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Research Article

Persistent homology based Wasserstein distance for graph networks

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Abstract

The technique of measuring similarity between topological spaces using Wasserstein distance between persistence diagrams is extended to graph networks in this paper. A relationship between the Wasserstein distance of the Cartesian product of topological spaces and the Wasserstein distance of individual spaces is found to ease the comparative study of the Cartesian product of topological spaces. The Cartesian product and the strong product of weighted graphs are defined, and the relationship between the Wasserstein distance between graph products and the Wasserstein distance between individual graphs is determined. For this, clique complex filtration and the Vietoris- Rips filtration are used.

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Keywords. persistent homology, Wasserstein distance, cartesian product of graphs, strong product of graphs

1. Introduction

In the past few decades, the process of extracting useful information from large and intricate data sets has become increasingly challenging. To address the issue of analysing the structure of such data, there has been significant progress in the field of topological data analysis. This rapidly evolving area offers a range of powerful techniques that aid in identifying important topological characteristics within the data. To categorize objects based on their topological characteristics, an effective mathematical tool known as homology is implemented. Homology helps identify the presence of "holes" in objects of different dimensions, aiding in their classification. The simplicial approach to homology has gained popularity due to its computational efficiency and ability to store data effectively. Persistent homology is an innovative method used to identify topological characteristics within data. It accomplishes this by converting the given data into simplicial complexes, providing a comprehensive understanding of the space's topology at different levels of spatial resolution. Through persistent homology, a multitude of persistent homology classes is identified across a broad range of spatial resolutions. These classes represent significant

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features of the underlying space, offering valuable insights into its structure. Persistent homology groups can be effectively visualized using two tools: persistent barcodes (bcd_n) [8, 14] and persistence diagrams. These visualizations helps to understand the persistent Betti numbers [22], which provide information about the existence and lifespan of topological features. The concept of persistent homology and the idea of using size functions to identify 0-dimensional persistent homology by counting connected components were initially introduced by Patrizio Frosini and his collaborators in 1990 [12]. Building upon this work, Vanessa Robbins explored the homology of sample spaces by defining persistent homology groups as the images of homomorphisms induced by inclusion [23]. A more widely accepted definition of persistent homology, based on simplicial complexes, was later proposed by Edelsbrunner, Letscher, and Zomorodian [11]. This definition has gained significant recognition in the field, providing a solid foundation for studying persistent homology and its applications.

The study and exploration of information networks have emerged as a rapidly advancing field, unveiling the intricate connections and relationships present in data [5, 20]. These networks can be found in various domains, such as biological networks [4, 6], social networks [5], communication networks [9], and more. Graphs serve as a valuable means of representing information networks, and they can also be viewed as topological entities. Comparing graphs can be a challenging and labor-intensive task [3]. Therefore, it would be a significant breakthrough if any technique designed for comparing topological objects could also be applied to compare graphs . This would enable researchers and analysts to leverage existing methods and tools developed for studying topological objects in the analysis and comparison of graphs. Such an innovation would greatly simplify and enhance the process of graph comparison, offering valuable insights into the similarities and differences between complex networks. The determination of topological equivalence between structures can provide valuable insights as the number of connected components and the presence of holes or voids are topological invariants [24]. Recent studies demonstrated that persistent homology, a mathematical tool that quantifies the uniqueness of data, can be effectively applied to study the topological properties of networks by identifying structural holes [1, 19]. These structural holes enable the comparison of networks and the assessment of their similarity. In the context of practical network science, a comprehensive resource provides a theoretical overview of significant advancements in the application of persistent homology (PH). This resource explores how PH has been utilized to analyze and understand complex networks [2].

Indeed, Hitesh Gakhar and Jose A. Perea [13] have proposed an approach to analyze the filtration of the Cartesian product of topological spaces by utilizing the concept of categorical product. They have also developed a method for determining the barcodes, which provide valuable information about the topological features, of the Cartesian product of topological spaces. Since we can evaluate the similarity of topological spaces to some extent using persistence diagrams, it is feasible to extend this notion to measure the similarity of Cartesian products of topological spaces using persistence diagrams. In this context, the similarity is assessed by quantifying the distance between the persistence diagrams. The Wasserstein distance is a well-established metric frequently employed to calculate the distance between persistence diagrams [10, 22]. By leveraging the Wasserstein distance, it becomes possible to effectively measure the similarity between Cartesian products of topological spaces. Establishing a relationship between the Wasserstein distance of the Cartesian product of topological spaces and the Wasserstein distance of the individual spaces would indeed be a remarkable innovation. Such a relationship would provide valuable insights into the interactions and dependencies between the spaces. This innovative approach has the potential to deepen our understanding of how different spaces interact and influence each other, ultimately leading to novel applications in topological analysis and similarity measurement.

Comparing graphs using common methods can be challenging [2]. However, graphs can be compared using the Wasserstein distance with the aid of various filtering methods. These filtering techniques enable the transformation of complex graphs into simpler representations, such as products of smaller graphs. This approach offers the advantage of simplifying the analysis of complex graphs by decomposing them into more manageable components [15]. In this regard, it is worth exploring whether a relationship exists between the distance between complex networks and the distance between the smaller graphs when the complex networks can be expressed as a product of these smaller graphs. That is precisely what we are doing here.

Section 2 of the paper provides the necessary background information, including preliminary definitions, results, theorems, and notations. It establishes the foundation for the subsequent sections by introducing the key concepts and frameworks required for the analysis. In Section 3, the paper presents major results related to the comparison of persistence diagrams. Specifically, it explores the relationship between the Wasserstein distances of persistence diagrams of Cartesian products of topological spaces and the Wasserstein distances of the individual spaces. Additionally, the paper investigates the relationship between the Wasserstein distances of persistence diagrams of Cartesian products and strong products of graphs, focusing on the case of smaller graphs. These relationships are examined for both weighted and unweighted graphs by defining the Cartesian product and the strong product operations on these graph structures.

For fundamental definitions, notations, and terminologies associated with Homology, Persistent homology, and homology of product spaces we can refer to [8, 10, 13, 16, 21]. Further, for graph theoretical concepts we refer [15, 26, 27].

2. Preliminaries

Indeed, determining the homology of any arbitrary topological object can be extremely hard. Consequently, an alternative approach involves approximating the topological object with a simplicial complex. By applying this method, we can compute the homology, a technique referred to as simplicial homology. While defining simplicial complexes, specific rules must be followed. These rules are clearly outlined in the subsequent definitions.

Definition 2.1 ([10, 25])**.** A simplex is a generalization of the notion of a triangle or tetrahedron to arbitrary dimensions. Specifically, a s simplex is a s -dimensional polytope which is the convex hull of $s+1$ affinely independent points.

Definition 2.2 ([10, 25])**.** A simplicial complex S is a collection of simplices such that

(1) If S contains a simplex s_1, then S also contains every face of s_1
(2) If two simplices in S intersect, then their intersection is a face of each of them.

Before delving into the definition of simplicial homology, which is an effective method of identifying voids or holes within a system, it is essential to understand the concept of chains, cycles, and boundaries. This fundamental concept serves as a foundation for working with simplicial complexes and their homology.

Definition 2.3 ([10, 25])**.** Let S be a simplicial complex and p a dimension. A p-chain is a formal sum of p-simplices in S with integer coefficients. The standard notation for this is $c_p = \sum a_i\sigma_i$, where the σ_i are the p-simplices and the a_i are the coefficients. The set of all p-chains form a group C_p under addition.

Definition 2.4 ([10,25]). The boundary of a p-simplex is the sum of its $(p-1)$-dimensional faces. If $\sigma = [u_0, u_1, ..., u_p]$ for the simplex spanned by the listed vertices, then its boundary is $\partial_p \sigma = \sum_{j=0}^{p} (-1)^j [u_0, ..., \hat{u_j}, ..., u_p]$, where ∂_p is called boundary operator and the hat indicates that u_j is omitted.

Definition 2.5 ([10, 25]). A p-cycle c is a p-chain with empty boundary, $\partial_p c = 0$. A p-boundary b is a p-chain that is the boundary of a $(p+1)$-chain, $b = \partial_{p+1} d$ with $d \in C_{p+1}$. And the set of all p- cycles and p-boundaries will form subgroups of chain group C_p.

Definition 2.6 ([10, 25]). Let C_p be a chain group whose elements are the p chains and $\partial_p : C_p \to C_{p-1}$ maps each p-chain to the sum of the $(p-1)$dimensional faces of its p cells which is a $(p-1)$ chain.
Writing the groups and maps in sequence, we get the chain complex,

$$\ldots \xrightarrow{\partial_{p+2}} C_{p+1} \xrightarrow{\partial_{p+1}} C_p \xrightarrow{\partial_p} C_{p-1} \xrightarrow{\partial_{p-1}} \ldots$$

Then the n^{th}homology group is defined as

$$H_n = Ker(\partial_n)/Im(\partial_{n+1}).$$

For point cloud data, homology might not provide relevant insights. To overcome this, we can use a different approach by constructing a family of simplicial complexes through a process called filtration. Filtration involves gradually building simplicial complexes from the given points based on certain criteria or distances, creating a sequence of complexes that captures the underlying topological features of the data at various scales. This method allows for a more nuanced analysis of the data's topological structure, providing valuable insights that might not be apparent when examining individual points. Certainly, during the construction of simplicial complexes and the computation of homology, certain holes might emerge and then vanish. The persistence of these homological features, capturing how long these holes persist across different scales, can be regarded as significant features of the data set. This concept of persistence is fundamental in topological data analysis and helps in understanding the robust topological features of the data.

Definition 2.7 ([10]). Consider a real valued function $g' : T \to R$ is defined on a topological space T. Let $T_a = g'^{-1}(-\infty, a]$ denote the sublevel set for the function value a. So we have inclusions:

$$T_a \subseteq T_b \text{ for } a \leq b$$

This inclusion induces a map in the homology groups. So, if $i : T_a \to T_b$ denotes the inclusion map, we have induced map

$$\mathrm{f} = i_* : H_p(T_a) \to H_p(T_b)$$

Consider the sequence a sequence of distinct values $a_1 < a_2 < \ldots$ corresponding to which we have the sequence of homomorphisms induced by inclusions.

$$0 \to H_p(T_{a_1}) \to H_p(T_{a_2}) \to H_p(T_{a_3}) \to \ldots \to H_p(T_{a_n}) \to H_p(T)$$

Then the homomorphism

$$\mathrm{f}^{ij} : H_p(T_{a_i}) \to H_p(T_{a_j})$$

for all p and $1 \leq i \leq j \leq n$ takes the homology classes of the sublevel set T_{a_i} to those of the sublevel sets of T_{a_j}.
The p^{th} persistent homology groups are the images of the homomorphisms:

$$H_p^{ij} = im\mathrm{f}_p^{ij} \text{ for } 1 \leq i \leq j.$$

Topological persistence may be introduced with the observation that a nested sequence of topological spaces

$$T_0 \subseteq T_1 \subseteq T_2 \ldots \subseteq T_n$$

gives a sequence of vector spaces and linear maps

$$H_p(T_0) \to H_p(T_1) \to \ldots \to H_p(T_n)$$

upon computing homology with coefficients in a field $\mathbb{F}$. In general, a diagram of vector spaces and linear maps $V_0 \to V_1 \to \ldots \to V_n$ is called a persistent module indexed by $0, 1, 2, \ldots, n$. We can write persistent homology module :

$$M_h = H_p(K_1) \oplus H_P(K_2) \oplus \ldots \oplus H_p(K_n)$$

Module M_h decomposes in to a direct sum of interval modules M^p_{hj}, each of which corresponds to a bar in the barcode (bcd_n). In the study of Hitesh Gakhar and Jose A. Perea, they have developed methods to determine the barcode of the product of topological spaces by using the barcodes of the individual spaces. This innovative approach allows for a more efficient and comprehensive analysis of the topological features present in the product space. The categorical approach to persistence homology gives a deeper understanding by revealing relationships with other mathematical structures. Understanding these relationships involves utilizing categorical concepts such as functors and natural transformations. Functors enable mapping between different homology theories, which is extremely useful when dealing with various sorts of data. Indeed, the study of persistent homology in the context of products of topological spaces has been studied by Hitesh Gakhar and Jose A. Perea, especially from a categorical point of view. Since our specific focus on topological and graph products, it will be beneficial to consider relevant results from their work.

Definition 2.8 ([7])**.** A category, $\mathcal{C}$, consists of a class of objects, $\mathcal{C}_0$, and for each pair of objects $\mathcal{X}_1$,$\mathcal{X}_2 \in \mathcal{C}_0$, a set of morphisms, $\mathcal{C}(\mathcal{X}_1, \mathcal{X}_2)$. We often write $f : \mathcal{X}_1 \to \mathcal{X}_2$ if $f \in \mathcal{C}(\mathcal{X}_1, \mathcal{X}_2)$. For every triple $\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3 \in \mathcal{C}_0$, there is a set mapping,

$$\mathcal{C}(\mathcal{X}_2, \mathcal{X}_3) \times \mathcal{C}(\mathcal{X}_1, \mathcal{X}_2) \to \mathcal{C}(\mathcal{X}_1, \mathcal{X}_3), \quad (g, f) \to gf,$$

called composition. Composition must be associative, in the sense that $(hg)f = h(gf)$. Finally, for all $\mathcal{W} \in \mathcal{C}$, there is an identity morphism, $\mathcal{I}d_{\mathcal{W}} : \mathcal{W} \to \mathcal{W}$, that satisfies $\mathcal{I}d_{\mathcal{W}} f = f$ and $g\mathcal{I}d_{\mathcal{W}} = g$ for all $f : \mathcal{Z} \to \mathcal{W}$ and all $g : \mathcal{W} \to \mathcal{Y}$. The identity morphism is unique.

Theorem 2.9 ([13])**.** *Let P_c be the poset category of a separable(with respect to the order topology) totally ordered set. Let $K_1, K_2 \in S^{P_c}$ be P_c-indexed diagrams of spaces, and assume that $H_i(K_1; \mathbb{F})$ and $H_i(K_2; \mathbb{F})$ are pointwise finite for each $0 \le i, j \le n$ where H_i is the i^{th} persistence homology group. Then $H_n(K_1 \times K_2; \mathbb{F})$ is pointwise finite, and its barcode satisfies:*

$$bcd_n(K_1 \times K_2; \mathbb{F}) = \bigcup_{i+j=n} \{I \cap J \mid I \in \mathrm{bcd}_i(K_1), J \in \mathrm{bcd}_j(K_2)\}$$

where the union on the right is of multisets.

Corollary 2.10 ([13])**.** *Let $K_1, \ldots, K_m \in S^{P_c}$. Assume that for each $1 \le j \le m$ and $0 \le n_j \le n$, then $H_n(K_1 \times K_2 \times \ldots \times K_m)$ is pointwise finite, and its barcode satisfies:*

$$bcd_n(K_1 \times K_2 \times \ldots \times K_m) = \{I_1 \cap I_2 \cap \ldots I_k | I_j \in \mathrm{bcd}_{n_j}(K_j), \sum_{j=1}^{m} n_j = n\}$$

where the union on the right is of multisets.

A graph L is defined as a pair of sets $L = (U(L), J(L))$ simply $L = (U, J)$, where $U(L)$ represents the set of vertices and $J(L)$ represents the set of edges in the graph. Graphs are a common representation for networks, where vertices are used to represent objects or entities, and edges denote the relationships or connections between these objects. When edges connect two vertices symmetrically, the graph is called an undirected graph. If edges connect two vertices asymmetrically, the graph is called a directed graph. In the context of a weighted graph, a weight function $\mathbf{W} : J(L) \to \mathbb{R}$ is introduced to assign a specific weight to each edge. Otherwise it is unweighted. Simply a graph refers to an unweighted graph. Graph products and relationships between their individual graphs are useful to some extent to facilitate the study of complicated networks. Therefore, our study focuses on properties between graph products and individual graphs especially cartesian product and strong product.

Definition 2.11 ([15])**.** If $L_1 = (U(L_1), J(L_1)), L_2 = (U(L_2), J(L_2)), \ldots, L_k = (U(L_k), J(L_k))$ are k unweighted undirected graphs, then their Cartesian product is the graph $L_1 \Box L_2 \ldots \Box L_k$ with vertex set $\{(x_1, x_2, \ldots, x_k)| \; xi \in U(Li)\}$, and for which two vertices $(x_1, x_2, \ldots, x_k)$, $(x'_1, x'_2, \ldots, x'_k)$ in $L_1 \Box L_2 \ldots \Box L_k$ with $x_i, x'_i \in U(Li)$ are adjacent whenever $x_i x'_i \in J(L_i)$ for exactly one index $1 \leq i \leq k$, and $x_j = x'_j$ for each index $j \neq i$.

Example 2.12. Consider the complete graphs K_3 with vertices a, b, c, K_2 with vertices $0, 1$ and p, q. Then the Cartesian product of these three unweighted undirected graphs is then

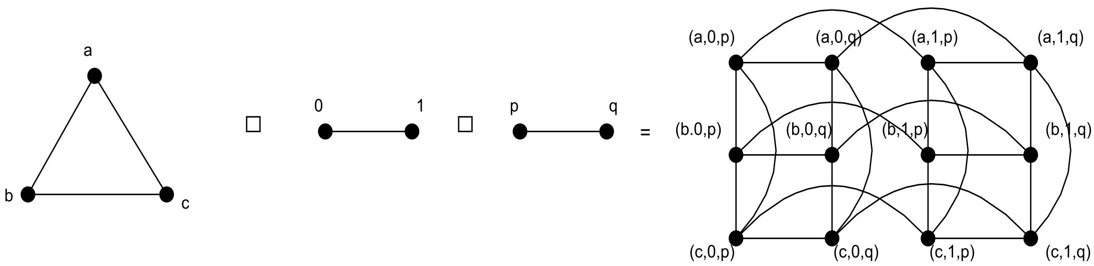

Figure 1. Cartesian product of graphs

Definition 2.13 ([15])**.** If $L_1 = (U(L_1), J(L_1)), L_2 = (U(L_2), J(L_2)), \ldots, L_k = (U(L_k), J(L_k))$ are k unweighted undirected graphs, then their Strong product is the graph $L_1 \boxtimes L_2 \ldots \boxtimes L_k$ with vertex set $\{(x_1, x_2, \ldots, x_k)| \; xi \in U(Li)\}$, and for which two vertices $(x_1, x_2, \ldots, x_k)$, $(x'_1, x'_2, \ldots, x'_k)$ in $L_1 \boxtimes L_2 \ldots \boxtimes L_k$ with $x_i, x'_i \in U(Li)$ are adjacent whenever $x_i x'_i \in J(L_i)$ or $x_i = x'_i$ for each index $1 \leq i \leq k$.

Example 2.14. The strong product of K_3 with vertices a, b, c, K_2 with vertices $0, 1$ and p, q is

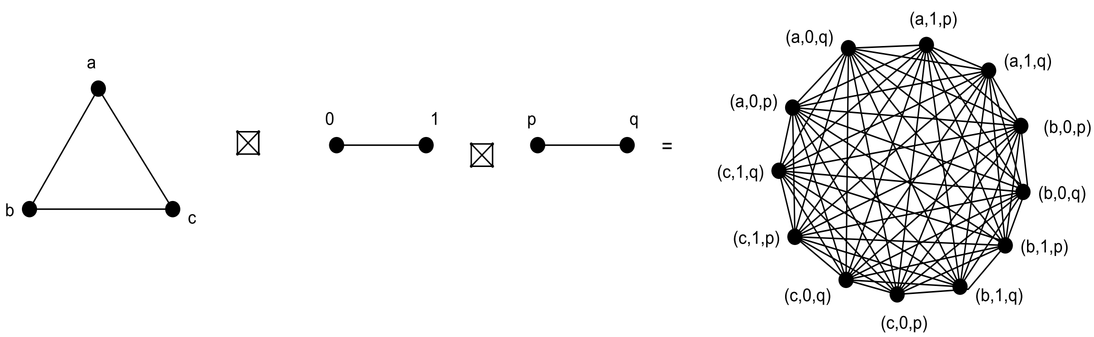

Figure 2. Strong product of graphs

In the realm of graph theory, two graphs L_1 and L_2 are equal in other words L_1 and L_2 are isomorphic if there exist a bijection from vertex set of L_1 to vertex set of L_2 that preserves adjacence and nonadjacence. The work of Mehmet E Aktas, Esra Akbas, and Ahmed El Fatmaoui [2] has shown that because direct comparison can be relatively difficult, graphs can be effectively compared through topological filtrations. For unweighted graph we are considering clique complex filtration.

Definition 2.15 ([2]). The clique complex $\mathfrak{CL}(L)$ of an undirected graph $L = (U, J)$ is a simplicial complex where vertices of L are its vertices and each k-clique, i.e. the complex sub graphs with k vertices, in L corresponds to a $(k-1)$-simplex in $\mathfrak{CL}(L)$.

An alternative approach used for visualizing persistence homology is the use of persistence diagrams. In this method, the intervals in the barcode are depicted as points on the extended real plane $\mathbb{R}^2$. If so, we can define a distance function using persistence diagrams to compare two topological structures. One of these distance metrics is the Wasserstein distance. Here we make effective use of the distance function in the comparative study of network products and their individual networks.

Definition 2.16 ([10]). Let P_H and Q_H are two persistence diagrams of filtrations of topological spaces M and N. Then the p-th Wasserstein distance between P_H and Q_H is defined as

$$d_{wp}(P_H, Q_H) = \left[\inf_{\Gamma} \sum_{x \in P_H} \|x - \Gamma(x)\|_{\infty}{}^{p} \right]^{1/p}$$

where Γ ranges over all matchings from P_H to Q_H and $\|p-q\|_\infty = \max(|p_1 - q_1|, |p_2 - q_2|)$ for $p = (p_1, p_2), q = (q_1, q_2) \in \mathbb{R}^2$ with $\|\infty - \infty\| = 0$.

3. New Results

The Cartesian product of topological spaces is a fundamental concept in mathematics with diverse applications. Its study enables mathematicians and scientists to understand the structure and behavior of multidimensional systems, paving the way for deeper insights and discoveries in various fields. While considering the Cartesian product of two topological spaces, it becomes clear that the Wasserstein distance between two Cartesian products relates to the Wasserstein distances between the persistence diagrams of the individual topological spaces. If the individual spaces are homeomorphic, which means they have the same topological structure, the Wasserstein distance between their persistence diagrams will be around zero. Based on this observation, we can conclude that the Cartesian product is equivalent, only if related Wasserstein distance is close to zero. This relationship is confirmed by the next theorem and corresponding corollary by defining Wasserstein distance between cartesian product of topological spaces, which further reinforces the relation between the Wasserstein distances of particular spaces and the Cartesian product.

Here for any topological spaces M_1, M_2, N_1, N_2, $bcd_i(M_1)$ be the i-dim persistence barcode of topological space M_1, $bcd_i(N_1)$ be the i-dim persistence barcode of topological space N_1, $bcd_j(M_2)$ be the j-dim persistence barcode of topological space M_2 and $bcd_j(N_2)$ be the i-dim persistence barcode of topological space N_2. For any $i + j = k$, consider

$$\max\{|a_{il} - c_{ir}|, |b_{il} - d_{ir}|, ||a'_{js} - c'_{jt}|, |b'_{js} - d'_{jt}|; r, l = 1, 2, \ldots, m_i, s, t = 1, 2, \ldots, m_j, \\ i, j = 0, 1, 2, \ldots, k, i + j = k\}$$

for any interval $(a_{il}, b_{il}) \in bcd_i(M_1)$,$(c_{ir}, d_{ir}) \in bcd_i(N_1)$, $(a'_{js}, b'_{js}) \in bcd_j(M_2)$ and (c'_{jt}, d'_{jt}) $\in bcd_j(N_2), r, l = 1, 2, \ldots, m_i, s, t = 1, 2, \ldots, m_j, m_i, m_j \in \mathbb{N}$.

Let,

$$d_{kmax}((M_1,N_1),(M_2,N_2)) = \max\{|a_{il}-c_{ir}|,|b_{il}-d_{ir}|,||a'_{js}-c'_{jt}|,|b'_{js}-d'_{jt}|; r,l=1,2,\dots,m_i,\\ s,t=1,2,\dots,m_j, i,j=0,1,2,\dots,k, i+j=k\}.$$

In the case of cartesian product of n topological spaces $M_1,M_2,\dots,M_n$ and $N_1,N_2,\dots,N_n$, we are representing this maximum value as $d_{kmax}((M_1,N_1),(M_2,N_2),(M_3,N_3),\dots,(M_n,N_n))$.

Definition 3.1. Let $P^k_{M_1\times M_2\dots\times M_n}$ and $Q^k_{N_1\times N_2\dots\times N_n}$ are two persistence diagrams of filtrations of Cartesian product of topological spaces $M_1,M_2,\dots,M_n$ and $N_1,N_2,\dots,N_n$. Then the p-th Wasserstein distance between $P^k_{M_1\times M_2\dots\times M_n}$ and $Q^k_{N_1\times N_2\dots\times N_n}$ is defined as

$$d_{wp}(P^k_{M_1\times M_2\times\dots\times M_n},Q^k_{N_1\times N_2\times\dots\times N_n}) = \left[\inf_{\Gamma}\sum_{x\in P^k_{M_1\times M_2\times\dots\times M_n}}\|x-\Gamma(x)\|_\infty{}^p\right]^{1/p}$$

where Γ ranges over all matchings from $P^k_{M_1\times M_2\times\dots\times M_n}$ to $Q^k_{N_1\times N_2\times\dots\times N_n}$ and $\|p-q\|_\infty = \max(|p_1-q_1|,|p_2-q_2|)$ for $p=(p_1,p_2), q=(q_1,q_2)\in\mathbb{R}^2$ with $\|\infty-\infty\|=0$.

Theorem 3.2. *Let $\mathfrak{M}_1,\mathfrak{M}_2,\mathfrak{N}_1,\mathfrak{N}_2$ be the filtrations of topological spaces M_1,M_2,N_1,N_2 respectively. Also let $P^k_{M_1\times M_2}$ be the k-dimensional persistence diagram of Cartesian product $M_1\times M_2$ and let $Q^k_{N_1\times N_2}$ be the k-dimensional persistence diagram of Cartesian product $N_1\times N_2$. Then the p-th Wasserstein distance $d_{wp}(P^k_{M_1\times M_2},Q^k_{N_1\times N_2}) \leq (\Upsilon_k)^{1/p} d_{kmax}((M_1,N_1),(M_2,N_2))$ where $\Upsilon_k = \sum_{i,j=0,i+j=k}^{k} m_i m_j$, m_i is the number of elements in $bcd_i(\mathfrak{M}_1)$ or $bcd_i(\mathfrak{N}_1)$ and m_j is the number of elements in $bcd_j(\mathfrak{M}_2)$ or $bcd_i(\mathfrak{N}_2)$.*

Proof. Let $P^i_{M_1}$ be the i-th persistent diagram of M_1, $Q^i_{N_1}$ be the i-th persistent diagram of N_1, $P^j_{M_2}$ be the j-th persistent diagram of M_2 and $Q^j_{N_2}$ be the j-th persistent diagram of N_2. If $d_{wp}(P^i_{M_1},Q^i_{N_1})$ be the p-th Wasserstein distance between $P^i_{M_1}$ and $Q^i_{N_1}$ and $d_{wp}(P^j_{M_2},Q^j_{N_2})$ be the p-th Wasserstein distance between $P^j_{M_2}$ and $Q^j_{N_2}$,then there will be bijections Γ_i and Γ_j which are best matching maps elements of $P^i_{M_1}$ to $Q^i_{N_1}$ and $P^j_{M_2}$ to $Q^j_{N_2}$ respectively. We have any point $(\mathfrak{m},\mathfrak{m}')\in P^k_{M_1\times M_2}$ and $(\mathfrak{n},\mathfrak{n}')\in Q^k_{N_1\times N_2}$ will represent intervals in $bcd_k(\mathfrak{M}_1\times\mathfrak{M}_2)$ and $bcd_k(\mathfrak{N}_1\times\mathfrak{N}_2)$. For some $i,j=1,2,\dots,k$ with $i+j=k$, let

$$\begin{aligned} bcd_i(\mathfrak{M}_1) &= \{(a_{i1},b_{i1}),(a_{i2},b_{i2}),\dots,(a_{imi},b_{imi})\}\\ bcd_j(\mathfrak{M}_2) &= \{(a'_{j1},b'_{j1}),(a'_{j2},b'_{j2}),\dots,(a'_{jmj},b'_{jmj})\}\\ bcd_i(\mathfrak{N}_1) &= \{(c_{i1},d_{i1}),(c_{i2},d_{i2}),\dots,(c_{imi},d_{imi})\} \end{aligned}$$

and

$$bcd_j(\mathfrak{N}_2) = \{(c'_{j1},d'_{j1}),(c'_{j2},d'_{j2}),\dots,(c'_{jmj},d'_{jmj}).$$

Now for any $(\mathfrak{m},\mathfrak{m}')\in P^k_{M_1\times M_2}$ and $(\mathfrak{n},\mathfrak{n}')\in Q^k_{N_1\times N_2}$,

$$(\mathfrak{m},\mathfrak{m}') = (a_{il},b_{il})\cap(a'_{js},b'_{js}) \text{ for some } l=1,2,\dots,m_i, s=1,2,\dots,m_j$$

and $(\mathfrak{n},\mathfrak{n}')=(c_{ir},d_{ir})\cap(c'_{jt},d'_{jt})$ for some $r=1,2,\dots,m_{i,}$ $t=1,2,\dots,m_j$. For a particular i,j with $i+j=k$, we have

$$\max\{|\mathfrak{m}-\mathfrak{n}|,|\mathfrak{m}'-\mathfrak{n}'|\}\leq\max\{|a_{il}-c_{ir}|,|b_{il}-d_{ir}|,|a'_{js}-c'_{jt}|,|b'_{js}-d'_{jt}|\}.$$

Then for any element $\mathfrak{p}' = (\mathfrak{p}_1', \mathfrak{p}_2')$ in $P^k_{M_1\times M_2}$ and $\mathfrak{q}' = (\mathfrak{q}_1', \mathfrak{q}_2')$ in $Q^k_{N_1\times N_2}$, we have

$$|\mathfrak{p}' - \mathfrak{q}'\|^p_\infty \leq \max\{|a_{il} - c_{ir}|^p, |b_{il} - d_{ir}|^p, ||a'_{js} - c'_{jt}|^p, |b'_{js} - d'_{jt}|^p; \\ r, l = 1, 2, \dots, m_i,\ s, t = 1, 2, \dots, m_j,\ i, j = 0, 1, 2, \dots, k,\ i + j = k\}.$$

Consider the bijection $\Gamma : P^k_{M_1\times M_2} \to Q^k_{N_1\times N_2}$ which maps the element $(\mathfrak{m}, \mathfrak{m}')$ to $(\mathfrak{n}, \mathfrak{n}')$ if and only if Γ_i maps (a_{il}, b_{il}) to (c_{ir}, d_{ir}) and Γ_j maps (a'_{js}, b'_{js}) to (c'_{jt}, d'_{jt}) then,

$$d_{wp}(P^k_{M_1\times M_2}, Q^k_{N_1\times N_2}) \leq (\Upsilon_k)^{1/p} d_{kmax}((M_1, N_1), (M_2, N_2))$$

where $\Upsilon_k = \sum\limits_{i,j=0, i+j=k}^{k} m_i m_j$, m_i is the number of elements in $bcd_i(\mathfrak{M}_1)$ or $bcd_i(\mathfrak{N}_1)$ and m_j is the number of elements in $bcd_j(\mathfrak{M}_2)$ or $bcd_i(\mathfrak{N}_2)$. □

Corollary 3.3. *Let $\mathfrak{M}_1, \mathfrak{M}_2, \dots, \mathfrak{M}_n$ be the filtration of topological spaces $M_1, M_2, \dots, M_n$ and $\mathfrak{N}_1, \mathfrak{N}_2, \dots, \mathfrak{N}_n$ be the filtration of topological spaces $N_1, N_2, \dots, N_n$. If $P^k_{M_1\times M_2\times\ldots\times M_n}$ be the k-dim persistence diagram of $bcd_k(\mathfrak{M}_1 \times \mathfrak{M}_2 \times \ldots \times \mathfrak{M}_n)$ and $Q^k_{N_1\times N_2\times\ldots\times N_n}$ be the k-dim persistence diagram of $bcd_k(\mathfrak{N}_1 \times \mathfrak{N}_2 \times \ldots \times \mathfrak{N}_n)$ then*

$$d_{wp}(P^k_{M_1\times M_2\times\ldots\times M_n}, Q^k_{N_1\times N_2\times\ldots\times N_n}) \leq (\Upsilon'_k)^{1/p} d_{kmax}((M_1, N_1), (M_2, N_2), \dots, (M_n, N_n))$$

where $\Upsilon'_k = \sum\limits_{i_j=0, i_1+i_2+\cdots+i_n=k}^{k} m_{i_1} m_{i_2}, \dots, m_{i_n}$ where mi_j be the number of elements in $bcd_{i_j}(\mathfrak{M}_j)$ or $bcd_{i_j}(\mathfrak{N}_j)$ for $j = 1, 2, \dots, n$.

Proof. For $j = 1, 2, \dots, n$ Let $d_{wp}(P^{i_j}_{M_j}, Q^{i_j}_{N_j})$ be the Wasserstein i_j-dim persistence diagram of M_j and N_j $i_1, i_2, \dots, i_n \in \{0, 1, 2, \dots, k\}, i_1 + i_2 + \cdots + i_n = k$. Then there exist bijections Γ_{i_j} which are best matchings between $P^{i_j}_{M_j}$ and $Q^{i_j}_{M_j}$. Let

$$\begin{aligned}
bcd_{i_1}(\mathfrak{M}_1) &= \{(a_{i_{1_1}}, b_{i_{1_1}}), (a_{i_{1_2}}, b_{i_{1_2}}), \dots, (a_{i_{1m_{i_1}}}, b_{i_{1m_{i_1}}})\} \\
bcd_{i_2}(\mathfrak{M}_2) &= \{(a_{i_{2_1}}, b_{i_{2_1}}), (a_{i_{2_2}}, b_{i_{2_2}}), \dots, (a_{i_{2m_{i_2}}}, b_{i_{2mi_2}})\} \\
bcd_{i_n}(\mathfrak{M}_n) &= \{(a_{i_{n_1}}, b_{i_{n_1}}), (a_{i_{n_2}}, b_{i_{n_2}}), \dots, (a_{i_{nm_{in}}}, b_{i_{1m_{in}}})\} \\
bcd_{i_1}(\mathfrak{N}_1) &= \{(c_{i_{1_1}}, d_{i_{1_1}}), (c_{i_{1_2}}, d_{i_{1_2}}), \dots, (c_{i_{1m_{i_1}}}, d_{i_{1m_{i_1}}})\} \\
bcd_{i_2}(\mathfrak{N}_2) &= \{(c_{i_{2_1}}, d_{i_{2_1}}), (c_{i_{2_2}}, d_{i_{2_2}}), \dots, (c_{i_{2m_{i_2}}}, d_{i_{2m_{i_2}}})\} \\
bcd_{i_n}(\mathfrak{N}_n) &= \{(c_{i_{n_1}}, d_{i_{n_1}}), (c_{i_{n_2}}, d_{i_{n_2}}), \dots, (c_{i_{nm_{in}}}, d_{i_{nm_{in}}})\}, m_{i_1}, m_{i_2}, \dots, m_{i_n} \in \mathbb{N}
\end{aligned}$$

For any interval $(m_n, m'_n) \in bcd_k(\mathfrak{M}_1 \times \mathfrak{M}_2 \times \ldots \times \mathfrak{M}_\mathfrak{n})$ and $(n_n, n'_n) \in bcd_k(\mathfrak{N}_1 \times \mathfrak{N}_2 \times \ldots \times \mathfrak{N}_\mathfrak{n})$ can be written as

$$(m_n, m'_n) = (a_{i_{1_{s_1}}}, b_{i_{1_{s_1}}}) \cap (a_{i_{2_{s_2}}}, b_{i_{2_{s_2}}}) \cap \cdots \cap (a_{i_{n_{s_n}}}, b_{i_{n_{s_n}}})$$

and

$$(n_n, n'_n) = (c_{i_{1_{s_1}}}, d_{i_{1_{s_1}}}) \cap (c_{i_{2_{s_2}}}, d_{i_{2_{s_2}}}) \cap \cdots \cap (c_{i_{n_{s_n}}}, d_{i_{n_{s_n}}})$$

for some $s_1, r_1 = 1, 2, \dots, m_{i_1},\ s_2, r_2 = 1, 2, \dots, m_{i_2}, \dots,\ s_n, r_n = 1, 2, \dots, m_{i_n}$.
Consider the bijection $\Gamma' : P^k_{M_1\times M_2\times\ldots\times M_n} \to Q^k_{N_1\times N_2\times\ldots\times N_n}$ which maps (m_n, m'_n) to (n_n, n'_n) if and only if Γ_{i_j} maps $(a_{i_{j_{s_j}}}, b_{i_{j_{s_j}}})$ on to $(c_{i_{j_{r_j}}}, d_{i_{j_{r_j}}})$ for all $j = 1, 2, \dots, n$.
We have,

$$\max\{|m_n - n_n|, |m'_n - n'_n|\} \leq \max\{|a_{i_{1_{s_1}}} - c_{i_{1_{r_1}}}|, |a_{i_{2_{s_2}}} - c_{i_{2_{r_2}}}|, \cdots, |a_{i_{n_{s_n}}} - c_{i_{n_{r_n}}}|, \\ |b_{i_{1_{s_1}}} - d_{i_{1_{r_1}}}|, |b_{i_{2_{s_2}}} - d_{i_{2_{r_2}}}|, \cdots, |b_{i_{n_{s_n}}} - d_{i_{n_{r_n}}}|\}$$

Now for any element $\mathfrak{p}_{\mathfrak{k}} = (\mathfrak{p}_{\mathfrak{k}_1}, \mathfrak{p}_{\mathfrak{k}_2}) \in P^k_{M_1\times M_2\times\ldots\times M_n}$ and $\mathfrak{q}_{\mathfrak{k}} = (\mathfrak{q}_{\mathfrak{k}_1}, \mathfrak{q}_{\mathfrak{k}_2}) \in Q^k_{M_1\times M_2\times\cdots\times M_n}$,

$$\max\{|\mathfrak{p}_{\mathfrak{k}_1} - \mathfrak{q}_{\mathfrak{k}_1}|^p, |\mathfrak{p}_{\mathfrak{k}_2} - \mathfrak{q}_{\mathfrak{k}_2}|^p\} \leq \max\{|a_{i_{1s_1}} - c_{i_{1r_1}}|, |a_{i_{2s_2}} - c_{i_{2r_2}}|, \ldots, |a_{i_{ns_n}} - c_{i_{nr_n}}|,$$
$$|b_{i_{1s_1}} - d_{i_{1r_1}}|, |b_{i_{2s_2}} - d_{i_{2r_2}}|, \ldots, |b_{i_{ns_n}} - d_{i_{nr_n}}|,$$

$s_1, r_1 = 1, 2, \ldots, m_{i_1},\ s_2, r_2 = 1, 2, \ldots, m_{i_2}, \ldots,\ s_n, r_n = 1, 2 \ldots m_{i_n}, i_1 + i_2 + \ldots + i_n = k,$ $i_j = 0, 1 \ldots k, j = 1, 2, \ldots, n\}$
So

$$d_{wp}(P^k_{M_1\times M_2\times\cdots\times M_n}, Q^k_{N_1\times N_2\times\cdots\times N_n}) \leq (\Upsilon'_k)^{1/p} d_{kmax}((M_1, N_1), (M_2, N_2), \cdots, (M_n, N_n))$$

where $\Upsilon'_k = \sum_{i_j=0, i_1+i_2+\cdots+i_n=k}^{k} m_{i_1} m_{i_2}, \ldots, m_{i_n}$ where mi_j is the number of elements in $bcd_{i_j}(\mathfrak{M}_j)$ or $bcd_{i_j}(\mathfrak{N}_j)$ for $j = 1, 2, \ldots, n$. □

In the context of Graph theory, an n-clique is defined as a graph with n vertices, where each vertex is connected to every other vertex in the clique. It is important to note that in our study, all the graphs considered are connected and simple, meaning there are no self-loops or multiple edges between the same pair of vertices.

The dimension of $H_0(L)$ in a network context represents the number of disconnected subgraphs or connected components in the graph. $H_1(L)$ indicates the presence of 1-dimensional holes or cycles within the graph. As graphs do not contain two or higher-dimensional simplices, the higher-level homology dimensions are always zero.

When comparing two graphs, it is customary to consider graphs that have the same number of vertices. This ensures a fair comparison. Consequently, the number of elements in the barcode of dimension zero, which corresponds to the connected components, will be the same for these graphs. Additionally, it is worth mentioning that our analysis is focused solely on finite graphs.

The Cartesian product of graphs is a powerful concept with numerous applications in different fields. It provides a way to explore the interactions and relationships between different entities in a systematic manner, making it a valuable tool in graph theory and related disciplines. Also it is used across disciplines to model interactions, solve complex problems, and gain insights into different systems and their behaviors. Indeed, when it comes to graphs, the analysis can be simplified by focusing on the 0-dimensional persistence diagram and 1-dimensional persistence diagram. These persistence diagrams are obtained from various filtrations applied to the graphs, such as the clique-complex filtration [17] and the Vietoris-Rips filtration [18]. The 0-dimensional persistent homology captures the information about connected components in both topological spaces and graphs. Whether we consider a product of topological spaces or a Cartesian product of graphs, the 0-dimensional barcodes will be the same. This is because the 0-dimensional persistent homology solely focuses on the presence and connectivity of connected components, regardless of the underlying structure. In the case of the Cartesian product of unweighted graphs, we can define a clique complex filtration to extract the corresponding barcodes and persistence diagrams.

Definition 3.4. Let $L_1 = (U_1, J_1), L_2 = (U_2, J_2)$ be two unweighted undirected graphs where U_1, U_2 are the set of vertices in L_1, L_2 with $|U_1| = |U_2| = n$. Consider the Cartesian product $L_1 \square L_2$ of L_1 and L_2. The clique complex filtration $\mathcal{L}_{c1} \square \mathcal{L}_{c2}$ of $L_1 \square L_2$ is defined as

$\mathfrak{CL}_0(L_1 \square L_2) \to \mathfrak{CL}_1(L_1 \square L_2) \to \mathfrak{CL}_2(L_1 \square L_2) \to \ldots \to \mathfrak{CL}_n(L_1 \square L_2) = \mathfrak{CL}_(L_1 \square L_2)$ such that $\mathfrak{CL}_0(L_1 \square L_2) \subset \mathfrak{CL}_1(L_1 \square L_2) \subset \mathfrak{CL}_2(L_1 \square L_2) \subset \ldots \subset \mathfrak{CL}_n(L_1 \square L_2) = \mathfrak{CL}(L_1 \square L_2)$ where $\mathfrak{CL}_i(L_1 \square L_2) = \sum_{j=1}^{i} \mathcal{S}_{cj}$ which is the i-th filtration and $\mathcal{S}_{cj}$ is the j-th skeleton of the clique complex. Here the filtration is based on the threshold ς. So adding the vertices at $\varsigma = 0$, edges at $\varsigma = 1$, triangles at $\varsigma = 2$ and so on.

The systematic study of Cartesian products allows for a deeper comprehension of the interconnectedness between different components within a system. Here, analyzing the topological aspects of Cartesian products of graphs involves detecting holes or voids within the structure. These voids can be classified as either zero-dimensional holes or one-dimensional holes in the case of graphs. specifically, when evaluating the Cartesian products of graphs based on distance, it is important to take their dimensions into account. The following two theorems provide a comparison of persistent diagrams in each dimension. Both of these dimensions are essential for the comparison of graphs.

Theorem 3.5. *If $P^0_{L_1}, P^0_{L_2}, Q^0_{L_3}, Q^0_{L_4}$ are 0-dimensional persistence diagrams of L_1, L_2, L_3 and L_4 respectively. If $P^0_{L_1\square L_2}$ be the 0-dimensional persistence diagram of $L_1\square L_2$ and $Q^0_{L_3\square L_4}$ be the 0-dimensional persistence diagram of $L_3\square L_4$, then*

$$d_{wp}(P^0_{L_1\square L_2}, Q^0_{L_3\square L_4}) \leq (mm')^{1/p} d_{0max}((L_1, L_3), (L_2, L_4))$$

where m is the number of elements in $P^0_{L_1}$ or $P^0_{L_3}$ and m' is the number of elements in $P^0_{L_2}$ or $P^0_{L_4}$.

Proof. Consider the Cartesian product $L_1\square L_2$ of the graphs L_1 and L_2 with clique complex filtration $\mathcal{L}_{c1}\square\mathcal{L}_{c2}$ and 0-dimensional persistent diagram $P^0_{L_1\square L_2}$ and Cartesian product $L_3\square L_4$ of the graphs L_3 and L_4 with clique complex filtration $\mathcal{L}_{c3}\square\mathcal{L}_{c4}$ and 0-dimensional persistent diagram $P^0_{L_1\square L_2}$. While considering the cartesian product of two topological spaces and cartesian product of two graphs 0- dimensional persistence barcode will contain same elements. Then, by theorem 3.2, $d_{wp}(P^0_{L_1\square L_2}, Q^0_{L_3\square L_4}) \leq (mm')^{1/p} d_{0max}((L_1, L_3), (L_2, L_4))$ where m is the number of elements in $P^0_{L_1}$ or $P^0_{L_3}$ and m' is the number of elements in $P^0_{L_2}$ or $P^0_{L_4}$. □

Theorem 3.6. *Let $P^1_{L_1}, P^1_{L_2}, Q^1_{L_3}, Q^1_{L_4}$ are 1-dimensional and $P^0_{L_1}, P^0_{L_2}, Q^0_{L_3}, Q^0_{L_4}$ are 0- dimensional persistence diagrams of L_1, L_2, L_3, L_4 respectively. If $P^1_{L_1\square L_2}$ be the 1-dimensional persistence diagram of $L_1\square L_2$ and $Q^1_{L_3\square L_4}$ be the 1-dimensional persistence diagram of $L_3\square L_4$. Then,*

$$d_{wp}(P^1_{L_1\square L_2}, Q^1_{L_3\square L_4}) \leq n(d_{wp}(P^1_{L_1}, Q^1_{L_3}) + d_{wp}(P^1_{L_2}, Q^1_{L_4})) + (mm')^{1/p} d_{0max}((L_1, L_3), (L_2, L_4))$$

where n is the number of vertices in each graph.

Proof. Let $d_{wp}(P^0_{L_1}, Q^0_{L_3})$ and $d_{wp}(P^0_{L_2}, Q^0_{L_4})$ are the Wassestein distance between 0- dimensional persistence diagrams $P^1_{L_1}$,$Q^0_{L_3}$ and $P^0_{L_2}$,$Q^0_{L_4}$ respectively. Then there exist bijection $\Lambda_0 : P^0_{L_1} \to Q^0_{L_3}$ and $\Lambda'_0 : P^0_{L_2} \to Q^0_{L_4}$ which are best matchings. Also let $d_{wp}(P^1_{L_1}, Q^1_{L_3})$ and $d_{wp}(P^1_{L_2}, Q^1_{L_4})$ are Wasserstein distance between $P^1_{L_1}$,$Q^1_{L_3}$ and $P^1_{L_2}$,$Q^1_{L_4}$ respectively. Then there exist bijection $\Lambda_1 : P^1_{L_1} \to Q^1_{L_3}$ and $\Lambda'_1 : P^1_{L_2} \to Q^1_{L_4}$ which are best matchings.

If $\{l_{11}, l_{12}, \ldots, l_{1n}\}$ and $\{l'_{11}, l'_{12}, \ldots, l'_{1n}\}$ are the vertex sets ofL_1 and L_2 then for any l_{2j} in L_2, we can define the L_1 fibre in $L_1\square L_2$ as

$$L_1 l_{2j} = \{(l_{1i}, l_{2j}) | l_{1i} \in L_1, i = 1, 2, \ldots, n\},\ j = 1, 2, \ldots, n$$

and for any l_{1j} in L_1, L_2 fibre in $L_1\square L_2$ is defined as

$$l_{1j} L_2 = \{(l_{1j}, l_{2i}) | l_{2i} \in L_2, i = 1, 2, \ldots, n\},\ j = 1, 2, \ldots, n.$$

Clearly L_1 fibre is isomorphic to L_1 and L_2 fibre is isomorphic to L_2. So there are n copies of L_1 and n copies of L_2 will be there in $L_1\square L_2$. Similarly there are n copies of L_3 and n copies of L_4 will be there in $L_3\square L_4$. Some other loops also there which are formed by the edges of L_1 and L_2 in $L_1\square L_2$ and L_3 and L_4 in $L_3\square L_4$. Intervals representing these types of 1-dimensional holes will be of the form $(\theta, \infty), 0 \leq \theta < \infty$ with $\theta = \max\{\theta_1, \theta_2\}$ where

$(0, \theta_1)$ and $(0, \theta_2)$ are the intervals in $bcd_0(\mathcal{L}_{c1}), bcd_0(\mathcal{L}_{c2})$ and (θ', ∞) with $\theta' = \max\{\theta'_1, \theta'_2\}$ where $(0, \theta'_1)$ and $(0, \theta'_2)$ are the intervals in $bcd_0(\mathcal{L}_{c3}), bcd_0(\mathcal{L}_{c4})$ respectively. Since we are taking the graphs with same number of vertices, number of these type of will be the same.

Now consider the bijection $\Lambda : P^1_{L_1 \square L_2} \to Q^1_{L_3 \square L_4}$ which maps n copies of L_1 fibre loops to n copies of L_3 fibre loops as Λ_1 maps and n copies of L_2 fibre loops to n copies of L_2 fibre loops as Λ_2 maps. The loops of the form (θ, ∞) maps on to (θ', ∞) if and only of Λ_0 maps $(0, \theta_1)$ to $(0, \theta'_1)$ and Λ'_0 maps $(0, \theta_2)$ to $(0, \theta'_2)$. Then we can conclude that $d_{wp}(P^1_{L_1 \square L_2}, Q^1_{L_3 \square L_4}) \leq n(d_{wp}(P^1_{L_1}, Q^1_{L_3}) + d_{wp}(P^1_{L_2}, Q^1_{L_4})) + (mm')^{1/p} d_{0max}((L_1, L_3), (L_2, L_4))$, where n is the number of vertices in each graph. □

The subsequent two theorems point out that it is possible to compare the Cartesian products of graphs by with the help of their individual spaces through distance, even when considering the Cartesian product of more than just two, but rather a finite number of graphs. This can be seen as an extension of the preceding two theorems. Filtration is essential in this case yet again. Thus, we proceed to compare persistence diagrams by introducing clique filtration on the Cartesian product of finitely many graphs.

Definition 3.7. Let $L_1 = (U_1, J_1), L_2 = (U_2, J_2), \ldots, L_n = (U_n, J_n)$ be n unweighted undirected graphs with $|U_1| = |U_2|, \ldots, = |U_n| = n$, Consider the Cartesian product $L_1 \square L_2 \square \ldots \square L_n$ of $L_1, L_2, \ldots, L_n$. The clique complex filtration $\mathcal{L}_{c1} \square \mathcal{L}_{c2} \square \ldots \square \mathcal{L}_{cn}$ of $L_1 \square L_2 \square \ldots \square L_n$ is defined as

$$\mathfrak{CL}_0(L_1 \square L_2 \square \ldots \square L_n) \to \mathfrak{CL}_1(L_1 \square L_2 \square \ldots \square L_n) \to \mathfrak{CL}_2(L_1 \square L_2 \square \ldots \square L_n) \to \ldots \to \mathfrak{CL}_n(L_1 \square L_2 \square \ldots \square L_n) = \mathfrak{CL}(L_1 \square L_2 \square \ldots \square L_n)$$

such that

$$\mathfrak{CL}_0(L_1 \square L_2 \square \ldots \square L_n) \subset \mathfrak{CL}_1(L_1 \square L_2 \square \ldots \square L_n) \subset \mathfrak{CL}_2(L_1 \square L_2 \square \ldots \square L_n) \subset \ldots \subset \mathfrak{CL}_n(L_1 \square L_2 \square \ldots \square L_n) = \mathfrak{CL}(L_1 \square L_2 \square \ldots \square L_n)$$

where $\mathfrak{CL}_i(L_1 \square L_2 \square \ldots \square L_n) = \sum_{j=1}^{i} \mathcal{S}'_{cj}$ where $\mathcal{S}'_{cj}$ is the j-th skeleton of the clique complex. Here the filtration is based on the threshold ν. So adding the vertices at $\nu = 0$, edges at$\nu = 1$, triangles at $\nu = 2$ and so on.

Theorem 3.8. *If $P^0_{L_1}, P^0_{L_2}, \ldots, P^0_{L_n}$ and $Q^0_{L'_1}, Q^0_{L'_2}, \ldots, Q^0_{L'_n}$ are 0-dimensional persistence diagrams of $L_1, L_2, \ldots, L_n$ and $L'_1, L'_2, \ldots, L'_n$ respectively. If $P^0_{L_1 \square L_2 \square \cdots \square L_n}$ be the 0-dim persistence diagram of $L_1 \square L_2 \square \cdots \square L_n$ and $Q^0_{L'_1 \square L'_2 \square \cdots \square L'_n}$ be the 0-dim persistence diagram of $L'_1 \square L'_2 \square \cdots \square L'_n$ then, the p-Wassestein distance*

$$d_{wp}(P^0_{L_1 \square L_2 \square \cdots \square L_n}, Q^0_{L'_1 \square L'_2 \square \cdots \square L'_n}) \leq (\sum_{i,j=1}^{n} (m_i m_j))^{1/p} d_{0max}((L_1, L'_1), (L_2, L'_2), \cdots, (L_n, L'_n))$$

where m_i is the number of elements in $P^0_{L_i}$ or $Q^0_{L'_i}$ and m_j is the number of elements in $P^0_{L_j}$ or $Q^0_{Lj'}$.

Proof. Let $P^0_{L_1 \square L_2 \square \cdots \square L_n}$,$Q^0_{L'_1 \square L'_2 \square \cdots \square L'_n}$ are 0- dimensional persistence diagrams of $L_1 \square L_2 \square \cdots \square L_n$ and $L'_1 \square L'_2 \square \cdots \square L'_n$ respectively. The 0-dim persistence diagram of Cartesian product of graphs will be the same as the Cartesian product of topological spaces since it represents the connected components. Then by theorem 3.2

$$d_{wp}(P^0_{L_1 \square L_2 \square \cdots \square L_n}, Q^0_{L'_1 \square L'_2 \square \cdots \square L'_n}) \leq (\sum_{i,j=1}^{n} (m_i m_j))^{1/p} d_{0max}((L_1, L'_1), (L_2, L'_2), \ldots, (L_n, L'_n))$$

where m_i is the number of elements in $P^0_{L_i}$ or $Q^0_{L'_i}$ and m_j is the number of elements in $P^0_{L_j}$ or $Q^0_{L'_j}$. □

Theorem 3.9. *If $P^1_{L_1}, P^1_{L_2}, \ldots, P^1_{L_n}$ and $Q^1_{L'_1}, Q^1_{L'_2}, \ldots, Q^1_{L'_n}$ are 1-dimensional persistence diagrams of $L_1, L_2, \ldots, L_n$ and $L'_1, L'_2, \ldots, L'_n$ respectively. If $P^1_{L_1\square L_2\square \ldots \square L_n}$ be the 1-dim persistence diagram of $L_1\square L_2\square \ldots \square L_n$ and $Q^1_{L'_1\square L'_2\square \ldots \square L'_n}$ be the 1-dim persistence diagram of $L'_1\square L'_2\square \ldots \square L'_n$ then*

$$d_{wp}(P^1_{L_1\square L_2\square \ldots \square L_n}, Q^1_{L'_1\square L'_2\square \ldots \square L'_n}) \leq n^{n-1}\{d_{wp}(P^1_{L_1}, Q^1_{L_1}) + d_{wp}(P^1_{L_2}, Q^1_{L_2}) + \cdots + d_{wp}(P^1_{L_n}, Q^1_{L_n})\} + (\sum_{i,j=1}^{n}(m_i m_j))^{1/p} d_{0max}((L_1, L'_1), (L_2, L'_2), \ldots, (L_n, L'_n))$$

where m_i is the number of elements in $P^0_{L_i}$ or $Q^0_{L'_i}$ and m_j is the number of elements in $P^0_{L_j}$ or $Q^0_{L'_j}$.

Proof. Let $P^1_{L_1}, P^1_{L_2}, \ldots, P^1_{L_n}$ and $Q^1_{L'_1}, Q^1_{L'_2}, \ldots, Q^1_{L'_n}$ are 1-dimensional persistence diagrams of $L_1, L_2, \ldots, L_n$ and $L'_1, L'_2, \ldots, L'_n$ respectively. Also let $P^0_{L_1}, P^0_{L_2}, \ldots, P^0_{L_n}$ and $Q^0_{L'_1}, Q^0_{L'_2}, \ldots,$
$Q^0_{L'_n}$ are 0-dimensional persistence diagrams of $L_1, L_2, \ldots, L_n$ and $L'_1, L'_2, \ldots, L'_n$ respectively. Here we can find n^{n-1} copies of L_1 fibres, n^{n-1} copies of L_2fibres, $\ldots, n^{n-1}$ copies of L_n fibres in $L_1\square L_2\square \ldots \square L_n$. In similar way we can find n^{n-1} copies of L'_1 fibres, n^{n-1} copies of L'_2 fibres, $\ldots, n^{n-1}$ copies of L'_n fibres in $L'_1\square L'_2\square \ldots \square L'_n$. Also some more one dimensional holes(loops) will be there formed by the edges which represents the interval in 0 dimensional barcodes of the filtration of each L_i and L'_i which are of the form $(\upsilon, \infty), \upsilon = \max\{\upsilon_1, \upsilon_2, \ldots, \upsilon_n\}$ where $(0, \upsilon_1) \in bcd_0(L_1), (0, \upsilon_2) \in bcd_0(L_2), \ldots, (0, \upsilon_n) \in bcd_0(L_n)$ and $(\upsilon', \infty), \upsilon' = \max\{\upsilon'_1, \upsilon'_2, \ldots, \upsilon'_n\}$ where $(0, \upsilon'_1) \in bcd_0(L_1), (0, \upsilon'_2) \in bcd_0(L_2), \ldots, (0, \upsilon'_n) \in bcd_0(L_n)$ respectively. If we are defining the bijection as per theorem 3.6,

$$d_{wp}(P^1_{L_1\square L_2\square \ldots \square L_n}, Q^1_{L'_1\square L'_2\square \ldots \square L'_n}) \leq n^{n-1}\{d_{wp}(P^1_{L_1}, Q^1_{L_1}) + d_{wp}(P^1_{L_2}, Q^1_{L_2}) + \cdots + d_{wp}(P^1_{L_n}, Q^1_{L_n})\} + (\sum_{i,j=1}^{n}(m_i m_j))^{1/p} d_{0max}((L_1, L'_1), (L_2, L'_2), \ldots, (L_n, L'_n))$$

where m_i be the number of elements in $P^0_{L_i}$ or $Q^0_{L'_i}$ and m_j be the number of elements in $P^0_{L_j}$ or $Q^0_{L'_j}$. □

The strong product of graphs is a vital concept in graph theory. It plays a crucial role in understanding graph isomorphism and connectivity patterns. Its versatility in modeling real-world systems makes it a powerful tool, providing insights into complex relationships and aiding problem-solving across diverse domains. Therefore, comparing strong product of graphs using wasserstein distances is certainly a significant step forward in entire graph theory. In the case of the strong product of unweighted graphs, it is interesting to note that the 0-dimensional barcodes and persistence diagrams remain the same as those obtained from the Cartesian product of topological spaces. This means that the connectivity patterns and the number of connected components in the strong product can be directly related to the Cartesian product. By defining the clique complex filtration in the strong product of graphs, we can establish a relationship between the Wasserstein distance of the persistence diagrams of the strong product and the persistence diagrams of the individual graphs.

Definition 3.10. Let $L_1 = (U_1, J_1), L_2 = (U_2, J_2)$ be two unweighted undirected graphs with $|U_1| = |U_2| = n$, consider the strong product $L_1 \boxtimes L_2$ of L_1 and L_2. The clique complex filtration $\mathcal{L}_{c1} \boxtimes \mathcal{L}_{c2}$ of $L_1 \boxtimes L_2$ is defined as

$\mathfrak{CL}_0(L_1 \boxtimes L_2) \to \mathfrak{CL}_1(L_1 \boxtimes L_2 \to \mathfrak{CL}_0(L_1 \boxtimes L_2) \to \ldots \to \mathfrak{CL}_n(L_1 \boxtimes L_2) = \mathfrak{CL}(L_1 \boxtimes L_2)$ such that $\mathfrak{CL}_0(L_1 \boxtimes L_2) \subset \mathfrak{CL}_1(L_1 \boxtimes L_2 \subset \mathfrak{CL}_0(L_1 \boxtimes L_2) \subset \ldots \subset \mathfrak{CL}_n(L_1 \boxtimes L_2) = \mathfrak{CL}(L_1 \boxtimes L_2)$, $\mathfrak{CL}_i(L_1 \boxtimes L_2) = \sum_{j=1}^{i} \mathcal{S}_{bj}$ where $\mathcal{S}_{bj}$ is the j-th skeleton of the clique complex. Here the filtration is based on the threshold σ. So adding the vertices at $\sigma = 0$, edges at $\sigma = 1$, triangles at $\sigma = 2$ and so on.

When computing the strong product of graphs, it is also necessary to consider both 0-dimensional and 1-dimensional holes. This implies that we need to look into both 0-dimensional and 1-dimensional persistence diagrams. Such a comprehensive approach is essential for obtaining an accurate understanding of the graph's intricate structure. The subsequent two theorems yield valuable insights in comparing the strong product of graphs with individual graphs, yielding important outcomes in the study.

Theorem 3.11. *If $P^0_{L_1}, P^0_{L_2}, Q^0_{L_3}, Q^0_{L_4}$ are 0-dimensional persistence diagrams of L_1, L_2, L_3 and L_4 respectively. If $P^0_{L_1 \boxtimes L_2}$ be the 0-dimensional persistence diagram of $L_1 \boxtimes L_2$ and $Q^0_{L_3 \boxtimes L_4}$ be the 0-dimensional persistence diagram of $L_3 \boxtimes L_4$ then $d_{wp}(P^0_{L_1 \boxtimes L_2}, Q^0_{L_3 \boxtimes L_4}) \leq (mm')^{1/p} d_{0max}((L_1, L_3), (L_2, L_4))$ where m is the number of elements in $P^0_{L_1}$ or $P^0_{L_3}$ and m' is the number of elements in $P^0_{L_2}$ or $P^0_{L_4}$.*

Proof. Consider the Strong product $L_1 \boxtimes L_2$ of the graphs L_1 and L_2 with clique complex filtration $\mathcal{L}_{c1} \boxtimes \mathcal{L}_{c2}$ and 0-dimensional persistent diagram $P^0_{L_1 \boxtimes L_2}$ and Cartesian product $L_3 \boxtimes L_4$ of the graphs L_3 and L_4 with clique complex filtration $\mathcal{L}_{c3} \boxtimes \mathcal{L}_{c4}$ and 0- dimensional persistent diagram $P^0_{L_3 \boxtimes L_4}$. While considering the cartesian product of two topological spaces and cartesian product of two graphs 0 dimensional persistence barcode will contain same elements since it represents the connected components. Then by theorem 3.2

$$d_{wp}(P^0_{L_1 \boxtimes L_2}, Q^0_{L_3 \boxtimes L_4}) \leq (mm')^{1/p} d_{0max}((L_1, L_3), (L_2, L_4))$$

where m is the number of elements in $P^0_{L_1}$ or $P^0_{L_3}$ and m' is the number of elements in $P^0_{L_2}$ or $P^0_{L_4}$. □

Theorem 3.12. *Let $P^1_{L_1}, P^1_{L_2}, Q^1_{L_3}, Q^1_{L_4}$ are 1-dimensional persistence diagrams of L_1, L_2, L_3, L_4 respectively. If $P^1_{L_1 \boxtimes L_2}$ be the 1-dimensional persistence diagram of $L_1 \boxtimes L_2$ and $Q^1_{L_3 \boxtimes L_4}$ be the 1-dimensional persistence diagram of $L_3 \boxtimes L_4$, then,*

$$d_{wp}(P^1_{L_1 \boxtimes L_2}, Q^1_{L_3 \boxtimes L_4}) \leq n\{d_{wp}(P^1_{L_1}, Q^1_{L_3}) + d_{wp}(P^1_{L_2}, Q^1_{L_4})\}$$

where n is the number of vertices in each graph.

Proof. Let $d_{wp}(P^1_{L_1}, Q^1_{L_3})$ and $d_{wp}(P^1_{L_2}, Q^1_{L_4})$ are Wasserstein distance between $P^1_{L_1}$,$Q^1_{L_3}$ and $P^1_{L_2}$,$Q^1_{L_4}$ respectively. In the case of strong product if $\{l_{11}, l_{12}, \ldots, l_{1n}\}$ and $\{l'_{11}, l'_{12}, \ldots, l'_{1n}\}$ are the vertex sets ofL_1 and L_2 then for any l_{2j} in L_2, we can define the L_1 fibre in $L_1 \boxtimes L_2$ as

$$L_1 l_{2j} = \{(l_{1i}, l_{2j}) | l_{1i} \in L_1, i = 1, 2, \ldots, n\},\ j = 1, 2, \ldots, n$$

and for any l_{1j} in L_1, L_2 fibre in $L_1 \boxtimes L_2$ is defined as

$$l_{1j} L_2 = \{(l_{1j}, l_{2i}) | l_{2i} \in L_2, i = 1, 2, \ldots, n\},\ j = 1, 2, \ldots, n.$$

Clearly L_1 fibre is isomorphic to L_1 and L_2 fibre is isomorphic to L_2. So there are n copies of L_1 and n copies of L_2 will be there in $L_1 \boxtimes L_2$. Similarly there are n copies of L_3 and n

copies of L_4 will be there in $L_3 \boxtimes L_4$. In the case of strong product there wont be intervals represents one dimensional loops which are formed by the zero dimensional barcodes. So the only 1- dimensional loops will be the loops present in L_1, L_1 and L_3, L_4 n times. Now consider the bijection $\Psi : P^1_{L_1 \square L_2} \to Q^1_{L_3 \square L_4}$ which maps n copies of L_1 fibre loops to n copies of L_3 fibre loops as Λ_1 maps and n copies of L_2 fibre loops to n copies of L_4 fibre loops as Λ_2 maps. Then

$$d_{wp}(P^1_{L_1 \boxtimes L_2}, Q^1_{L_3 \boxtimes L_4}) \leq n\{d_{wp}(P^1_{L_1}, Q^1_{L_3}) + d_{wp}(P^1_{L_2}, Q^1_{L_4})\}.$$

□

The compelling aspect here lies in the versatility of strong product results, which transcend the limitation of pairing just two graphs. This intriguing extension allows for a broad application across a multitude of graphs, not confined to pairs but encompassing a finite assortment. This means the principles and insights gained from strong products can be extrapolated to complex networks involving multiple interconnected graphs. so in next two theorems we are extending our concepts to strong product of finitely many graphs.

Definition 3.13. Let $L_1 = (U_1, J_1), L_2 = (U_2, J_2), \ldots, L_n = (U_n, J_n)$ be n unweighted undirected graphs with $|U_1| = |U_2| \ldots = |U_n| = n$, Consider the Strong product $L_1 \boxtimes L_2 \boxtimes \ldots \boxtimes L_n$ of L_1 $L_2, \ldots, L_n$. The clique complex filtration $\mathcal{L}_{c1} \boxtimes \mathcal{L}_{c2} \boxtimes \ldots \boxtimes \mathcal{L}_{cn}$ of $L_1 \boxtimes L_2 \boxtimes \ldots \boxtimes L_n$ is defined as
$\mathfrak{CL}_0(L_1 \boxtimes L_2 \boxtimes \ldots \boxtimes L_n) \to \mathfrak{CL}_1(L_1 \boxtimes L_2 \boxtimes \ldots \boxtimes L_n) \to \mathfrak{CL}_2(L_1 \boxtimes L_2 \boxtimes \ldots \boxtimes L_n) \to, \ldots, \to \mathfrak{CL}_n(L_1 \boxtimes L_2 \boxtimes \ldots \boxtimes L_n) = \mathfrak{CL}(L_1 \boxtimes L_2 \boxtimes \ldots \boxtimes L_n)$ such that $\mathfrak{CL}_0(H_1 \boxtimes L_2 \boxtimes \ldots \boxtimes L_n) \subset \mathfrak{CL}_1(L_1 \boxtimes L_2 \boxtimes \ldots \boxtimes L_n) \subset \mathfrak{CL}_2(L_1 \boxtimes L_2 \boxtimes \ldots \boxtimes L_n) \subset, \ldots, \subset \mathfrak{CL}_n(L_1 \boxtimes L_2 \boxtimes \ldots \boxtimes L_n) = \mathfrak{CL}(L_1 \boxtimes L_2 \boxtimes \ldots \boxtimes L_n)$ where $\mathfrak{CL}_i(L_1 \boxtimes L_2 \boxtimes \ldots \boxtimes L_n) = \sum_{j=1}^{i} \mathcal{S}'_{bj}$ $\mathcal{S}'_{bj}$ is the j-th skeleton of the clique complex. Here the filtration is based on the threshold σ'. So adding the vertices at $\sigma' = 0$, edges at $\sigma' = 1$, triangles at $\sigma' = 2$ and so on.

Theorem 3.14. *If $P^0_{L_1}, P^0_{L_2}, \ldots, P^0_{L_n}$ and $Q^0_{L'_1}, Q^0_{L'_2}, \ldots, Q^0_{L'_n}$ are 0-dimensional persistence diagrams of $L_1, L_2, \ldots, L_n$ and $L'_1, L'_2, \ldots, L'_n$ respectively. If $P^0_{L_1 \boxtimes L_2 \boxtimes \ldots \boxtimes L_n}$ be the 0-dim persistence diagram of $L_1 \boxtimes L_2 \boxtimes \ldots \boxtimes L_n$ and $Q^0_{L'_1 \boxtimes L'_2 \boxtimes \ldots \boxtimes L'_n}$ be the 0-dimensional persistence diagram of $L'_1 \boxtimes L'_2 \boxtimes \ldots \boxtimes L'_n$ then*

$$d_{wp}(P^0_{L_1 \boxtimes L_2 \boxtimes \ldots \boxtimes L_n}, Q^0_{L'_1 \boxtimes L'_2 \boxtimes \ldots \boxtimes L'_n}) \leq (\sum_{i,j=1}^{n} (m_i m_j))^{1/p} d_{0max}((L_1, L'_1), (L_2, L'_2), \ldots, (L_n, L'_n))$$

where m_i is the number of elements in $P^0_{L_i}$ or $Q^0_{L'_i}$ and m_j is the number of elements in $P^0_{L_j}$ or $Q^0_{L'_j}$ for all $i = 1, 2, \ldots, n, j = 1, 2, \ldots, n$.

Proof. Let $P^0_{L_1 \boxtimes L_2 \boxtimes \ldots \boxtimes L_n}$,$Q^0_{L'_1 \boxtimes L'_2 \boxtimes \ldots \boxtimes L'_n}$ are 0- dimensional persistence diagrams of $L_1 \boxtimes L_2 \boxtimes \ldots \boxtimes L_n$ and $L'_1 \boxtimes L'_2 \boxtimes \ldots \boxtimes L'_n$ respectively. Then by theorem 3.2

$$d_{wp}(P^0_{L_1 \boxtimes L_2 \boxtimes \ldots \boxtimes L_n}, Q^0_{L'_1 \boxtimes L'_2 \boxtimes \ldots \boxtimes L'_n}) \leq (\sum_{i,j=1}^{n} (m_i m_j))^{1/p} d_{0max}((L_1, L'_1), (L_2, L'_2), \ldots, (L_n, L'_n))$$

where m_i is the number of elements in $P^0_{L_i}$ or $Q^0_{L'_i}$ and m_j is the number of elements in $P^0_{L_j}$ or $Q^0_{L'_j}$for all $i = 1, 2, \ldots, n, j = 1, 2, \ldots, n$. □

Theorem 3.15. *If $P^1_{L_1}, P^1_{L_2}, \ldots, P^1_{L_n}$ and $Q^1_{L'_1}, Q^1_{L'_2}, \ldots, Q^1_{L'_n}$ are 1-dimensional persistence diagrams of $L_1, L_2, \ldots, L_n$ and $L'_1, L'_2, \ldots, L'_n$ respectively. If $P^1_{L_1 \boxtimes L_2 \boxtimes \ldots \boxtimes L_n}$ be the 1-dim persistence diagram of $L_1 \boxtimes L_2 \boxtimes \ldots \boxtimes L_n$ and $Q^1_{L'_1 \boxtimes L'_2 \boxtimes \ldots \boxtimes L'_n}$ be the 1-dim persistence diagram of $L'_1 \boxtimes L'_2 \boxtimes \ldots \boxtimes L'_n$ then*

$$d_{wp}(P^1_{L_1 \boxtimes L_2 \boxtimes \ldots \boxtimes L_n}, Q^1_{L'_1 \boxtimes L'_2 \boxtimes \ldots \boxtimes L'_n}) \leq n^{n-1}\{d_{wp}(P^1_{L_1}, Q^1_{L_1}) + d_{wp}(P^1_{L_2}, Q^1_{L_2}) + \ldots + d_{wp}(P^1_{L_n}, Q^1_{L_n})\}$$

where n is the number of vertices in each graph.

Proof. Let $P^1_{L_1}, P^1_{L_2}, \ldots, P^1_{L_n}$ and $Q^1_{L'_1}, Q^1_{L'_2}, \ldots, Q^1_{L'_n}$ are 1-dimensional persistence diagrams of $L_1, L_2, \ldots, L_n$ and $L'_1, L'_2, \ldots, L'_n$ respectively. Also let $P^0_{L_1}, P^0_{L_2}, \ldots, P^0_{L_n}$ and $Q^0_{L'_1}, Q^0_{L'_2}, \ldots,$
$Q^0_{L'_n}$ are 0-dimensional persistence diagrams of $L_1, L_2, \ldots, L_n$ and $L'_1, L'_2, \ldots, L'_n$ respectively. Here we can find n^{n-1} copies of L_1 fibres, n^{n-1} copies of L_2 fibres, $\ldots, n^{n-1}$ copies of L_n fibres in $L_1 \boxtimes L_2 \boxtimes \ldots \boxtimes L_n$. In similar way we can find n^{n-1} copies of L'_1 fibres, n^{n-1} copies of L'_2 fibres, $\ldots, n^{n-1}$ copies of L'_n fibres in $L'_1 \boxtimes L'_2 \boxtimes \ldots \boxtimes L'_n$. So the only one dimensional loops in the strong product will be the loops present in $L_1, L_2, \ldots, L_n$ and $L'_1, L'_2, \ldots, L'_n$ in n^{n-1} copies. If we are taking the bijection as per theorem 3.12 we can say that

$$d_{wp}(P^1_{L_1 \boxtimes L_2 \boxtimes, \ldots, \boxtimes L_n}, Q^1_{L'_1 \boxtimes L'_2 \boxtimes \ldots \boxtimes L'_n}) \leq n^{n-1}\{d_{wp}(P^1_{L_1}, Q^1_{L_1}) + d_{wp}(P^1_{L_2}, Q^1_{L_2}) + \cdots + d_{wp}(P^1_{L_n}, Q^1_{L_n})\}$$

where n is the number of vertices in each graph. □

Weighted graphs, a fundamental concept in graph theory, enhance traditional graphs by assigning numerical values, or weights, to edges or vertices. Each weight represents a specific cost, distance, or capacity associated with the connection between nodes, providing a more detailed representation of real-world networks. Firstly, it is necessary to establish the definition of the Cartesian product within the context of weighted graphs. Following that, defining the Cartesian product specifically for weighted graphs which is a significant and valuable step forward in this area of study.

Definition 3.16. Let $\mathbf{L}_1 = (\mathbf{U}_1, \mathbf{J}_1)$, $\mathbf{L}_2 = (\mathbf{U}_2, \mathbf{J}_2)$ be two edge weighted undirected graphs with weight function U_1 and U_2 respectively. Then the Cartesian product of $\mathbf{L}_1$ and $\mathbf{L}_2$ be $\mathbf{L}_1 \Box \mathbf{L}_2 = (\mathbf{U}, \mathbf{J})$, where $\mathbf{U} = \mathbf{U}_1 \times \mathbf{U}_2$ and $\mathbf{J} = \{(\mathbf{u}, \mathbf{v}_1), (\mathbf{u}, \mathbf{v}_2) | \mathbf{u} \in \mathbf{U}_1, (\mathbf{v}_1, \mathbf{v}_2) \in \mathbf{J}_2\} \bigcup \{(\mathbf{u}_1, \mathbf{v}), (\mathbf{u}_2, \mathbf{v}) | (\mathbf{u}_1, \mathbf{u}_2) \in \mathbf{J}_1, \mathbf{v} \in \mathbf{U}_2\}$ with the weight function $W : \mathbf{U}_1 \times \mathbf{U}_2 \to \mathbb{R}$ defined by

$$U((\mathbf{u}, \mathbf{v}_1), (\mathbf{u}, \mathbf{v}_2)) = U_2(\mathbf{v}_1, \mathbf{v}_2), \mathbf{u} \in \mathbf{U}_1, (\mathbf{v}_1, \mathbf{v}_2) \in \mathbf{J}_2$$

$$U((\mathbf{u}_1, \mathbf{v}), (\mathbf{u}_2, \mathbf{v})) = U_1(\mathbf{u}_1, \mathbf{u}_2), \mathbf{v} \in \mathbf{U}_2, (\mathbf{u}_1, \mathbf{u}_2) \in \mathbf{J}_1.$$

Example 3.17. Let us consider the weighted graphs K_3 with the vertices a, b, c with the edges $w1, w2, w3$, K_2 with the vertices $0, 1$ and an edge $w4$ and K_2 with the vertices p, q and an edge $w5$. The Cartesian product of these three weighted graphs is then

The Cartesian product of weighted graphs combines the vertex sets and edge weights of the individual graphs to create a new weighted graph. To further analyze the topological properties of the Cartesian product of weighted graphs, the Vietoris-Rips filtration can be applied. The Vietoris-Rips filtration constructs a sequence of simplicial complexes by

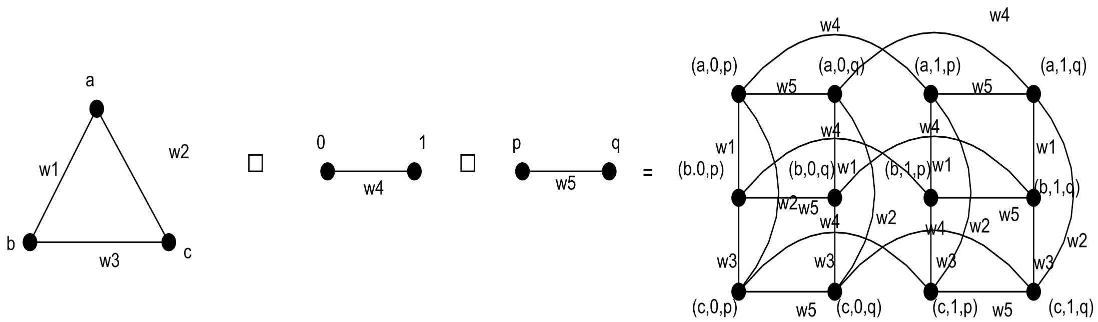

Figure 3. Cartesian product of weighted graphs

gradually adding higher-dimensional simplices based on the proximity of vertices in the Cartesian product.

Definition 3.18. Let $\mathbf{L}_1 = (\mathbf{U}_1, \mathbf{J}_1)$, $\mathbf{L}_2 = (\mathbf{U}_2, \mathbf{J}_2)$ be two edge weighted undirected graphs. For any $\varepsilon \in \mathbb{R}$, the $1-$skeleton $(\mathbf{L}_1\Box\mathbf{L}_2)_\varepsilon = (\mathbf{U}_\varepsilon, \mathbf{J}_\varepsilon)$ of the Cartesian product $\mathbf{L}_1\Box\mathbf{L}_2 = (\mathbf{U}, \mathbf{J})$ is defined as the subgraph of $\mathbf{L}_1\Box\mathbf{L}_2$ where $\mathbf{U}_\varepsilon = \mathbf{U} = \mathbf{U}_1 \times \mathbf{U}_2$ and its edge set $\mathbf{J}_\varepsilon \in \mathbf{J}$ only includes the edges whose weight is less than or equal to ε. Then, for any $\varepsilon \in \mathbb{R}$, we define the Vietoris-Rips complex $\mathcal{L}_{v1}\Box\mathcal{L}_{v2}$ as the clique complex of the $1-$skeleton $(\mathbf{L}_1\Box\mathbf{L}_2)_\varepsilon$, $\mathfrak{Cl}(\mathbf{L}_1\Box\mathbf{L}_2)_\varepsilon$, and the Vietoris-Rips filtration is defined as

$$\{\mathfrak{Cl}(\mathbf{L}_1\Box\mathbf{L}_2)_\varepsilon \to \mathfrak{Cl}(\mathbf{L}_1\Box\mathbf{L}_2)_{\varepsilon'}\}_{0\leq\varepsilon\leq\varepsilon'}$$

Filtration starts with vertex and the edge weight is assumed to be 0 to ∞. For each step, edges are added and the corresponding complex is found.

Although the isomorphism of weighted graphs depends on their weights, their topological structure is essential. Therefore, we should have an understanding of that topological structure. Here we have some results that are helpful for the qualitative study of topological properties of Cartesian products of weighted graphs. Just by looking at the topological structure, it is obvious that we need to consider 0-dimensional persistence diagram and 1-dimensional persistence diagram exactly as we did in the unweighted case.

Theorem 3.19. *If* $\mathbf{P^0_{L_1}}, \mathbf{P^0_{L_2}}, \mathbf{Q^0_{L_3}}, \mathbf{Q^0_{L_4}}$ *are* 0*-dimensional persistence diagrams of weighted graphs* $\mathbf{L}_1, \mathbf{L}_2, \mathbf{L}_3$ *and* $\mathbf{L}_4$ *respectively. If* $\mathbf{P^0_{L_1\Box L_2}}$ *be the* 0*-dimensional persistence diagram of* $\mathbf{L_1\Box L_2}$ *and* $\mathbf{Q^0_{L_3\Box L_4}}$ *be the* 0*-dimensional persistence diagram of* $\mathbf{L_3\Box L_4}$*, then*

$$d_{wp}(\mathbf{P^0_{L_1\Box L_2}}, \mathbf{Q^0_{L_3\Box L_4}}) \leq (\mathbf{mm'})^{\mathbf{1/p}} d_{0max}((\mathbf{L_1}, \mathbf{L_3}), (\mathbf{L_2}, \mathbf{L_4}))$$

where $\mathbf{m}$ *be the number of elements in* $\mathbf{P^0_{L_1}}$ *or* $\mathbf{P^0_{L_3}}$ *and* $\mathbf{m'}$ *be the number of elements in* $\mathbf{P^0_{L_2}}$ *or* $\mathbf{P^0_{L_4}}$.

Proof. Consider the Cartesian product $\mathbf{L_1\Box L_2}$ of the weighted graphs $\mathbf{L_1}$ and $\mathbf{L_2}$ with clique complex filtration $\mathcal{L}_{v1}\Box\mathcal{L}_{v2}$ and 0-dimensional persistent diagram $\mathbf{P^0_{L_1\Box L_2}}$ and Cartesian product $\mathbf{L_3\Box L_4}$ of the graphs $\mathbf{L_3}$ and $\mathbf{L_4}$ with clique complex filtration $\mathcal{L}_{v3}\Box\mathcal{L}_{v4}$ and 0- dimensional persistent diagram $\mathbf{P^0_{L_3\Box L_4}}$. While considering the cartesian product of two topological spaces and cartesian product of two graphs 0- dimensional persistence barcode will contain same elements. Then by theorem 3.2,

$$d_{wp}(\mathbf{P^0_{L_1\Box L_2}}, \mathbf{Q^0_{L_3\Box L_4}}) \leq (\mathbf{mm'})^{\mathbf{1/p}} d_{0max}((\mathbf{L_1}, \mathbf{L_3}), (\mathbf{L_2}, \mathbf{L_4}))$$

where $\mathbf{m}$ be the number of elements in $\mathbf{P^0_{L_1}}$ or $\mathbf{P^0_{L_3}}$ and $\mathbf{m'}$ be the number of elements in $\mathbf{P^0_{L_2}}$ or $\mathbf{P^0_{L_4}}$. $\Box$

Theorem 3.20. *Let* $\mathbf{P^1_{L_1}}, \mathbf{P^1_{L_2}}, \mathbf{Q^1_{L_3}}, \mathbf{Q^1_{L_4}}$ *are* 1*-dimensional and* $\mathbf{P^0_{L_1}}, \mathbf{P^0_{L_2}}, \mathbf{Q^0_{L_3}}, \mathbf{Q^0_{L_4}}$ *are* 0*- dimensional persistence diagrams of wighted graphs* $\mathbf{L_1}, \mathbf{L_2}, \mathbf{L_3}, \mathbf{L_4}$ *respectively. If* $\mathbf{P^1_{L_1 \Box L_2}}$ *be the* 1*-dim persistence diagram of* $\mathbf{L_1 \Box L_2}$ *and* $\mathbf{Q^1_{L_3 \Box L_4}}$ *be the* 1*-dim persistence diagram of* $\mathbf{L_3 \Box L_4}$*, then*

$$\begin{aligned} d_{wp}(\mathbf{P^1_{L_1 \Box L_2}}, \mathbf{Q^1_{L_3 \Box L_4}}) \leq n(d_{wp}(\mathbf{P^1_{L_1}}, \mathbf{Q^1_{L_3}}) + d_{wp}(\mathbf{P^1_{L_2}}, \mathbf{Q^1_{L_4}}))+ \\ (\mathbf{mm'})^{1/p} d_{0max}((\mathbf{L_1}, \mathbf{L_3}), (\mathbf{L_2}, \mathbf{L_4})) \end{aligned}$$

where n *is the number of vertices in each graph.*

Proof. Let $d_{wp}(\mathbf{P^0_{L_1}}, \mathbf{Q^0_{L_3}})$ and $d_{wp}(\mathbf{P^0_{L_2}}, \mathbf{Q^0_{L_4}})$ are the Wassestein distance between 0-dimensional persistence diagrams $\mathbf{P^1_{L_1}}$,$\mathbf{Q^0_{L_3}}$ and $\mathbf{P^0_{L_2}}$,$\mathbf{Q^0_{L_4}}$ respectively. Then there exist bijection $\varrho_0 : \mathbf{P^0_{L_1}} \to \mathbf{Q^0_{L_3}}$ and $\varrho'_0 : \mathbf{P^0_{L_2}} \to \mathbf{Q^0_{L_4}}$ which are best matchings. Also let $d_{wp}(\mathbf{P^1_{L_1}}, \mathbf{Q^1_{L_3}})$ and $d_{wp}(\mathbf{P^1_{L_2}}, \mathbf{Q^1_{L_4}})$ are Wasserstein distance between $\mathbf{P^1_{L_1}}$,$\mathbf{Q^1_{L_3}}$ and $\mathbf{P^1_{L_2}}$,$\mathbf{Q^1_{L_4}}$ respectively. Then there exist bijection $\varrho_1 : \mathbf{P^1_{L_1}} \to \mathbf{Q^1_{L_3}}$ and $\varrho'_1 : \mathbf{P^1_{L_2}} \to \mathbf{Q^1_{L_4}}$ which are best matchings.

If $\{\mathbf{l_{11}}, \mathbf{l_{12}} \ldots \mathbf{l_{1n}}\}$ and $\{\mathbf{l'_{11}}, \mathbf{l'_{12}} \ldots \mathbf{l'_{1n}}\}$ are the vertex sets of $\mathbf{L_1}$ and $\mathbf{L_2}$ then for any $\mathbf{l_{2j}}$ in $\mathbf{L_2}$, we can define the $\mathbf{L_1}$ fibre in $\mathbf{L_1 \Box L_2}$ as

$$\mathbf{L_1 l_{2j}} = \{(\mathbf{l_{1i}}, \mathbf{l_{2j}}) | \mathbf{l_{1i}} \in \mathbf{L_1}, i = 1, 2, \ldots, n\} \; j = 1, 2, \ldots, n$$

and for any $\mathbf{l_{1j}}$ in $\mathbf{L_1}$, $\mathbf{L_2}$ fibre in $\mathbf{L_1 \Box L_2}$ is defined as

$$\mathbf{l_{1j} L_1} = \{(\mathbf{l_{1j}}, \mathbf{l_{2i}}) | \mathbf{l_{2i}} \in \mathbf{L_2}, \mathbf{i} = \mathbf{1}, \mathbf{2}, \ldots, \mathbf{n}\} \; j = 1, 2, \ldots, n.$$

Clearly $\mathbf{L_1}$ fibre is isomorphic to $\mathbf{L_1}$ and $\mathbf{L_2}$ fibre is isomorphic to $\mathbf{L_2}$. So there are n copies of $\mathbf{L_1}$ and n copies of $\mathbf{L_2}$ will be there in $\mathbf{L_1 \Box L_2}$. Similarly there are n copies of $\mathbf{L_3}$ and n copies of $\mathbf{L_4}$ will be there in $\mathbf{L_3 \Box L_4}$. Some other loops also there which are formed by the edges of $\mathbf{L_1}$ and $\mathbf{L_2}$ in $\mathbf{L_1 \Box L_2}$ and $\mathbf{L_3}$ and $\mathbf{L_4}$ in $\mathbf{L_3 \Box L_4}$. Intervals representing these types of 1-dimensional holes will be of the form $(\varphi, \infty), 0 \leq \varphi < \infty$ with $\varphi = \max\{\varphi_1, \varphi_2\}$ where $(0, \varphi_1)$ and $(0, \varphi_2)$ are the intervals in $bcd_0(\mathcal{L}_{v1}), bcd_0(\mathcal{L}_{v2})$ and (φ', ∞) with $\varphi' = \max\{\varphi'_1, \varphi'_2\}$ where $(0, \varphi'_1)$ and $(0, \varphi'_2)$ are the intervals in $bcd_0(\mathcal{L}_{v3}), bcd_0(\mathcal{L}_{v4})$ respectively. Since we are taking the graphs with same number of vertices, number of these type of will be the same.

Now consider the bijection $\varrho : \mathbf{P^1_{L_1 \Box L_2}} \to \mathbf{Q^1_{L_3 \Box L_4}}$ which maps n copies of $\mathbf{L_1}$ fibre loops to n copies of $\mathbf{L_3}$ fibre loops as ϱ_1 maps and n copies of $\mathbf{L_2}$ fibre loops to n copies of $\mathbf{L_2}$ fibre loops as ϱ_2 maps. The loops of the form (φ, ∞) maps on to (φ', ∞) if and only of ϱ_0 maps $(0, \varphi_1)$ to $(0, \varphi'_1)$ and ϱ'_0 maps $(0, \varphi_2)$ to $(0, \varphi'_2)$. Then we can conclude that

$$\begin{aligned} d_{wp}(\mathbf{P^1_{L_1 \Box L_2}}, \mathbf{Q^1_{L_3 \Box L_4}}) \leq n\{d_{wp}(\mathbf{P^1_{L_1}}, \mathbf{Q^1_{L_3}}) + d_{wp}(\mathbf{P^1_{L_2}}, \mathbf{Q^1_{L_4}})\}+ \\ (\mathbf{mm'})^{1/p} d_{0max}((\mathbf{L_1}, \mathbf{L_3}), (\mathbf{L_2}, \mathbf{L_4})) \end{aligned}$$

where n is the number of vertices in each graph. □

The following theorems show how to extend the results found in the product of two graphs to the caseăof weighted graphs. By defining the product for n weighted graphs, we use the Vietoris-Rips filtration to obtain the filtered structures of Cartesian product of a finite set of weighted graphs. This approach relates the Cartesian product of finitely many graphs and the individual graphs, pointing out the relationship between of these concepts in the context of weighted graphs.

Definition 3.21. Let $\mathbf{L}_1 = (\mathbf{U}_1, \mathbf{J}_1)$, $\mathbf{L}_2 = (\mathbf{U}_2, \mathbf{J}_2), \ldots, \mathbf{L}_n = (\mathbf{U}_n, \mathbf{J_n})$ be edge weighted undirected graphs with weight functions $U_1, U_2, \ldots, U_n$ respectively. Then the Cartesian product of $\mathbf{L}_1, \mathbf{L}_2, \ldots, \mathbf{L}_n$ is defined as $\mathbf{L}_1 \Box \mathbf{L}_2 \Box \ldots \Box \mathbf{L}_n$ =$(\mathbf{U}, \mathbf{J})$, where $\mathbf{U} = \mathbf{U}_1 \times \mathbf{U}_2 \times \ldots \times \mathbf{U}_n$ with weight function $U : \mathbf{U}_1 \times \mathbf{U}_2 \times \ldots \times \mathbf{U}_n \to \mathbb{R}$ defined by and $\mathbf{J}$ is the edge set in which two vertices $(\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n)$ and $(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n)$ are adjacent whenever $(\mathbf{u}_i, \mathbf{v}_i) \in \mathbf{F}_i$ for exactly one index $1 \le i \le n$, and $\mathbf{u}_j = \mathbf{v}_j$ for each index $i \neq j$ with

$$U((\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_i, \ldots, \mathbf{u}_n), (\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_i, \ldots, \mathbf{v}_n)) = U_i(\mathbf{u}_i, \mathbf{v}_i).$$

Definition 3.22. Let $\mathbf{L}_1 = (\mathbf{U}_1, \mathbf{J}_1)$, $\mathbf{L}_2 = (\mathbf{U}_2, \mathbf{J}_2), \ldots, \mathbf{L}_n = (\mathbf{U}_n, \mathbf{J}_n)$ be weighted undirected graphs. For any $\xi \in \mathbb{R}$, the $1-$skeleton $(\mathbf{L}_1 \square \mathbf{L}_2 \square \ldots \square \mathbf{L}_n)_\xi = (\mathbf{U}_\xi, \mathbf{J}_\xi)$ of the Cartesian product $\mathbf{L}_1 \square \mathbf{L}_2 \square \ldots \square \mathbf{L}_n = (\mathbf{U}, \mathbf{J})$ is defined as the subgraph of $\mathbf{L}_1 \square \mathbf{L}_2 \square \ldots \square \mathbf{L}_n$ where $\mathbf{U}_\xi = \mathbf{U} = \mathbf{U}_1 \times \mathbf{U}_2 \times \ldots \times \mathbf{U}_n$ and its edge set $\mathbf{J}_\zeta \in \mathbf{J}$ only includes the edges whose weight is less than or equal to ζ. Then, for any $\xi \in \mathbb{R}$, we define the Vietoris-Rips complex $\mathcal{L}_{v1} \square \mathcal{L}_{v2} \square \ldots \mathcal{L}_{vn}$ as the clique complex of the $1-$ skeleton $(\mathbf{L}_1 \square \mathbf{L}_2 \square \ldots \square \mathbf{L}_n)_\xi$, $\mathfrak{Cl}(\mathbf{L}_1 \square \mathbf{L}_2 \square \ldots \square \mathbf{L}_n)_\xi$, and the Vietoris-Rips filtration is defined as

$$\{\mathfrak{Cl}(\mathbf{L}_1 \square \mathbf{L}_2 \ldots \square \mathbf{L}_n)_\xi \longrightarrow \mathfrak{Cl}(\mathbf{L}_1 \square \mathbf{L}_2 \ldots \square \mathbf{L}_n)_{\xi'}\}_{0 \le \xi \le \xi'}$$

Filtration starts with vertex and the edge weight is assumed to be 0 to ∞. For each step, edges are added and the corresponding complex is found.

Theorem 3.23. *Let* $\mathbf{P^0_{L_1}}, \mathbf{P^0_{L_2}}, \ldots, \mathbf{P^0_{L_n}}$ *and* $\mathbf{Q^0_{L'_1}}, \mathbf{Q^0_{L'_2}}, \ldots, \mathbf{Q^0_{L'_n}}$ *are* 0*-dimensional persistence diagrams of weighted undirected graphs* $\mathbf{L_1}, \mathbf{L_2}, \ldots, \mathbf{L_n}$ *and* $\mathbf{L_1}', \mathbf{L'_2}, \ldots, \mathbf{L_n}'$ *respectively. If* $\mathbf{P^0_{L_1 \square L_2 \square \square \ldots \square L_n}}$ *is the* 0*-dim persistence diagram of* $\mathbf{L_1 \square L_2 \square \ldots \square L_n}$ *and* $\mathbf{Q^0_{L'_1 \square L'_2 \square \ldots \square L'_n}}$ *is the* 0*-dim persistence diagram of* $\mathbf{L'_1 \square L'_2 \square \ldots \square L'_n}$ *then*

$$d_{wp}(\mathbf{P^0_{L_1 \square L_2 \square \ldots \square L_n}}, \mathbf{Q^0_{L'_1 \square L'_2 \square \ldots \square L'_n}}) \le (\sum_{i,j=1}^{n} (\mathbf{m_i m_j}))^{1/p} d_{0max}((\mathbf{L_1}, \mathbf{L'_1}), (\mathbf{L_2}, \mathbf{L'_2}), \ldots, (\mathbf{L_n}, \mathbf{L'_n}))$$

where $\mathbf{m_i}$ *is the number of elements in* $\mathbf{P^0_{L_i}}$ *or* $\mathbf{Q^0_{L'_i}}$ *and* $\mathbf{m_j}$ *is the number of elements in* $\mathbf{P^0_{L_j}}$ *or* $\mathbf{Q^0_{Lj'}}$.

Proof. Let $\mathbf{P^0_{L_1 \square L_2 \square \ldots \square L_n}}, \mathbf{Q^0_{L'_1 \square L'_2 \square \ldots \square L'_n}}$ are 0- dimensional persistence diagrams of weighted graphs $\mathbf{L_1 \square L_2 \square \ldots \square L_n}$ and $\mathbf{L'_1 \square L'_2 \square \ldots \square L'_n}$ respectively. Here we can define the bijection according to the theorem 3.2. Then

$$d_{wp}(\mathbf{P^0_{L_1 \square L_2 \square \ldots \square L_n}}, \mathbf{Q^0_{L'_1 \square L'_2 \square \ldots \square L'_n}}) \le (\sum_{i,j=1}^{n} (\mathbf{m_i m_j}))^{1/p} d_{0max}((\mathbf{L_1}, \mathbf{L'_1}), (\mathbf{L_2}, \mathbf{L'_2}), \ldots, (\mathbf{L_n}, \mathbf{L'_n}))$$

where $\mathbf{m_i}$ is the number of elements in $\mathbf{P^0_{L_i}}$ or $\mathbf{Q^0_{L'_i}}$ and $\mathbf{m_j}$ is the number of elements in $\mathbf{P^0_{L_j}}$ or $\mathbf{Q^0_{Lj'}}$. □

Theorem 3.24. *If* $\mathbf{P^1_{L_1}}, \mathbf{P^1_{L_2}}, \ldots, \mathbf{P^1_{L_n}}$ *and* $\mathbf{Q^1_{L'_1}}, \mathbf{Q^1_{L'_2}}, \ldots, \mathbf{Q^1_{L'_n}}$ *are* 1*-dimensional persistence diagrams of weighted undirected graphs* $\mathbf{L_1}, \mathbf{L_2}, \ldots, \mathbf{L_n}$ *and* $\mathbf{L'_1}, \mathbf{L'_2}, \ldots, \mathbf{L'_n}$ *respectively. If* $\mathbf{P^1_{L_1 \square L_2 \square \ldots \square L_n}}$ *be the* 1*-dim persistence diagram of* $\mathbf{L_1 \square L_2 \square \ldots \square L_n}$ *and* $\mathbf{Q^1_{L'_1 \square L'_2 \square \ldots \square L'_n}}$ *be the* 1*-dim persistence diagram of* $\mathbf{L'_1 \square L'_2 \square \ldots \square L'_n}$ *then*

$$\begin{aligned} d_{wp}(\mathbf{P^1_{L_1 \square L_2 \square \ldots \square L_n}}, \mathbf{Q^1_{L'_1 \square L'_2 \square \ldots \square L'_n}}) \le n^{n-1} \{ d_{wp}(\mathbf{P^1_{L_1}}, \mathbf{Q^1_{L_1}}) + d_{wp}(\mathbf{P^1_{L_2}}, \mathbf{Q^1_{L_2}}) + \ldots + \\ d_{wp}(\mathbf{P^1_{L_n}}, \mathbf{Q^1_{L_n}}) \} + (\sum_{i,j=1}^{n} (\mathbf{m_i m_j}))^{1/p} d_{0max}((\mathbf{L_1}, \mathbf{L'_1}), (\mathbf{L_2}, \mathbf{L'_2}), \ldots, (\mathbf{L_n}, \mathbf{L'_n})) \end{aligned}$$

where $\mathbf{m_i}$ *is the number of elements in* $\mathbf{P^0_{L_i}}$ *or* $\mathbf{Q^0_{L'_i}}$ *and* $\mathbf{m_j}$ *is the number of elements in* $\mathbf{P^0_{L_j}}$ *or* $\mathbf{Q^0_{L'_j}}$.

Proof. Let $\mathbf{P^1_{L_1}}, \mathbf{P^1_{L_2}}, \dots, \mathbf{P^1_{L_n}}$ and $\mathbf{Q^1_{L'_1}}, \mathbf{Q^1_{L'_2}}, \dots, \mathbf{Q^1_{L'_n}}$ are 1-dimensional persistence diagrams of weighted graphs $\mathbf{L_1}, \mathbf{L_2}, \dots, \mathbf{L_n}$ and $\mathbf{L'_1}, \mathbf{L'_2}, \dots, \mathbf{L'_n}$ respectively. Also let $\mathbf{P^0_{L_1}}, \mathbf{P^0_{L_2}}, \dots, \mathbf{P^0_{L_n}}$ and $\mathbf{Q^0_{L'_1}}, \mathbf{Q^0_{L'_2}}, \dots, \mathbf{Q^0_{L'_n}}$ are 0-dimensional persistence diagrams of $\mathbf{L_1}, \mathbf{L_2}, \dots, \mathbf{L_n}$ and $\mathbf{L'_1}, \mathbf{L'_2}, \dots, \mathbf{L'_n}$ respectively. Here we can find n^{n-1} copies of $\mathbf{L_1}$ fibres, n^{n-1} copies of $\mathbf{L_2}$ fibres, $\dots, n^{n-1}$ copies of $\mathbf{L_n}$ fibres in $\mathbf{L_1}\square\mathbf{L_2}\square\dots\square\mathbf{L_n}$. In similar way we can find n^{n-1} copies of $\mathbf{L'_1}$ fibres, n^{n-1} copies of $\mathbf{L'_2}$ fibres, $\dots, n^{n-1}$ copies of $\mathbf{L'_n}$ fibres in $\mathbf{L'_1}\square\mathbf{L'_2}\square\dots\square\mathbf{L'_n}$. Also some more one dimensional holes(loops) will be there formed by the edges which represents the interval in 0 dimensional barcodes of the filtration of each $\mathbf{L_i}$ and $\mathbf{L'_i}$. If we are defining the bijection as per theorem 3.6, the p Wasserstein distance of 1-dimensional persistence diagrams $\mathbf{P^1_{L_1\square L_2\square\dots\square L_n}}$ and $\mathbf{Q^1_{L'_1\square L'_2\square\dots\square L'_n}}$,

$$d_{wp}(\mathbf{P^1_{L_1\square L_2\square\dots\square L_n}}, \mathbf{Q^1_{L'_1\square L'_2\square\dots\square L'_n}}) \leq n^{n-1}\{d_{wp}(\mathbf{P^1_{L_1}}, \mathbf{Q^1_{L_1}}) + d_{wp}(\mathbf{P^1_{L_2}}, \mathbf{Q^1_{L_2}}) + \dots + d_{wp}(\mathbf{P^1_{L_n}}, \mathbf{Q^1_{L_n}})\} + (\sum_{i,j=1}^{n}(\mathbf{m_i m_j}))^{1/p} d_{0max}((\mathbf{L_1}, \mathbf{L'_1}), (\mathbf{L_2}, \mathbf{L'_2}), \dots, (\mathbf{L_n}, \mathbf{L'_n}))$$

where $\mathbf{m_i}$ is the number of elements in $\mathbf{P^0_{L_i}}$ or $\mathbf{Q^0_{L'_i}}$ and $\mathbf{m_j}$ is the number of elements in $\mathbf{P^0_{L_j}}$ or $\mathbf{Q^0_{L'_j}}$. $\square$

We establish a definition by mandating a strong product on a weighted graph. Following the subsequent filtration definition, the theorems also elucidate the correlation between strong product of graphs with individual graphs in both 0-dimensional and 1-dimensional persistence diagrams, providing a comprehensive understanding of these relationships in the context of weighted graphs. By applying the Vietoris-Rips filtration to the strong product of weighted graphs, we can obtain results that are analogous to those observed in unweighted graphs. This allows us to analyze the topological properties and connectivity patterns of the product graph, exploring features such as connected components, loops, and higher-dimensional simplices.

Definition 3.25. Let $\mathbf{L}_1 = (\mathbf{U}_1, \mathbf{J}_1)$, $\mathbf{L}_2 = (\mathbf{U}_2, \mathbf{J}_2)$ be two edge weighted undirected graphs with weight function U_1 and U_2 respectively. Then the Strong product of $\mathbf{L}_1$ and $\mathbf{L}_2$ is $\mathbf{L}_1 \boxtimes \mathbf{L}_2 = (\mathbf{U}, \mathbf{J})$, where $\mathbf{U} = \mathbf{U}_1 \times \mathbf{U}_2$ and $\mathbf{J} = \{(\mathbf{u}, \mathbf{v}_1), (\mathbf{u}, \mathbf{v}_2) | \mathbf{u} \in \mathbf{U}_1, (\mathbf{v}_1, \mathbf{v}_2) \in \mathbf{J}_2\} \bigcup \{(\mathbf{u}_1, \mathbf{v}), (\mathbf{u}_2, \mathbf{v}) | (\mathbf{u}_1, \mathbf{u}_2) \in \mathbf{J}_1, \mathbf{v} \in \mathbf{U}_2\} \bigcup \{(\mathbf{u}_1, \mathbf{v}_1), (\mathbf{u}_2, \mathbf{v}_2) | (\mathbf{u}_1, \mathbf{u}_2) \in \mathbf{J}_1, (\mathbf{v}_1, \mathbf{v}_2) \in \mathbf{J}_2\}$ with weight function $U' : \mathbf{U}_1 \times \mathbf{U}_2 \to \mathbb{R}$ defined by

$$U'((\mathbf{u}, \mathbf{v}_1), (\mathbf{u}, \mathbf{v}_2)) = \mathbf{U_2}(\mathbf{v}_1, \mathbf{v}_2), \text{ if } \mathbf{u} \in \mathbf{U}_1, (\mathbf{v}_1, \mathbf{v}_2) \in \mathbf{J}_2$$
$$U'((\mathbf{u}_1, \mathbf{v}), (\mathbf{u}_2, \mathbf{v})) = \mathbf{U_1}(\mathbf{u}_1, \mathbf{u}_2), \text{ if } \mathbf{v} \in \mathbf{U}_2, (\mathbf{u}_1, \mathbf{u}_2) \in \mathbf{J}_1$$
$$U'((\mathbf{u}_1, \mathbf{v}_1), (\mathbf{u}_2, \mathbf{v}_2)) = \mathbf{U_1}(\mathbf{u}_1, \mathbf{u}_2) \wedge \mathbf{U_2}(\mathbf{v}_1, \mathbf{v}_2), (\mathbf{u}_1, \mathbf{u}_2) \in \mathbf{J}_1 \text{ and } (\mathbf{v}_1, \mathbf{v}_2) \in \mathbf{J}_2.$$

Example 3.26. Here we are considering two weighted graphs. First one is a complete graph K_3 with vertices a, b, c and edge weights $w1, w2, w3$. Second one is a weighted graph with vertices $0, 1, 2$ with edge weights $w4, w5$. Let

$$W1 = w1 \wedge w4 = \min\{w1, w4\}, W2 = \min\{w1, w5\}, W3 = \min\{w2, w4\}, W4 = \min\{w2, w5\},$$

$$W5 = \min\{w3, w4\}, W6 = \min\{w3, w5\}.$$

Then the strong product of these weighted graphs will be

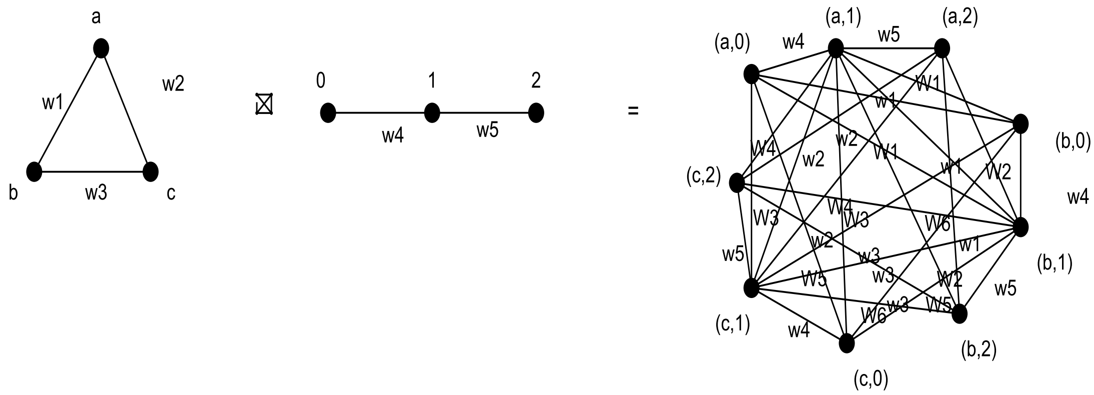

Figure 4. Strong product of weighted graphs

Definition 3.27. Let $\mathbf{L}_1 = (\mathbf{U}_1, \mathbf{J}_1)$, $\mathbf{L}_2 = (\mathbf{U}_2, \mathbf{J}_2)$ be two edge weighted undirected graphs. For any $\Xi \in \mathbb{R}$, the $1-$skeleton $(\mathbf{L}_1 \boxtimes \mathbf{L}_2)_\Xi = (\mathbf{U}_\Xi, \mathbf{J}_\Xi)$ of the Strong product $\mathbf{L}_1 \boxtimes \mathbf{L}_2 = (\mathbf{U}, \mathbf{J})$ is defined as the subgraph of $\mathbf{L}_1 \boxtimes \mathbf{L}_2$ where $\mathbf{U}_\Xi = \mathbf{U} = \mathbf{U}_1 \times \mathbf{U}_2$ and its edge set $\mathbf{J}_\Xi \in \mathbf{J}$ only includes the edges whose weight is less than or equal to Ξ. Then, for any $\Xi \in \mathbb{R}$, we define the Vietoris-Rips complex $\mathcal{L}_{v1} \boxtimes \mathcal{L}_{v2} \ldots \boxtimes \mathcal{L}_{vn}$ as the clique complex of the $1-$ skeleton $(\mathbf{L}_1 \boxtimes \mathbf{L}_2)_\Xi$, $\mathfrak{Cl}(\mathbf{L}_1 \boxtimes \mathbf{L}_2)_\Xi$, and the Vietoris-Rips filtration is defined as

$$\{\mathfrak{Cl}(\mathbf{L}_1 \boxtimes \mathbf{L}_2)_\Xi \to \mathfrak{Cl}(\mathbf{L}_1 \boxtimes \mathbf{L}_2)_{\Xi'}\}_{0 \le \Xi \le \Xi'}$$

Filtration starts with vertex and the edge weight is assumed to be 0 to ∞. For each step, edges are added and the corresponding complex is found.

Theorem 3.28. *Let* $\mathbf{P^0_{L_1}}, \mathbf{P^0_{L_2}}, \mathbf{Q^0_{L_3}}, \mathbf{Q^0_{L_4}}$ *are* 0*-dimensional persistence diagrams of weighted undirected graphs* $\mathbf{L_1}, \mathbf{L_2}, \mathbf{L_3}$ *and* $\mathbf{L_4}$ *respectively. If* $\mathbf{P^0_{L_1 \boxtimes L_2}}$ *be the* 0*-dimensional persistence diagram of* $\mathbf{L_1} \boxtimes \mathbf{L_2}$ *and* $\mathbf{Q^0_{L_3 \boxtimes L_4}}$ *be the* 0*-dim persistence diagram of* $\mathbf{L_3} \boxtimes \mathbf{L_4}$ *then,*

$$d_{wp}(\mathbf{P^0_{L_1 \boxtimes L_2}}, \mathbf{Q^0_{L_3 \boxtimes L_4}}) \le \mathbf{(mm')^{1/p} d_{0max}}((\mathbf{L_1}, \mathbf{L_3}), (\mathbf{L_2}, \mathbf{L_4}))$$

where $\mathbf{m}$ *is the number of elements in* $\mathbf{P^0_{L_1}}$ *or* $\mathbf{P^0_{L_3}}$ *and* $\mathbf{m'}$ *is the number of elements in* $\mathbf{P^0_{L_2}}$ *or* $\mathbf{P^0_{L_4}}$.

Proof. Let $\mathbf{P^0_{L_1}}, \mathbf{P^0_{L_2}}, \mathbf{Q^0_{L_3}}, \mathbf{Q^0_{L_4}}$ are 0-dimensional persistence diagrams of weighted undirected graphs $\mathbf{L_1}, \mathbf{L_2}, \mathbf{L_3}$ and $\mathbf{L_4}$ respectively. Also let $\mathbf{P^0_{L_1 \boxtimes L_2}}$ be the 0-dim persistence diagram of $\mathbf{L_1} \boxtimes \mathbf{L_2}$ and $\mathbf{Q^0_{L_3 \boxtimes L_4}}$ be the 0-dim persistence diagram of $\mathbf{L_3} \boxtimes \mathbf{L_4}$. In the case of strong product of weighted graphs, $bcd_0(L_1 \boxtimes L_2)$ contains some elements which are intersection of elements from $bcd_0(L_1)$ and $bcd_0(L_2)$ also remaining elements will be of the form $(0, \iota), \iota = min\{\iota_1, \iota_2\}$ for some $(0, \iota_1) \in bcd_0(L_1)$ and $(0, \iota_2) \in bcd_0(L_2)$. The number of elements in 0 dimensional barcodes of cartesian product and strong product of weighted graphs will be the same Now define a bijection $\chi : \mathbf{P^0_{L_1 \boxtimes L_2}} \to \mathbf{Q^0_{L_3 \boxtimes L_4}}$ which maps each element in $\mathbf{P^0_{L_1 \boxtimes L_2}}$ whic is the intersection of elements in $bcd_0(L_1)$ and $bcd_0(L_2)$ as per theorem 3.2 and elements of the form $(0, \iota)$ in $\mathbf{P^0_{L_1 \boxtimes L_2}}$ to $(0, \iota')$ in $\mathbf{Q^0_{L_3 \boxtimes L_4}}$ if and only if the best matching from $\mathbf{P^0_{L_1}}$ to $\mathbf{Q^0_{L_3}}$ maps $(0, \iota_1)$ on to $(0, \iota'_1)$ and the best matching from $\mathbf{P^0_{L_2}}$ to $\mathbf{Q^0_{L_4}}$ maps $(0, \iota_2)$ on to $(0, \iota'_2)$. Hence

$$d_{wp}(\mathbf{P^0_{L_1 \boxtimes L_2}}, \mathbf{Q^0_{L_3 \boxtimes L_4}}) \le \mathbf{(mm')^{1/p} d_{0max}}((\mathbf{L_1}, \mathbf{L_3}), (\mathbf{L_2}, \mathbf{L_4}))$$

where $\mathbf{m}$ is the number of elements in $\mathbf{P^0_{L_1}}$ or $\mathbf{P^0_{L_3}}$ and $\mathbf{m'}$ is the number of elements in $\mathbf{P^0_{L_2}}$ or $\mathbf{P^0_{L_4}}$.

□

Theorem 3.29. *Let $\mathbf{P^1_{L_1}}, \mathbf{P^1_{L_2}}, \mathbf{Q^1_{L_3}}, \mathbf{Q^1_{L_4}}$ are 1-dimensional persistence diagrams of weighted undirected graphs $\mathbf{L_1}, \mathbf{L_2}, \mathbf{L_3}, \mathbf{L_4}$ respectively. If $\mathbf{P^1_{L_1 \boxtimes L_2}}$ be the 1-dim persistence diagram of $\mathbf{L_1} \boxtimes \mathbf{L_2}$ and $\mathbf{Q^1_{L_3 \boxtimes L_4}}$ be the 1-dim persistence diagram of $\mathbf{L_3} \boxtimes \mathbf{L_4}$. then*

$$d_{wp}(\mathbf{P^1_{L_1 \boxtimes L_2}}, \mathbf{Q^1_{L_3 \boxtimes L_4}}) \leq n\{d_{wp}(\mathbf{P^1_{L_1}}, \mathbf{Q^1_{L_3}}) + d_{wp}(\mathbf{P^1_{L_2}}, \mathbf{Q^1_{L_4}})\}$$

where n is the number of vertices in each graph.

Proof. Let $d_{wp}(\mathbf{P^1_{L_1}}, \mathbf{Q^1_{L_3}})$ and $d_{wp}(\mathbf{P^1_{L_2}}, \mathbf{Q^1_{L_4}})$ are Wasserstein distance between $\mathbf{P^1_{L_1}}$,$\mathbf{Q^1_{L_3}}$ and $\mathbf{P^1_{L_2}}$,$\mathbf{Q^1_{L_4}}$ respectively. Then there exist bijection $\Pi_1 : \mathbf{P^1_{L_1}} \to \mathbf{Q^1_{L_3}}$ and $\Pi'_1 : \mathbf{P^1_{L_2}} \to \mathbf{Q^1_{L_4}}$ which are best matchings. In the case of strong product if $\{\mathbf{l_{11}}, \mathbf{l_{12}}, \ldots, \mathbf{l_{1n}}\}$ and $\{\mathbf{l'_{11}}, \mathbf{l'_{12}}, \ldots, \mathbf{l'_{1n}}\}$ are the vertex sets of weighted graphs $\mathbf{L_1}$ and $\mathbf{L_2}$ then for any $\mathbf{l_{2j}}$ in $\mathbf{L_2}$, we can define the $\mathbf{L_1}$ fibre in $\mathbf{L_1} \boxtimes \mathbf{L_2}$ as

$$\mathbf{L_1 l_{2j}} = \{(\mathbf{l_{1i}}, \mathbf{l_{2j}}) | \mathbf{l_{1i}} \in \mathbf{L_1}, \mathbf{i} = 1, 2, \ldots, n\}, \mathbf{j} = 1, 2, \ldots, n$$

and for any $\mathbf{l_{1j}}$ in $\mathbf{L_1}$, $\mathbf{L_2}$ fibre in $\mathbf{L_1} \boxtimes \mathbf{L_2}$ is defined as

$$\mathbf{l_{1j} L_1} = \{(\mathbf{l_{1j}}, \mathbf{l_{2i}}) | \mathbf{l_{2i}} \in \mathbf{L_2}, \mathbf{i} = 1, 2, \ldots, n\}, \mathbf{j} = 1, 2, \ldots, n.$$

Clearly $\mathbf{L_1}$ fibre is isomorphic to $\mathbf{L_1}$ and $\mathbf{L_2}$ fibre is isomorphic to $\mathbf{L_2}$. So there are n copies of $\mathbf{L_1}$ and n copies of $\mathbf{L_2}$ will be there in $\mathbf{L_1} \boxtimes \mathbf{L_2}$. Similarly there are n copies of $\mathbf{L_3}$ and n copies of $\mathbf{L_4}$ will be there in $\mathbf{L_3} \boxtimes \mathbf{L_4}$. So the 1- dimensional loops will be the loops present in $\mathbf{L_1}$, $\mathbf{L_1}$ and $\mathbf{L_3}$, $\mathbf{L_4}$ n times. Now consider the bijection $\Pi : \mathbf{P^1_{L_1 \square L_2}} \to \mathbf{Q^1_{L_3 \square L_4}}$ which maps n copies of $\mathbf{L_1}$ fibre loops to n copies of $\mathbf{L_3}$ fibre loops as Π_1 maps and n copies of $\mathbf{L_2}$ fibre loops to n copies of $\mathbf{L_2}$ fibre loops as Π_2 maps. Then

$$d_{wp}(\mathbf{P^1_{L_1 \boxtimes L_2}}, \mathbf{Q^1_{L_3 \boxtimes L_4}}) \leq n\{d_{wp}(\mathbf{P^1_{L_1}}, \mathbf{Q^1_{L_3}}) + d_{wp}(\mathbf{P^1_{L_2}}, \mathbf{Q^1_{L_4}})\}$$

where n is the number of vertices in each graph.

□

Analogous to unweighted graphs we can extend the results obtained in the case of weighted graphs to n products. For that the strong product of n weighted graphs is defined. Furthermore, the filtering and the desired results are achieved in order to compare the corresponding topological structure.

Definition 3.30. Let $\mathbf{L}_1 = (\mathbf{U}_1, \mathbf{J}_1)$, $\mathbf{L}_2 = (\mathbf{U}_2, \mathbf{J}_2), \ldots, \mathbf{L}_n = (\mathbf{U}_n, \mathbf{J_n})$ be edge weighted undirected graphs with weight functions $U_1, U_2, \ldots, U_3$ respectively. Then the Strong product of $\mathbf{L}_1, \mathbf{L}_2, \ldots, \mathbf{L}_n$ is defined as $\mathbf{L}_1 \boxtimes \mathbf{L}_2 \boxtimes \ldots \boxtimes \mathbf{L}_n$ =$(\mathbf{U}, \mathbf{J})$, where $\mathbf{U} = \mathbf{U}_1 \times \mathbf{U}_2 \times \ldots \times \mathbf{U}_n$ and $\mathbf{J}$ is the edge set in which two vertices $(\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n)$ and $(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n)$ are adjacent provided $(\mathbf{u}_i, \mathbf{v}_i) \in \mathbf{J}_i$ or $\mathbf{u}_j = \mathbf{v}_j$ for each $1 \leq i \leq n$ with weight function $U : \mathbf{U}_1 \times \mathbf{U}_2 \times \ldots \times \mathbf{U}_n \to \mathbb{R}$ defined by

$$U((\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_i, \ldots, \mathbf{u}_n), (\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_i, \ldots, \mathbf{v}_n)) = \min_{1 \leq i \leq n} \{U_i(\mathbf{u}_i, \mathbf{v}_i); (\mathbf{u}_i, \mathbf{v}_i) \in \mathbf{J}_i\}$$

.

Definition 3.31. Let $\mathbf{L}_1 = (\mathbf{U}_1, \mathbf{J}_1)$, $\mathbf{L}_2 = (\mathbf{U}_2, \mathbf{J}_2), \ldots, \mathbf{L}_n = (\mathbf{U}_n, \mathbf{J}_n)$ be edge weighted undirected graphs. For any $\eta \in \mathbb{R}$, the 1−skeleton $(\mathbf{L}_1 \boxtimes \mathbf{L}_2 \boxtimes \ldots \boxtimes \mathbf{L}_n)_\eta = (\mathbf{U}_\eta, \mathbf{J}_\eta)$ of the Cartesian product $\mathbf{L}_1 \boxtimes \mathbf{L}_2 \boxtimes \ldots \boxtimes \mathbf{L}_n = (\mathbf{U}, \mathbf{J})$ is defined as the subgraph of $\mathbf{L}_1 \boxtimes \mathbf{L}_2 \boxtimes \ldots \boxtimes \mathbf{L}_n$ where $\mathbf{U}_\eta = \mathbf{U} = \mathbf{U}_1 \times \mathbf{U}_2 \times \ldots \times \mathbf{U}_n$ and its edge set $\mathbf{J}_\eta \in \mathbf{J}$ only includes the edges whose weight is less than or equal to η. Then, for any $\eta \in \mathbb{R}$, we define the Vietoris-Rips complex $\mathcal{L}_{v1}\boxtimes\mathcal{L}_{v2}\boxtimes\ldots\boxtimes\mathcal{L}_{vn}$as the clique complex of the 1− skeleton $(\mathbf{L}_1\boxtimes\mathbf{L}_2\boxtimes\ldots\boxtimes\mathbf{L}_n)_\eta$, $\mathfrak{Cl}(\mathbf{L}_1\boxtimes\mathbf{L}_2\boxtimes\ldots\boxtimes\mathbf{L}_n)_\eta$, and the Vietoris-Rips filtration is defined as

$$\{\mathfrak{Cl}(\mathbf{L}_1\boxtimes\mathbf{L}_2\boxtimes\ldots\boxtimes\mathbf{L}_n)_\eta\to\mathfrak{Cl}(\mathbf{L}_1\boxtimes\mathbf{L}_2\boxtimes\ldots\boxtimes\mathbf{L}_n)_{\eta'}\}_{0\leq\eta\leq\eta'}.$$

Filtration starts with vertex and the edge weight is assumed to be 0 to ∞. For each step, edges are added and the corresponding complex is found.

Theorem 3.32. *If $\mathbf{P^0_{L_1}},\mathbf{P^0_{L_2}},\ldots,\mathbf{P^0_{L_n}}$ and $\mathbf{Q^0_{L'_1}},\mathbf{Q^0_{L'_2}},\ldots,\mathbf{Q^0_{L'_n}}$ are 0-dimensional persistence diagrams of $\mathbf{L_1},\mathbf{L_2},\ldots,\mathbf{L_n}$ and $\mathbf{L'_1},\mathbf{L'_2},\ldots,\mathbf{L'_n}$ respectively. If $\mathbf{P^0_{L_1\boxtimes L_2\boxtimes\ldots\boxtimes L_n}}$ be the 0-dim persistence diagram of $\mathbf{L_1}\boxtimes\mathbf{L_2}\boxtimes\ldots\boxtimes\mathbf{L_n}$ and $\mathbf{Q^0_{L'_1\boxtimes L'_2\boxtimes\ldots\boxtimes L'_n}}$ be the 0-dim persistence diagram of $\mathbf{L'_1}\boxtimes\mathbf{L'_2}\boxtimes\ldots\boxtimes\mathbf{L'_n}$ then*

$$d_{wp}(\mathbf{P^0_{L_1\boxtimes L_2\boxtimes\ldots\boxtimes L_n}},\mathbf{Q^0_{L'_1\boxtimes L'_2\boxtimes\ldots\boxtimes L'_n}})\leq(\sum_{i,j=1}^{n}(\mathbf{m_i m_j}))^{1/p}d_{0max}((\mathbf{L_1},\mathbf{L'_1}),(\mathbf{L_2},\mathbf{L'_2}),\ldots,(\mathbf{L_n},\mathbf{L'_n}))$$

where $\mathbf{m_i}$ is the number of elements in $\mathbf{P^0_{L_i}}$ or $\mathbf{Q^0_{L'_i}}$ and $\mathbf{m_j}$ is the number of elements in $\mathbf{P^0_{L_j}}$ or $\mathbf{Q^0_{L'_j}}$ for all $i=1,2,\ldots,n, j=1,2,\ldots,n$.

Proof. Let $\mathbf{P^0_{L_1}},\mathbf{P^0_{L_2}},\mathbf{P^0_{L_3}},\ldots,\mathbf{P^0_{L_n}}$ are 0-dimensional persistence diagrams of weighted undirected graphs $\mathbf{L_1},\mathbf{L_2},\mathbf{L_3},\ldots,\mathbf{L_n}$ and $\mathbf{Q^0_{L'_1}},\mathbf{Q^0_{L'_2}},\mathbf{Q^0_{L'_3}},\ldots,\mathbf{Q^0_{L'_n}}$ are 0-dimensional persistence diagrams of weighted undirected graphs $\mathbf{L'_1},\mathbf{L'_2},\mathbf{L_{3'}},\ldots,\mathbf{L'_n}$ respectively. Also let $\mathbf{P^0_{L_1\boxtimes L_2\boxtimes\ldots L_n}}$ be the 0-dim persistence diagram of $\mathbf{L_1}\boxtimes\mathbf{L_2}\boxtimes\ldots\boxtimes\mathbf{L_n}$ and $\mathbf{Q^0_{L'_1\boxtimes L'_2\boxtimes\ldots\boxtimes L'_n}}$ be the 0-dim persistence diagram of $\mathbf{L'_1}\boxtimes\mathbf{L'_2}\boxtimes\ldots\boxtimes\mathbf{L'_n}$. In the case of strong product of weighted graphs,
$bcd_0(\mathbf{L_1}\boxtimes\mathbf{L_2}\boxtimes\ldots\boxtimes\mathbf{L_n})$ contains some elements which are intersection of elements from $bcd_0(\mathbf{L_1})$,$bcd_0(\mathbf{L_2}),\ldots,bcd_0(\mathbf{L_n})$ also remaining elements will be of the form $(0,\kappa)$,
$\kappa=\min\{\kappa_1,\kappa_2,\ldots,\kappa_n\}$ for some $(0,\kappa_1)\in bcd_0(\mathbf{L_1})$ and $(0,\kappa_2)\in bcd_0(\mathbf{L_2}),\ldots,(0,\kappa_n)\in bcd_0(\mathbf{L_n})$. Similarly we can find $(0,\kappa'),\kappa'=\min\{\kappa'_1,\kappa'_2,\ldots,\kappa'_n\}$ for some $(0,\kappa'_1)\in bcd_0(\mathbf{L'_1})$ and $(0,\kappa'_2)\in bcd_0(\mathbf{L'_2}),\ldots,(0,\kappa'_n)\in bcd_0(\mathbf{L'_n})$. The number of elements in 0 dimensional barcodes of cartesian product and strong product of weighted graphs will be the same. Now define a bijection $\chi':\mathbf{P^0_{L_1\boxtimes L_2\boxtimes\ldots\boxtimes L_n}}\to\mathbf{Q^0_{L'_1\boxtimes L'_2\boxtimes\ldots\boxtimes Q_{L'_n}}}$ which maps each element in$\mathbf{P^0_{L_1\boxtimes L_2\boxtimes\ldots\boxtimes L_n}}$ which is the intersection of elements in $bcd_0(\mathbf{L_1})$,$bcd_0(\mathbf{L_2}),\ldots,bcd_0(\mathbf{L_n})$ as per theorem 3.2 and elements of the form $(0,\kappa)$ in $\mathbf{P^0_{L_1\boxtimes L_2\boxtimes\ldots\boxtimes L_n}}$ to $(0,\kappa')$ in $\mathbf{Q^0_{L'_1\boxtimes L'_2\boxtimes\ldots\boxtimes Q_{L'_n}}}$ if and only if the best matching from $\mathbf{P^0_{L_1}}$ to $\mathbf{Q'^0_{L_1}}$ maps $(0,\kappa_1)$ on to $(0,\kappa'_1)$ and the best matching from $\mathbf{P^0_{L_2}}$ to $\mathbf{Q^0_{L'_2}}$ maps $(0,\kappa_2)$ on to $(0,\kappa'_2)\ldots$the best matching from $\mathbf{P^0_{L_n}}$ to $\mathbf{Q'^0_{L_n}}$ maps $(0,\kappa_n)$ on to $(0,\kappa'_n)$. Hence

$$d_{wp}(\mathbf{P^0_{L_1\boxtimes L_2\boxtimes\ldots\boxtimes L_n}},\mathbf{Q^0_{L'_1\boxtimes L'_2\boxtimes\ldots\boxtimes L'_n}})\leq(\sum_{i,j=1}^{n}(\mathbf{m_i m_j}))^{1/p}d_{0max}((\mathbf{L_1},\mathbf{L'_1}),(\mathbf{L_2},\mathbf{L'_2}),\ldots,(\mathbf{L_n},\mathbf{L'_n}))$$

where $\mathbf{m_i}$ is the number of elements in $\mathbf{P^0_{L_i}}$ or $\mathbf{Q^0_{L'_i}}$ and $\mathbf{m_j}$ is the number of elements in $\mathbf{P^0_{L_j}}$ or $\mathbf{Q^0_{L'_j}}$ for all $\mathbf{i}=1,2,\ldots,n,\ \mathbf{j}=1,2,\ldots,n$. □

Theorem 3.33. *If $\mathbf{P^1_{L_1}},\mathbf{P^1_{L_2}},\ldots,\mathbf{P^1_{L_n}}$ and $\mathbf{Q^1_{L'_1}},\mathbf{Q^1_{L'_2}},\ldots,\mathbf{Q^1_{L'_n}}$ are 1-dimensional persistence diagrams of weighted undirected graphs $\mathbf{L_1},\mathbf{L_2},\ldots,\mathbf{L_n}$ and $\mathbf{L'_1},\mathbf{L'_2},\ldots,\mathbf{L'_n}$ respectively. If $\mathbf{P^1_{L_1\boxtimes L_2\boxtimes\ldots\boxtimes L_n}}$ be the 1-dim persistence diagram of $\mathbf{L_1}\boxtimes\mathbf{L_2}\boxtimes\ldots\boxtimes\mathbf{L_n}$ and*

$\mathbf{Q^1_{L'_1 \boxtimes L'_2 \boxtimes \ldots \boxtimes L'_n}}$ *be the* 1*-dim persistence diagram of* $\mathbf{L'_1} \boxtimes \mathbf{L'_2} \boxtimes \ldots \boxtimes \mathbf{L'_n}$ *then*

$$d_{wp}(\mathbf{P^1_{L_1 \boxtimes L_2 \boxtimes \ldots \boxtimes L_n}}, \mathbf{Q^1_{L'_1 \boxtimes L'_2 \boxtimes \ldots \boxtimes L'_n}}) \leq n^{n-1}\{d_{wp}(\mathbf{P^1_{L_1}}, \mathbf{Q^1_{L_1}}) + d_{wp}(\mathbf{P^1_{L_2}}, \mathbf{Q^1_{L_2}}) + \ldots + d_{wp}(\mathbf{P^1_{L_n}}, \mathbf{Q^1_{L_n}})\}$$

where n *is the number of vertices in each graph.*

Proof. Let $\mathbf{P^1_{L_1}}, \mathbf{P^1_{L_2}}, \ldots, \mathbf{P^1_{L_n}}$ and $\mathbf{Q^1_{L'_1}}, \mathbf{Q^1_{L'_2}}, \ldots, \mathbf{Q^1_{L'_n}}$ are 1-dimensional persistence diagrams of $\mathbf{L_1}, \mathbf{L_2}, \ldots, \mathbf{L_2}$ and $\mathbf{L'_1}, \mathbf{L'_2}, \ldots, \mathbf{L'_n}$ respectively. Also let $\mathbf{P^0_{L_1}}, \mathbf{P^0_{L_2}}, \ldots, \mathbf{P^0_{L_n}}$ and $\mathbf{Q^0_{L'_1}}, \mathbf{Q^0_{L'_2}}, \ldots, \mathbf{Q^0_{L'_n}}$ are 0-dimensional persistence diagrams of $\mathbf{L_1}, \mathbf{L_2}, \ldots, \mathbf{L_n}$ and $\mathbf{L'_1}, \mathbf{L'_2}, \ldots, \mathbf{L'_n}$ respectively. Here we can find n^{n-1} copies of $\mathbf{L_1}$ fibres, n^{n-1} copies of $\mathbf{L_2}$ fibres $\ldots n^{n-1}$ copies of $\mathbf{L_n}$ fibres in $\mathbf{L_1} \boxtimes \mathbf{L_2} \boxtimes \ldots \boxtimes \mathbf{L_n}$. In similar way we can find n^{n-1} copies of $\mathbf{L'_1}$ fibres, n^{n-1} copies of $\mathbf{L'_2}$ fibres $\ldots n^{n-1}$ copies of $\mathbf{L'_n}$ fibres in $\mathbf{L'_1} \boxtimes \mathbf{L'_2} \boxtimes \ldots \boxtimes \mathbf{L'_n}$. So the only one dimensional loops in the strong product will be the loops present in $\mathbf{L_1}, \mathbf{L_2}, \ldots, \mathbf{L_n}$ and $\mathbf{L'_1}, \mathbf{L'_2}, \ldots, \mathbf{L'_n}$ in n^{n-1} copies. If we are taking the bijection as per theorem 3.12 we can say that

$$d_{wp}(\mathbf{P^1_{L_1 \boxtimes L_2 \boxtimes \ldots \boxtimes L_n}}, \mathbf{Q^1_{L'_1 \boxtimes L'_2 \boxtimes \ldots \boxtimes L'_n}}) \leq n^{n-1}\{d_{wp}(\mathbf{P^1_{L_1}}, \mathbf{Q^1_{L_1}}) + d_{wp}(\mathbf{P^1_{L_2}}, \mathbf{Q^1_{L_2}}) + \cdots + d_{wp}(\mathbf{P^1_{L_n}}, \mathbf{Q^1_{L_n}})\}$$

where n is the number of vertices in each graph. □

4. Conclusions and future directions of research

In the study, the Cartesian product and the strong product of weighted and unweighted graphs were defined and analyzed. Additionally, the paper introduced the concepts of clique and Vietoris filtration for the Cartesian and strong products of these graphs. By finding the relationship between the Wasserstein distance of Cartesian products and the Wasserstein distance of individual graphs, the study revealed similar results for the strong product of weighted and unweighted graphs. In future research, it is suggested to explore the direct product and the lexicographic product as potential techniques for reducing the complexity of comparing large networks. These product operations could offer alternative approaches for analyzing and comparing complex networks, particularly those that can be represented as lexicographic or direct products.

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