frac{\partial}{\partial t} g(t) = -\lambda C_{g(t)}, \tag{1.3}$$

where $C_{g(t)}$ is the $(0, 2)$-Cotton tensor corresponding to the metric $g(t)$. A *Cotton soliton* is a metric defined on a three-dimensional smooth manifold which satisfies

$$L_V g + C - \sigma g = 0, \tag{1.4}$$

where V is a vector field, called potential vector field, σ is constant and L denotes the Lie derivative [19]. Cotton soliton is *trivial* if $C = 0$ (i.e. conformally flat). Also, Cotton soliton is said to be *shrinking*, *steady* and *expanding* according as σ is positive, zero and negative resp.

As in Ricci and Yamabe soliton, Cotton soliton is a fixed point of (1.3) up to diffeomorphism and rescaling.

Calvino-Louzao et.al. [3] studied compact Riemannian Cotton solitons and proved that compact Riemannian Cotton solitons are locally conformally flat in Riemannian structure. Moreover, they investigated left-invariant Cotton solitons on homogeneous manifolds in [4]. Cotton solitons on three-dimensional almost coKaehler manifolds such that the characteristic vector field ξ is an eigenvector field of the Ricci operator Q (i.e. $Q\xi = \rho\xi$, where ρ is a smooth function on M) were studied by Chen in [6]. Furthermore, the same author investigated Cotton solitons on three-dimensional contact metric manifolds [7]. Recently, Ozkan et al. [25] have studied Cotton solitons on three-dimensional paracontact metric manifolds.

In the light of previous works, the fact that there are only studies about Cotton solitons on contact geometry motivate us to study Cotton solitons on 3-dimensional almost α-paracosymplectic manifolds. The paper is organized in the following way. In section 2, we recall some notations needed for this paper. Section 3 deals with the computations of the components of the $(0, 2)-$Cotton Tensor. In the last section, we consider three-dimensional almost paracosymplectic and almost α-para-Kenmotsu manifolds with $\mathfrak{h}_1$ and $\mathfrak{h}_3$ types resp., such that the characteristic vector field ξ is harmonic (i.e. $Q\xi = \rho\xi$) and ρ is constant along the characteristic vector field ξ. Then we proved that if these manifolds admits a Cotton soliton with potential vector field being collinear with characteristic vector field ξ, then M is either a locally warped product $M_1 \times_{f^2} M_2$, where M_2 is an almost para-Kaehler manifold, M_1 is an open interval with coordinate t, and $f^2 = we^{2\alpha t}$ for some positive constant or locally conformally flat. The results for three-dimensional almost paracosymplectic manifolds with $\hbar_2$ type are different from three-dimensional almost paracosymplectic and almost α-para-Kenmotsu manifolds with $\hbar_1$ and $\hbar_3$ types, resp. We consider a three-dimensional almost paracosymplectic manifold with $\hbar_2$ type such that the characteristic vector field is harmonic. Then we proved that if M admits a Cotton soliton with potential vector field being collinear with characteristic vector field ξ, then M is locally conformally flat and Cotton soliton is steady. Finally, an example which satisfies our results is constructed.

2. Preliminaries

A $(2n+1)-$ dimensional manifold M is called *almost paracontact manifold* if it admits triple (ϕ, ξ, η) satisfying the followings:

- $\eta(\xi) = 1, \phi^2 = I - \eta \otimes \xi,$
- ϕ induces on almost paracomplex structure on each fiber of $\mathcal{D} = ker(\eta),$

where ϕ, ξ and η are $(1,1)-$tensor field, vector field and $1-$form, resp. One can easily checked that $\phi\xi = 0, \eta \circ \phi = 0$ and $rank\phi = 2n,$ by the definition. Here, ξ is a unique vector field (called *Reeb* or *characteristic vector field*) dual to η and satisfying $d\eta(\xi, X) = 0$ for all X. When the tensor field $N_\phi := [\phi, \phi] - 2d\eta \otimes \xi$ vanishes identically, the almost paracontact manifold is said to be *normal*. If the structure (M, ϕ, ξ, η) admits a pseudo-Riemannian metric such that

$$g(\phi X, \phi Y) = -g(X, Y) + \eta(X)\eta(Y),$$

then we say that (M, ϕ, ξ, η, g) is an *almost paracontact metric manifold*. Note that any pseudo-Riemannian metric with a given almost paracontact metric manifold structure is necessarily of signature $(n+1, n)$. For an almost paracontant metric manifold, one can always find an orthogonal basis $\{X_1, \dots, X_n, Y_1, \dots, Y_n, \xi\}$, namely $\phi-$basis, such that $g(X_i, X_j) = -g(Y_i, Y_j) = \delta_{ij}$ and $Y_i = \phi X_i$, for any $i, j \in \{1, \dots, n\}$. Further, we can define a skew-symmetric tensor field (2-form), usually called fundamental form, Φ by

$$\Phi(X, Y) = g(\phi X, Y).$$

On an almost paracontact manifold, one can defines the $(1, 2)$-tensor $N^{(1)}$ by

$$N^{(1)}(X, Y) = [\phi, \phi](X, Y) - 2d\eta(X, Y)\xi$$

where

$$[\phi, \phi](X, Y) = \phi^2[X, Y] + [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y]$$

is the Nijenhuis torsion of ϕ. If $N^{(1)}$ vanishes, then the almost paracontact manifold is said to be *normal* [28].

An almost paracontact metric manifold M^{2n+1} with a structure (ϕ, ξ, η, g) is said to be an *almost α-paracosymplectic manifold*, if

$$d\eta = 0, \qquad d\Phi = 2\alpha\eta \wedge \Phi,$$

where α may be a constant or function on M. We have the following subclasses for a special choices of the function α:

- almost α-para-Kenmotsu manifolds, $\alpha = \text{const.} \neq 0,$
- almost paracosymplectic manifolds, $\alpha = 0.$

Moreover, if normality condition is fulfilled, then manifolds are called α-para-Kenmotsu or paracosymplectic, resp.

In an almost α-paracosymplectic manifold, one defines a symmetric operator $h := \frac{1}{2} L_\xi \phi$. The operator h also satisfies the followings:

$$\begin{cases} h\xi = 0, & g(hX, Y) = g(X, hY), \\ \phi \circ h + h \circ \phi = 0, & \nabla\xi = \alpha\phi^2 + \phi \circ h, \end{cases} \tag{2.1}$$

where ∇ is the Levi-Civita connection of the pseudo-Riemannian manifold [20]. Küpeli Erken, Dacko and Murathan proved the following Theorem.

Theorem 2.1. *[20] Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be an almost α-para-Kenmotsu manifold. Characteristic vector field ξ is harmonic if and only if it is an eigenvector of the Ricci operator.*

Theorem 2.2. *[20] Let M^{2n+1} be an almost α-para-Kenmotsu manifold with $h = 0$. Then M^{2n+1} is a locally warped product $M_1 \times_{f^2} M_2$, where M_2 is an almost para-Kaehler manifold, M_1 is an open interval with coordinate t, and $f^2 = we^{2\alpha t}$ for some positive constant.*

Now, we give some information about the canonical forms of h.

The tensor h has the canonical form (I). Let (M, ϕ, ξ, η, g) be a three-dimensional almost α-paracosymplectic manifold and let

$$U_1 = \{p \in M | h(p) \neq 0\} \subset M$$

$$U_2 = \{p \in M | h(p) = 0, \text{ in a neighborhood of } p\} \subset M.$$

That h is a smooth function on M implies $U_1 \cup U_2$ is an open and dense subset of M, so any property satisfied in $U_1 \cup U_2$ is also satisfied in M. For any point $p \in U_1 \cup U_2$, there exists a local orthonormal ϕ-basis $\{e, \phi e, \xi\}$ of smooth eigenvectors of h in a neighborhood of p, where $-g(e, e) = g(\phi e, \phi e) = g(\xi, \xi) = 1$. On U_1, we put $he = \lambda e$, where λ is a non-vanishing smooth function. Since $trh = 0$, we have $h\phi e = -\lambda\phi e$. In this case, we will say the operator h is of $\hbar_1$ type.

Lemma 2.1. *[20] Let (M, ϕ, ξ, η, g) be a three-dimensional almost α-para-Kenmotsu manifold with h of $\hbar_1$ type. Then for the covariant derivative on U_1, the following equations are valid*

$$\begin{aligned}
&\text{(i)}\ \nabla_e e = a\phi e + \alpha\xi,\\
&\text{(ii)}\ \nabla_e \phi e = ae - \lambda\xi,\\
&\text{(iii)}\ \nabla_e \xi = \alpha e + \lambda\phi e,\\
&\text{(iv)}\ \nabla_{\phi e} e = c\phi e - \lambda\xi,\\
&\text{(v)}\ \nabla_{\phi e}\phi e = ce - \alpha\xi,\\
&\text{(vi)}\ \nabla_{\phi e}\xi = -\lambda e + \alpha\phi e,\\
&\text{(vii)}\ \nabla_\xi e = a_1\phi e,\\
&\text{(viii)}\ \nabla_\xi \phi e = a_1 e,\\
&\text{(ix)}\ \nabla_\xi \xi = 0,\\
&\text{(x)}\ \nabla_\xi h = \xi(\lambda)s - 2a_1 h\phi,\\
&\text{(xi)}\ h^2 - \alpha^2\phi^2 = \frac{1}{2}S(\xi, \xi)\phi^2
\end{aligned} \tag{2.2}$$

where $\omega = S(\xi, .)_{ker\eta}$, $a_1 = g(\nabla_\xi e, \phi e)$, $A = \omega(e)$, $B = \omega(\phi e)$ and

$$a = \frac{A - \phi e(\lambda)}{2\lambda}, \tag{2.3}$$

$$c = -\left(\frac{B + e(\lambda)}{2\lambda}\right). \tag{2.4}$$

From (2.2), we have

$$\begin{cases}
[e, \phi e] &= ae - c\phi e,\\
[e, \xi] &= \alpha e + (\lambda - a_1)\phi e,\\
[\phi e, \xi] &= -(\lambda + a_1)e + \alpha\phi e.
\end{cases} \tag{2.5}$$

Lemma 2.2. *[20] Let (M, ϕ, ξ, η, g) be a 3-dimensional almost α-para-Kenmotsu manifold with h of $\hbar_1$ type. Then the Ricci operator Q is given by*

$$\begin{aligned}
Q =& \left(\frac{r}{2} + \alpha^2 - \lambda^2\right)I + \left(-\frac{r}{2} + 3(\lambda^2 - \alpha^2)\right)\eta \otimes \xi - 2\alpha\phi h - \phi(\nabla_\xi h)\\
&+ \sigma(\phi^2) \otimes \xi - \sigma(e)\eta \otimes e + \sigma(\phi e)\eta \otimes \phi e.
\end{aligned}$$

Then the components of the Ricci operator Q for $\hbar_1$ type are given by

$$\begin{cases} Qe &= (\frac{1}{2}r + \alpha^2 - \lambda^2 - 2\lambda a_1)e - (2\alpha\lambda + Z)\phi e + A\xi, \\ Q\phi e &= (2\lambda\alpha + Z)e + (\frac{1}{2}r + \alpha^2 - \lambda^2 + 2\lambda a_1)\phi e + B\xi, \\ Q\xi &= -Ae + B\phi e + 2(\lambda^2 - \alpha^2)\xi, \end{cases} \tag{2.6}$$

where $Z = \xi(\lambda)$.

The tensor h has the canonical form (II). Let (M, ϕ, ξ, η, g) be a three-dimensional almost α-paracosymplectic manifold and p is a point of M. Then there exists a local pseudo-orthonormal basis $\{e_1, e_2, \xi\}$ in a neighborhood of p, where $g(e_1, e_1) = g(e_2, e_2) = g(e_1, \xi) = g(e_2, \xi) = 0$ and $g(e_1, e_2) = g(\xi, \xi) = 1$. Let U be the open subset of M, where $h \neq 0$. For every $p \in U$, there exists an open neighborhood of p such that $he_1 = e_2, he_2 = 0, h\xi = 0$ and $\phi e_1 = \pm e_1, \phi e_2 = \mp e_2$. In this case, we say h is of $\hbar_2$ type.

Lemma 2.3. *[20] Let (M, ϕ, ξ, η, g) be a three-dimensional almost α-para-Kenmotsu manifold with h of $\hbar_2$ type. Then for the covariant derivative on U, the following equations are valid*

$$\begin{aligned} &\text{(i)} && \nabla_{e_1} e_1 = -b_1 e_1 + \xi, \\ &\text{(ii)} && \nabla_{e_1} e_2 = b_1 e_2 - \alpha\xi, \\ &\text{(iii)} && \nabla_{e_1} \xi = \alpha e_1 - e_2, \\ &\text{(iv)} && \nabla_{e_2} e_1 = -b_2 e_1 - \alpha\xi, \\ &\text{(v)} && \nabla_{e_2} e_2 = b_2 e_2, \\ &\text{(vi)} && \nabla_{e_2} \xi = \alpha e_2, \\ &\text{(vii)} && \nabla_{\xi} e_1 = a_2 e_1, \\ &\text{(viii)} && \nabla_{\xi} e_2 = -a_2 e_2, \\ &\text{(ix)} && \nabla_{\xi} h = -2a_2 h\phi, \\ &\text{(x)} && h^2 = 0, \end{aligned} \tag{2.7}$$

where $A_2 = \omega(e_1), a_2 = g(\nabla_\xi e_1, e_2), b_1 = g(\nabla_{e_1} e_2, e_1)$ and $b_2 = g(\nabla_{e_2} e_2, e_1) = -\frac{1}{2}\omega(e_1)$.

From (2.7) we have

$$\begin{cases} [e_1, e_2] &= b_2 e_1 + b_1 e_2, \\ [e_1, \xi] &= (\alpha - a_2)e_1 - e_2, \\ [e_2, \xi] &= (\alpha + a_2)e_2. \end{cases} \tag{2.8}$$

Lemma 2.4. *[20] Let (M, ϕ, ξ, η, g) be a 3-dimensional almost α-para-Kenmotsu manifold with h of $\hbar_2$ type. Then the Ricci operator Q is given by*

$$\begin{aligned} Q =& (\frac{r}{2} + \alpha^2)I - (\frac{r}{2} + 3\alpha^2)\eta \otimes \xi - 2\alpha\phi h - \phi(\nabla_\xi h) \\ &+ \sigma(\phi^2) \otimes \xi + \sigma(e_1)\eta \otimes e_2. \end{aligned}$$

Then the components of the Ricci operator Q for $\hbar_2$ are given by

$$\begin{cases} Qe_1 &= (\frac{1}{2}r + \alpha^2)e_1 + 2(\alpha - a_2)e_2 + A_2\xi, \\ Qe_2 &= (\frac{1}{2}r + \alpha^2)e_2, \\ Q\xi &= A_2 e_2 - 2\alpha^2\xi. \end{cases} \tag{2.9}$$

The tensor h has the canonical form (III). Let (M, ϕ, ξ, η, g) be a three-dimensional almost α-paracosymplectic manifold and let p is a point of M. Then there exists a local orthonormal ϕ-basis $\{e, \phi e, \xi\}$ in a neighborhood of p, where $-g(e, e) = g(\phi e, \phi e) = g(\xi, \xi) = 1$. Now, let U_1 be the open subset of M where $h \neq 0$ and let U_2 be the open subset of points $p \in M$ such that $h = 0$ in a neighborhood of p. $U_1 \cup U_2$ is an open subset of M. For every $p \in U_1$ there exists an open neighborhood of p such that $he = \lambda\phi e, h\phi e = -\lambda e$ and $h\xi = 0$ where λ is a non-vanishing smooth function. In this case, we say that h is of $\hbar_3$ type.

Lemma 2.5. *[20] Let (M, ϕ, ξ, η, g) be a three-dimensional almost α-para-Kenmotsu manifold with h of $\hbar_3$ type. Then for the covariant derivative on U_1, the following equations are valid*

$$\begin{aligned}
&\text{(i)}\ \nabla_e e = b_3\phi e + (\alpha+\lambda)\xi,\\
&\text{(ii)}\ \nabla_e \phi e = b_3 e,\\
&\text{(iii)}\ \nabla_e \xi = (\alpha+\lambda)e,\\
&\text{(iv)}\ \nabla_{\phi e} e = b_4\phi e,\\
&\text{(v)}\ \nabla_{\phi e}\phi e = b_4 e + (\lambda-\alpha)\xi,\\
&\text{(vi)}\ \nabla_{\phi e}\xi = -(\lambda-\alpha)\phi e,\\
&\text{(vii)}\ \nabla_\xi e = a_3\phi e,\\
&\text{(viii)}\ \nabla_\xi \phi e = a_3 e,\\
&\text{(ix)}\ \nabla_\xi h = \xi(\lambda)s - 2a_3 h\phi,\\
&\text{(x)}\ h^2 - \alpha^2\phi^2 = \frac{1}{2}S(\xi,\xi)\phi^2
\end{aligned} \tag{2.10}$$

where $\omega = S(\xi, .)_{ker\eta}, a_3 = g(\nabla_\xi e, \phi e), b_3 = -\frac{1}{2\lambda}[\omega(\phi e) + \phi e(\lambda)]$ and $b_4 = \frac{1}{2\lambda}[\omega(e) - e(\lambda)]$.

From (2.10) we have

$$\begin{cases} [e, \phi e] &= b_3 e - b_4\phi e,\\ [e, \xi] &= (\lambda+\alpha)e - a_3\phi e,\\ [\phi e, \xi] &= -a_3 e - (\lambda-\alpha)\phi e. \end{cases} \tag{2.11}$$

Lemma 2.6. *[20] Let (M, ϕ, ξ, η, g) be a 3-dimensional almost α-para-Kenmotsu manifold with h of $\hbar_3$ type. Then the Ricci operator Q is given by*

$$Q = \alpha I + b\eta\otimes\xi - 2\alpha\phi h - \phi(\nabla_\xi h) + \sigma(\phi^2)\otimes\xi - \sigma(e)\eta\otimes e + \sigma(\phi e)\eta\otimes\phi e,$$

where a and b are smooth functions defined by $a = \alpha^2 + \lambda^2 + \frac{r}{2}$ and $b = -3(\lambda^2+\alpha^2) - \frac{r}{2}$, respectively.

Then the components of the Ricci operator Q for $\hbar_3$ are given by

$$\begin{cases} Qe &= (\alpha^2+\lambda^2+\frac{1}{2}r - 2\alpha\lambda - Z)e - 2a_3\lambda\phi e + A_3\xi,\\ Q\phi e &= 2a_3\lambda e + (\alpha^2+\lambda^2+\frac{1}{2}r + 2\alpha\lambda + Z)\phi e + B_3\xi,\\ Q\xi &= -A_3 e + B_3\phi e - 2(\lambda^2+\alpha^2)\xi, \end{cases} \tag{2.12}$$

where $A_3 = \omega(e), B_3 = \omega(\phi e)$ and $Z = \xi(\lambda)$.

3. Cotton Solitons

In this section, we give the components of the Cotton tensor and calculate the scalar curvature for each three-dimensional almost α-para-Kenmotsu manifold according to their h types.

Using the relations $S(X,Y) = \sum_{i=1}^{2n+1}\varepsilon_i g(R(e_i,X)Y, e_i)$ and $r = \sum_{i=1}^{2n+1}\varepsilon_i S(e_i,e_i)$. We derive a useful formula for the scalar curvature.

Lemma 3.1. *Let (M, ϕ, ξ, η, g) be a three-dimensional almost α-para-Kenmotsu manifold with $\hbar_1$ type. Then the scalar curvature r is given as follows:*

$$r = trace(Q) = 2[e(c) - \phi e(a) - 3\alpha^2 + \lambda^2 + c^2 - a^2]. \tag{3.1}$$

Proposition 3.1. *Let (M, ϕ, ξ, η, g) be a three-dimensional almost α-para-Kenmotsu manifold with $\hbar_1$ type. If the characteristic vector field ξ is a harmonic vector field in the open subset U_1, then the following relations are valid for* *the components of Cotton tensor C.*

$$C_{11} = C(e,e) = \lambda(\frac{1}{2}r - 3\lambda^2 + \alpha^2 - 2a_1\lambda) - 3\alpha Z - \xi(Z) - 4a_1^2\lambda, \tag{3.2}$$

$$C_{12} = C(e,\phi e) = -\alpha(\frac{1}{2}r - \lambda^2 + 3\alpha^2 - 6a_1\lambda) + 2\xi(a_1)\lambda + (4a_1+\lambda)Z - \frac{1}{4}\xi(r), \tag{3.3}$$

$$C_{13} = C(e,\xi) = -2\alpha e(\lambda) - e(Z) - 4aa_1\lambda + \phi e(\alpha^2 - \lambda^2 - 2a_1\lambda) + \frac{1}{4}\phi e(r) - 2c(2\alpha\lambda + Z), \tag{3.4}$$

$$C_{22} = C(\phi e,\phi e) = \lambda(\frac{1}{2}r - 3\lambda^2 + \alpha^2 + 2a_1\lambda) - 3\alpha Z - \xi(Z) - 4a_1^2\lambda, \tag{3.5}$$

$$C_{23} = C(\phi e,\xi) = e(\alpha^2 - \lambda^2 + 2a_1\lambda) + 2aZ + \phi e(Z) + 4a_1c\lambda + \frac{1}{4}e(r) + 4a\alpha\lambda + 2\alpha\phi e(\lambda), \tag{3.6}$$

$$C_{33} = C(\xi,\xi) = -4a_1\lambda^2. \tag{3.7}$$

Proof. Well-known Cotton tensor equation is defined as

$$C(X,Y)Z = (\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z) - \frac{1}{4}[X(r)g(Y,Z) - Y(r)g(X,Z)] \tag{3.8}$$

for all vector fields X, Y, Z, where S is the Ricci curvature tensor and r is the scalar curvature. From (1.2) and using the notation $C_{ijk} = C(e_i,e_j)e_k$ for all $i,j = 1,2,3$, we get

$$\begin{aligned} C_{11} &= \frac{1}{2}[C_{nm1}\epsilon^{nml}g_{l1}] = \frac{1}{2}[-C_{nm1}\epsilon^{nm1}] = -\frac{1}{2}[C_{2m1}\epsilon^{2m1} + C_{3m1}\epsilon^{3m1}] \\ &= -C_{231}. \end{aligned}$$

Using similar calculations we have

$$C_{12} = C_{311}, \quad C_{13} = C_{121}, \quad C_{22} = C_{312}, \quad C_{23} = C_{122}, \quad C_{33} = C_{123}.$$

From the assumption of ξ is a harmonic vector field, using Theorem 2.1 and (2.6), we have $A = B = 0$. By using (1.2) and (2.6) after a long but straightforward calculations we compute the components C_{ij} as follows:

$$\begin{aligned} C_{11} &= -C_{231} = -[C(\phi e,\xi)e] \\ &= -[(\nabla_{\phi e}S)(\xi,e) - (\nabla_\xi S)(\phi e,e)] \\ &= -2\alpha^2\lambda - \alpha Z + \lambda(\frac{1}{2}r + \alpha^2 - \lambda^2 - 2a_1\lambda) - 2\lambda(\lambda^2 - \alpha^2) \\ &\quad - 2\xi(\alpha\lambda) - \xi(Z) - 4a_1^2\lambda \\ &= \lambda(\frac{1}{2}r - 3\lambda^2 + \alpha^2 - 2a_1\lambda) - 3\alpha Z - \xi(Z) - 4a_1^2\lambda, \end{aligned}$$

$$\begin{aligned} C_{12} &= C_{311} = [C(\xi,e)e] \\ &= [(\nabla_\xi S)(e,e) - (\nabla_e S)(\xi,e)] + \frac{1}{4}\xi(r) \\ &= -\xi(\frac{1}{2}r + \alpha^2 - \lambda^2 - 2a_1\lambda) + 2a_1(2\alpha\lambda + Z) \\ &\quad - \alpha(\frac{1}{2}r + \alpha^2 - \lambda^2 - 2a_1\lambda) - \lambda(2\alpha\lambda + Z) + 2\alpha(\lambda^2 - \alpha^2) + \frac{1}{4}\xi(r) \\ &= -\alpha(\frac{1}{2}r - \lambda^2 + 3\alpha^2 - 6a_1\lambda) + 2\xi(a_1)\lambda + (4a_1+\lambda)Z - \frac{1}{4}\xi(r), \end{aligned}$$

$$\begin{aligned} C_{13} = C_{121} &= [C(e,\phi e)e] \\ &= [(\nabla_e S)(\phi e, e) - (\nabla_{\phi e} S)(e,e)] - \frac{1}{4}\phi e(r) \\ &= -e(2\alpha\lambda + Z) - 4aa_1\lambda + \phi e(\frac{1}{2}r + \alpha^2 - \lambda^2 - 2a_1\lambda) - 2c(2\alpha\lambda + Z) \\ &\quad - \frac{1}{4}\phi e(r) \\ &= -2\alpha e(\lambda) - e(Z) - 4aa_1\lambda + \phi e(\alpha^2 - \lambda^2 - 2a_1\lambda) \\ &\quad + \frac{1}{4}\phi e(r) - 2c(2\alpha\lambda + Z), \end{aligned}$$

$$\begin{aligned} C_{22} = C_{312} &= [C(\xi, e)\phi e] \\ &= [(\nabla_\xi S)(e,\phi e) - (\nabla_e S)(\xi, \phi e)] \\ &= -\xi(2\alpha\lambda + Z) - 4a_1^2\lambda - \alpha(2\alpha\lambda + Z) - 2\lambda(\lambda^2 - \alpha^2) \\ &\quad + \lambda(\frac{1}{2}r + \alpha^2 - \lambda^2 + 2a_1\lambda) \\ &= \lambda(\frac{1}{2}r - 3\lambda^2 + \alpha^2 + 2a_1\lambda) - 3\alpha Z - \xi(Z) - 4a_1^2\lambda, \end{aligned}$$

$$\begin{aligned} C_{23} = C_{122} &= [C(e,\phi e)\phi e] \\ &= [(\nabla_e S)(\phi e, \phi e) - (\nabla_{\phi e} S)(e, \phi e)] - \frac{1}{4}e(r) \\ &= e(\frac{1}{2}r + \alpha^2 - \lambda^2 + 2a_1\lambda) + 2a(2\alpha\lambda + Z) + \phi e(2\alpha\lambda + Z) \\ &\quad + 4a_1 c\lambda - \frac{1}{4}e(r) \\ &= e(\alpha^2 - \lambda^2 + 2a_1\lambda) + 2aZ + \phi e(Z) + 4a_1 c\lambda \\ &\quad + \frac{1}{4}e(r) + 4a\alpha\lambda + 2\alpha\phi e(\lambda), \end{aligned}$$

$$\begin{aligned} C_{33} = C_{123} &= [C(e,\phi e)\xi] \\ &= [(\nabla_e S)(\phi e, \xi) - (\nabla_{\phi e} S)(e,\xi)] \\ &= -\lambda(\frac{1}{2}r + \alpha^2 - \lambda^2 + 2a_1\lambda) + \lambda(\frac{1}{2}r + \alpha^2 - \lambda^2 - 2a_1\lambda) \\ &= -4a_1\lambda^2. \end{aligned}$$

□

To calculate r for $\hbar_2$ type we construct a new pseudo-orthonormal frame $\{\tilde{e_1}, \tilde{e_2}, \tilde{e_3}\}$ such as $\tilde{e_1} = \frac{e_1+e_2}{\sqrt{2}}, \tilde{e_2} = \frac{e_1-e_2}{\sqrt{2}}$ and $\tilde{e_3} = \xi$. So, we get $g(\tilde{e_1}, \tilde{e_1}) = 1 = -g(\tilde{e_2}, \tilde{e_2}), g(\tilde{e_1}, \tilde{e_2}) = 0$ and $h\tilde{e_1} = h\tilde{e_2} = e_2$. Then we give the following lemma.

Lemma 3.2. *Let (M, ϕ, ξ, η, g) be a three-dimensional almost α-para-Kenmotsu manifold with $\hbar_2$ type. Then the scalar curvature r is given as follows:*

$$r = trace(Q) = 2[-e_1(b_2) + e_2(b_1) + 2b_1b_2 - 3\alpha^2]. \tag{3.9}$$

Since the proof of the following proposition is quite similar to Proposition 3.1, so we don't give the proof of it.

Proposition 3.2. *Let (M, ϕ, ξ, η, g) be a three-dimensional almost α-para-Kenmotsu manifold with $\mathfrak{h}_2$ type. If the characteristic vector field ξ is a harmonic vector field in the open subset U, then the following relations are valid for the components of Cotton tensor C.*

$$C_{11} = C_{311} = C(e_1, e_1) = -2\xi(a_2) + 2a_2(2a_2 - 3\alpha) - \alpha^2 - \frac{1}{2}r, \tag{3.10}$$

$$C_{12} = C_{231} = C(e_1, e_2) = -3\alpha^3 - \alpha\frac{1}{2}r - \frac{1}{4}\xi(r), \tag{3.11}$$

$$C_{13} = C_{121} = C(e_1, \xi) = 2e_2(a_2) - 4b_2(\alpha - a_2) + \frac{1}{4}e_1(r), \tag{3.12}$$

$$C_{22} = -C_{322} = C(e_2, e_2) = 0, \tag{3.13}$$

$$C_{23} = C_{122} = C(e_2, \xi) = -\frac{1}{4}e_2(r), \tag{3.14}$$

$$C_{33} = C_{123} = C(\xi, \xi) = 0. \tag{3.15}$$

Lemma 3.3. *Let (M, ϕ, ξ, η, g) be a three-dimensional almost α-para-Kenmotsu manifold with $\mathfrak{h}_3$ type. Then the scalar curvature r is given as follows:*

$$r = trace(Q) = 2[e(b_4) - \phi e(b_3) - 3\alpha^2 - \lambda^2 - b_3^2 + b_4^2]. \tag{3.16}$$

Since the proof of the following proposition is quite similar to Proposition 3.1, so we don't give the proof of it.

Proposition 3.3. *Let (M, ϕ, ξ, η, g) be a three-dimensional almost α-para-Kenmotsu manifold with $\mathfrak{h}_3$ type. If the characteristic vector field ξ is a harmonic vector field in the open subset U_1, then the following relations are valid for the components of Cotton tensor C.*

$$C_{11} = -C_{231} = C(e, e) = 2\lambda(a_3\lambda - \xi(a_3)) - 2a_3(3\alpha\lambda + 2Z), \tag{3.17}$$

$$C_{12} = C_{311} = C(e, \phi e) = -(\alpha + \lambda)(3\alpha^2 + 3\lambda^2 + \frac{1}{2}r - 2\alpha\lambda - Z) - \frac{1}{4}\xi(r) + \xi(Z) + 4a_3^2\lambda + 2\alpha Z - 2\lambda Z, \tag{3.18}$$

$$C_{13} = C_{121} = C(e, \xi) = -2e(a_3\lambda) + \phi e(\lambda^2 - 2\alpha\lambda - Z) + \frac{1}{4}\phi e(r) - 4a_3b_4\lambda - 2b_3(2\alpha\lambda + Z), \tag{3.19}$$

$$C_{22} = C_{312} = C(\phi e, \phi e) = -2\lambda\xi(a_3) - 2a_3(2Z + 3\alpha\lambda + \lambda^2), \tag{3.20}$$

$$C_{23} = C_{122} = C(\phi e, \xi) = e(\lambda^2 + 2\alpha\lambda + Z) + 2\phi e(a_3\lambda) + 4a_3b_3\lambda + \frac{1}{4}e(r) + 2b_4(2\alpha\lambda + Z), \tag{3.21}$$

$$C_{33} = C_{123} = C(\xi, \xi) = 4a_3\lambda^2. \tag{3.22}$$

4. 3-dimensional almost α-para-Kenmotsu manifolds with harmonic vector field ξ

Theorem 4.1. *Let (M, ϕ, ξ, η, g) be a three-dimensional almost paracosymplectic manifold with $\mathfrak{h}_1$ type such that the characteristic vector field is harmonic (i.e. $Q\xi = \rho\xi$) and ρ is constant along the characteristic vector field ξ. If M admits a Cotton soliton with potential vector field being collinear with characteristic vector field ξ, then M is either*

(i) a locally warped product $M_1 \times_{f^2} M_2$, where M_2 is an almost para-Kaehler manifold, M_1 is an open interval with coordinate t, and $f^2 = we^{2\alpha t}$ for some positive constant

or

(ii) locally conformally flat.

Proof. Firstly, we denote U_1 and U_2 as follows:

$$U_1 = \{p \in M : \lambda = 0 \; in \; a \; neighborhood \; of \; p\}$$

and

$$U_2 = \{p \in M : \lambda \neq 0 \; in \; a \; neighborhood \; of \; p\}.$$

If we only study on U_1, then the result (i) comes from Theorem 2.2. Now, assume that U_2 is a non-empty set and let $\{e, \phi e, \xi\}$ is a ϕ-basis in U_2.

From the characteristic vector field is harmonic and (2.6), we have $\rho = 2(\lambda^2 - \alpha^2), \xi(\rho) = \xi(\lambda) = Z = 0$ and $A = B = 0$.

If $V = 0$ (1.4) returns to $C = \sigma g$. It could be shown obviously that the tensor C is trace-free. So, σ is equal to zero. Hence, M is locally conformally flat.

Now, we assume that $V = f\xi$, where f is a non-vanishing constant function. Substituting V by $f\xi$ and using (2.1), equation (1.4) becomes:

$$\sigma g(X, Y) = 2fg(\phi hX, Y) + C(X, Y) \tag{4.1}$$

Putting $X = Y = e$ in (4.1) and using (3.2) we obtain

$$\sigma = -\lambda(\frac{1}{2}r - 3\lambda^2 - 2a_1\lambda) + 4a_1^2\lambda. \tag{4.2}$$

Similarly, letting $X = Y = \phi e$ in (4.1) and using (3.5) we get

$$\sigma = \lambda(\frac{1}{2}r - 3\lambda^2 + 2a_1\lambda) - 4a_1^2\lambda. \tag{4.3}$$

On the other hand, if we put $X = e$ and $Y = \phi e$ in (4.1) and use (3.3) we have

$$2f\lambda = -2\xi(a_1)\lambda + \frac{1}{4}\xi(r). \tag{4.4}$$

If we add (4.2) and (4.3) we have

$$\sigma = 2a_1\lambda^2. \tag{4.5}$$

Comparing (4.2) with (4.5) after some calculations, we get

$$\frac{1}{2}r - 3\lambda^2 = 4a_1^2. \tag{4.6}$$

Differentiating (4.5) along the vector field ξ, and from the fact that σ is constant and $\xi(\lambda) = 0$ we find $\xi(a_1) = 0$. Similarly, we have $\xi(r) = 0$ from differentiating (4.6) along the vector field ξ. Hence, from (4.4) we get $2f\lambda = 0$.

This completes the proof of the theorem. □

Theorem 4.2. *Let (M, ϕ, ξ, η, g) be a three-dimensional almost paracosymplectic manifold with $\mathfrak{h}_2$ type such that the characteristic vector field is harmonic (i.e. $Q\xi = \rho\xi$). If M admits a Cotton soliton with potential vector field being collinear with characteristic vector field ξ, then M is locally conformally flat or Cotton soliton is steady.*

Proof. Now, assume that U is an open set of M where $h \neq 0$ and let $\{e_1, e_2, \xi\}$ is a pseudo-orthonormal basis in U. From the characteristic vector field is harmonic and (2.9), we have $A_2 = 0$.

If $V = 0$ (1.4) returns to $C = \sigma g$. It could be shown obviously that the tensor C is trace-free. So, σ is equal to zero. Hence, M is locally conformally flat. Now, assume that $V = f\xi$, where f is a non-vanishing constant function. The equation (4.1) is also valid for $\mathfrak{h}_2$ type. Letting $X = Y = e_1$ in (4.1) and using (3.10) we get

$$0 = -2f - 2\xi(a_2) + 4a_2^2 - \frac{1}{2}r \tag{4.7}$$

Putting $X = Y = \xi$ in (4.1) and using (3.15) we have

$$\sigma = 2\xi(f) = 0. \tag{4.8}$$

This completes the proof of the theorem. □

Theorem 4.3. *Let (M, ϕ, ξ, η, g) be a three-dimensional almost α-para-Kenmotsu manifold with $\mathfrak{h}_3$ type such that the characteristic vector field is harmonic (i.e. $Q\xi = \rho\xi$) and ρ is constant along the characteristic vector field ξ. If M admits a Cotton soliton with potential vector field being collinear with characteristic vector field ξ, then M is either* *(i) a locally warped product $M_1 \times_{f^2} M_2$, where M_2 is an almost para-Kaehler manifold, M_1 is an open interval with coordinate t, and $f^2 = we^{2\alpha t}$ for some positive constant*

or

(ii) locally conformally flat.

Proof. The proof of (i) is similar to the proof of the Theorem 4.1. If $V = 0$, then M is locally conformally flat. Now, assume that $V = f\xi$, where f is a non-vanishing constant function. Substituting V by $f\xi$ and using (2.1), equation (1.4) becomes:

$$\sigma g(X,Y) = 2f\alpha g(X,Y) - 2\alpha f\eta(X)\eta(Y) + 2fg(\phi hX,Y) + C(X,Y). \tag{4.9}$$

Letting $X = Y = e$ in (4.9) and using (3.17) we get

$$\sigma = 2f\alpha + 2f\lambda - 2\lambda(\lambda a_3 - \xi(a_3)) + 6a_3\alpha\lambda. \tag{4.10}$$

Again, putting $X = Y = \phi e$ in (4.9) and by the help of (3.20), we obtain

$$\sigma = 2f\alpha - 2f\lambda - 2\lambda\xi(a_3) - 6a_3\alpha\lambda - 2a_3\lambda^2. \tag{4.11}$$

On the other hand, if we put $X = e$ and $Y = \phi e$ in (4.9) and use (3.18), we have

$$0 = -\frac{1}{4}\xi(r) + 4a_3^2\lambda - (\alpha+\lambda)(3\alpha^2 + 3\lambda^2 + \frac{1}{2}r - 2\alpha\lambda). \tag{4.12}$$

By adding (4.10) with (4.11) we get

$$\sigma = 2f\alpha - 2a_3\lambda^2. \tag{4.13}$$

Comparing (4.10) with (4.13), we conclude that

$$f = -\xi(a_3) - 3a_3\alpha. \tag{4.14}$$

By differentiating the equation (4.13) along the vector field ξ, we obtain $\xi(a_3) = 0$. Letting $X = Y = \xi$ in (4.9) and using (3.22) we have

$$\sigma = 4a_3\lambda^2. \tag{4.15}$$

From (4.13) and (4.15) we obtain

$$f\alpha = 3a_3\lambda^2. \tag{4.16}$$

From (4.14) and (4.16), we get $f(\lambda^2 + \alpha^2) = 0$. There are two possibilities. The first one is $f = 0$, it is impossible because f is non-vanishing. The second one is $\alpha^2 + \lambda^2 = 0$, again it is impossible.

Hence, we complete the proof of the theorem. □

Now, we will give an example which satisfies Theorem 4.2.

Example 4.1. We consider the three dimensional manifold M and the vector fields

$$e_1 = \frac{\partial}{\partial x}, \quad e_2 = \frac{\partial}{\partial y}, \quad e_3 = -x\frac{\partial}{\partial x} + (y-x)\frac{\partial}{\partial y} + 2\frac{\partial}{\partial z}$$

$$g(e_1,e_1) = g(e_1,e_3) = g(e_2,e_2) = g(e_2,e_3) = 0, \quad g(e_1,e_2) = g(e_3,e_3) = 1,$$

$$h = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The 1-form $\eta = \frac{1}{2}dz$ and the fundamental 2-form $\Phi = dx \wedge dy + \frac{x-y}{2}dx \wedge dz - \frac{x}{2}dy \wedge dz$ defines an almost paracosymplectic manifold.

Let g be the pseudo-Riemannian metric and the $(1,1)$-tensor field ϕ given by

$$g = \begin{pmatrix} 0 & \frac{1}{2} & \frac{x-y}{4} \\ \frac{1}{2} & 0 & \frac{x}{4} \\ \frac{x-y}{4} & \frac{x}{4} & \frac{1+2x(x-y)}{4} \end{pmatrix},$$

$$\phi = \begin{pmatrix} 1 & 0 & \frac{x}{2} \\ 0 & -1 & \frac{y-x}{2} \\ 0 & 0 & 0 \end{pmatrix}.$$

We get

$$\phi e_1 = e_1, \quad \phi e_2 = -e_2, \quad \phi e_3 = 0$$

and

$$\begin{aligned} [e_1, e_3] &= -e_1 - e_2 \\ [e_2, e_3] &= e_2 \\ [e_1, e_2] &= 0. \end{aligned}$$

Using the equation (1.4), (4.7) and Proposition 3.2, we can see that the manifold admits Cotton soliton for $V = 2\xi, b_1 = b_2 = 0$ and $a_2 = 1$. From Lemma 3.2, we can see that the scalar curvature $r = 0$ and the Cotton soliton is steady.

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