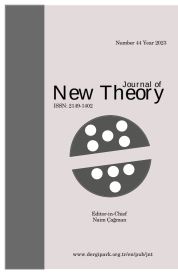
Journal of
New Theory

ISSN: 2149-1402

44 (2023) 20-30
Journal of New Theory
https://dergipark.org.tr/en/pub/jnt
Open Access



An Extension of the UEHL Distribution Based on the DUS Transformation

Murat Genç[1], Ömer Özbilen[2]

Article Info
Received: 20 Jun 2023
Accepted: 18 Sep 2023
Published: 30 Sep 2023
doi:10.53570/jnt.1317652
Research Article

Abstract − In this study, we propose a new distribution based on the Dinesh, Umesh, and Sanjay (DUS) transformation by using the Unit Exponentiated Half-Logistic (UEHL) distribution as the baseline distribution, a member of the family of proportional hazard rate models. Moreover, we study several properties, such as moments, skewness, kurtosis, stress-strength reliability, and likelihood ratio ordering. Further, we discuss the statistical inference on the parameters of the proposed distribution by the maximum likelihood estimation (MLE) method. Besides, we conduct a simulation based on the new distribution to investigate the behavior of the maximum likelihood estimates in various conditions. Furthermore, we present a numerical example to show the performance of the distribution on a real-life data set. Finally, we discuss the need for further research.

Keywords *DUS transformation, UEHL distribution, maximum likelihood estimator, data analysis*

Mathematics Subject Classification (2020) 62E10, 60E05

1. Introduction

A variety of models exist in the literature for examining lifetime data. The Weibull distribution originated as an extension of the exponential distribution. The distribution is more flexible in shape than the exponential distribution. This property makes the Weibull distribution useful for lifetime modeling and modeling data from economics and business administration [1-3]. In reliability theory, the Weibull distribution is convenient for modeling monotonic hazard rates. However, it does not provide a reasonable parametric fit for non-monotonic failure rates [4, 5]. To improve the ability to model non-monotonic failure rates, several generalizations and modifications of the Weibull distribution have been introduced in the literature [6, 7].

As an alternative to the Weibull distribution, the omega distribution was introduced as a new probability distribution, and its applications in reliability theory were discussed [8]. Dombi et al. [8] showed that the asymptotic omega hazard rate function is the Weibull hazard rate function, which implies that the asymptotic omega distribution is just the Weibull distribution. Moreover, the omega distribution belongs to the class of proportional hazard rate models. Özbilen and Genç [9] introduced the unit exponentiated-half logistic (UEHL) distribution corresponding to the omega distribution with a special case of the parameter value of the distribution. The distribution corresponds to the exponentiated half-logistic distribution with a simple transformation, which has many applications in reliability theory [10, 11].

In statistical literature, there are various ways to generate new distributions using some baseline distributions [12, 13]. In this context, Kumar et al. [14] proposed the Dinesh, Umesh, and Sanjay (DUS) transformation to

[1]muratgenc@tarsus.edu.tr (Corresponding Author); [2]ozbilen@mersin.edu.tr
[1]Department of Management Information Systems, Faculty of Economics and Administrative Sciences, Tarsus University, Mersin, Türkiye
[2]Department of Geomatic Engineering, Faculty of Engineering, Mersin University, Mersin, Türkiye

obtain a new distribution family. Let $f(x)$ and $F(x)$ denote the probability density function (PDF) and cumulative distribution function (CDF) of the baseline distribution, respectively. Then, the PDF and CDF of the DUS family are given by

$$f_{DUS}(x) = \frac{f(x)e^{F(x)}}{e-1}, \quad x \in D \tag{1}$$

and

$$F_{DUS}(x) = \frac{e^{F(x)}-1}{e-1} \tag{2}$$

respectively. In Equation (1), D is the domain of the baseline distribution. As observed from Equations (1) and (2) that the DUS family does not contain any new parameters in addition to those used in the baseline distribution. Hence, the DUS family results in a parsimonious distribution in terms of computation and interpretation.

There exist several studies in the literature where the DUS transformation is used. Kumar et al. [14] applied the DUS transformation to an exponential CDF. Deepthi and Chacko [15] proposed the DUS-Lomax distribution, considering the Lomax distribution as the baseline distribution for the DUS transformation. Kavya and Manoharan [16] obtained the DUS-Weibull distribution with the same approach. Maurya et al. [17] proposed a generalization of the DUS transformation and studied the exponential baseline in the proposed method. Karakaya et al. [18] considered the DUS transformation with the Kumaraswamy distribution and investigated the properties of the distribution.

The performance of the UEHL distribution can be improved by applying the DUS transformation to the CDF of the UEHL distribution. In this paper, we use the DUS transformation to the UEHL distribution to enhance distribution in the scope of modeling proportional data.

The rest of the paper is organized as follows: Section 2 introduces the UEHL distribution with DUS transformation and obtains the survival and hazard rate functions. Section 3 discusses some analytical characteristics, such as the moments, quantile function, stress-strength reliability, likelihood ratio ordering, and the maximum likelihood estimation (MLE) of the method. Section 4 performs a simulation study to validate the performance of the maximum likelihood estimates of the distribution. Section 5 provides a numerical example to illustrate the performance of the proposed distribution in modeling the real data. The last section concludes the paper.

2. DUS-UEHL Distribution

Recently, the UEHL distribution [9] has been proposed as a special case of omega distribution [8], a member of the class of proportional hazard rate models. The distribution corresponds to the transformation $Z = e^{-X}$ where X has the exponentiated half-logistic distribution. The UEHL distribution has many applications in reliability theory through exponentiated half-logistic distribution [10, 19, 20]. This section applies the DUS transformation to the UEHL distribution to introduce a new class of distributions. The PDF and CDF of the UEHL distribution are given as follows, respectively:

$$f_{UEHL}(x) = 2\lambda\theta x^{\theta-1}\frac{\left(1-x^{\theta}\right)^{\lambda-1}}{(1+x^{\theta})^{\lambda+1}}, \quad 0 < x < 1 \tag{3}$$

and

$$F_{UEHL}(x) = 1 - \left(\frac{1-x^{\theta}}{1+x^{\theta}}\right)^{\lambda}, \quad 0 < x < 1 \tag{4}$$

where $\theta > 0$ and $\lambda > 0$ are the scale and shape parameters of the distribution, respectively. By applying the transformations given in Equations (1) and (2) to Equations (3) and (4), the PDF and CDF of the new distribution are obtained as follows, respectively:

$$f_{D-UEHL}(x) = \frac{1}{e-1} 2\lambda\theta x^{\theta-1} \frac{\left(1-x^{\theta}\right)^{\lambda-1}}{(1+x^{\theta})^{\lambda+1}} e^{1-\left(\frac{1-x^{\theta}}{1+x^{\theta}}\right)^{\lambda}} \tag{5}$$

and

$$F_{D-UEHL}(x) = \frac{1}{e-1}\left(e^{1-\left(\frac{1-x^{\theta}}{1+x^{\theta}}\right)^{\lambda}} - 1\right) \tag{6}$$

The random variable X with the CDF in Equation (6) is said to follow the two-parameter DUS-UEHL distribution and is denoted by $D\text{-}UEHL(\theta,\lambda)$. Considering the derivative of PDF in Equation 5, we obtain

$$f'_{D-UEHL}(x) = \frac{2\lambda\theta x^{\theta-2}\left(-2\lambda\theta x^{\theta} + (\theta+1)x^{2\theta} + \theta - 1\right)\left(\frac{2}{x^{\theta}+1} - 1\right)^{\lambda}}{(x^{2\theta}-1)^2}$$

which has the critical points $x_{1,2} = \left(\frac{(\lambda\theta \pm \sqrt{\lambda^2\theta^2 - \theta^2 + 1})}{(\theta+1)}\right)^{1/\theta}$ and the extreme points 0 and 1. Given $\theta > 1$, this indicates that the $D\text{-}UEHL(\theta,\lambda)$ distribution is unimodal with the unique mode at x_1 or x_2 depending on the scale and shape parameters of the distribution. Moreover, the PDF can be an increasing, decreasing, or U-shaped function of x depending on the values of the PDF at the extreme points.

Figure 1 shows the PDF of the $D\text{-}UEHL(\theta,\lambda)$ distribution for various values of the parameters θ and λ. According to Figure 1, the $D\text{-}UEHL(\theta,\lambda)$ distribution can take different shapes, including unimodal, increasing, decreasing, and U-shaped functions.

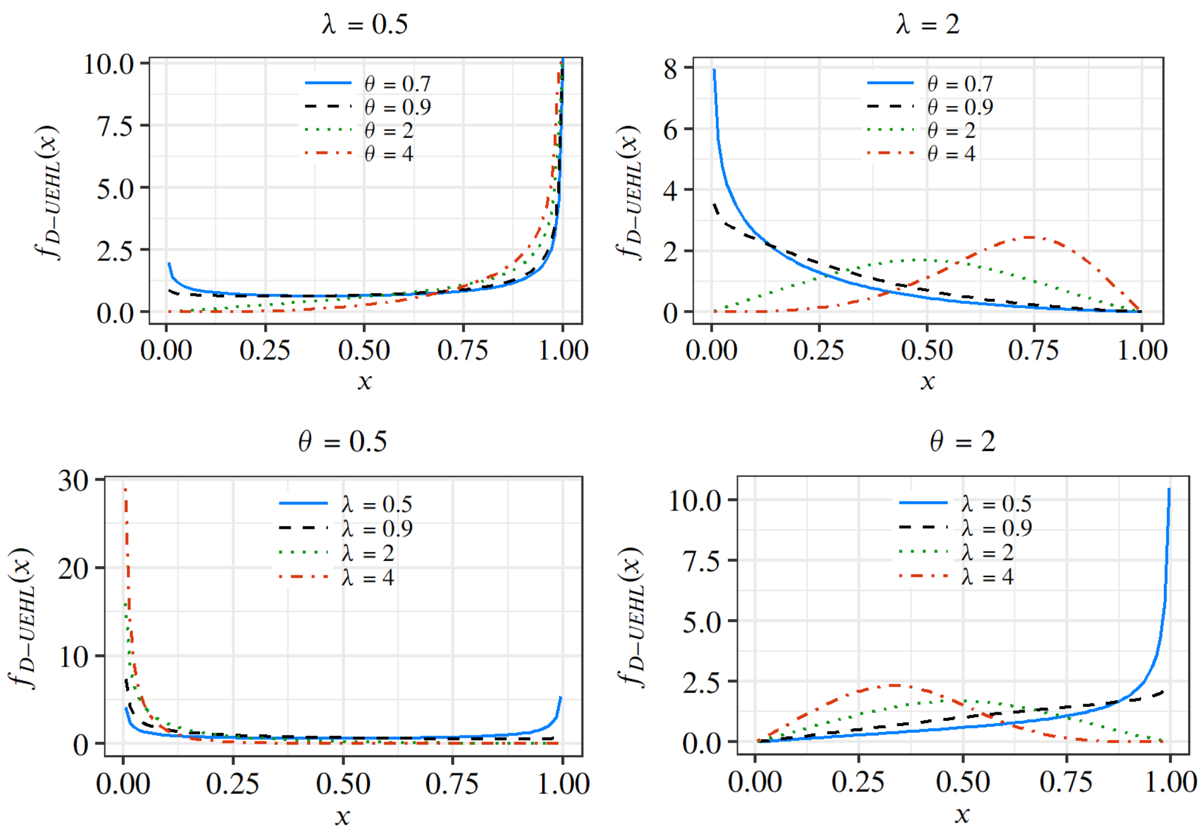


Figure 1. PDF plots of the $D\text{-}UEHL(\theta,\lambda)$ distribution for various values of the parameters θ and λ

The survival function and the hazard rate functions of the $D\text{-}UEHL$ distribution are provided as follows, respectively:

$$S_{D-UEHL}(x) = \frac{e - e^{1-\left(\frac{1-x^\theta}{1+x^\theta}\right)^\lambda}}{e-1} \tag{7}$$

and

$$h_{D-UEHL}(x) = \frac{2\lambda\theta x^{\theta-1}\frac{(1-x^\theta)^{\lambda-1}}{(1+x^\theta)^{\lambda+1}} e^{1-\left(\frac{1-x^\theta}{1+x^\theta}\right)^\lambda}}{e - e^{1-\left(\frac{1-x^\theta}{1+x^\theta}\right)^\lambda}} \tag{8}$$

Figure 2 manifests the hazard rate function for selected values of the parameters θ and λ. According to Figure 2, the hazard function of $D\text{-}UEHL(\theta,\lambda)$ is an increasing function.

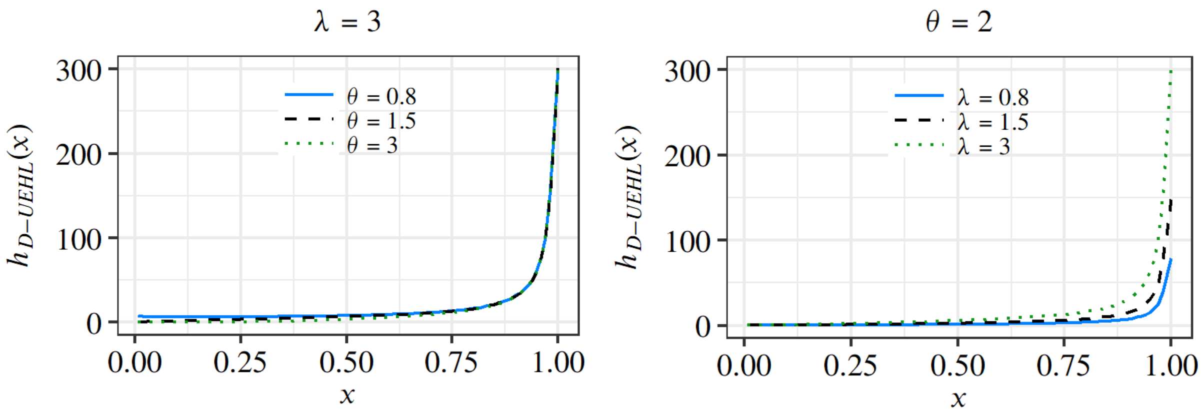


Figure 2. Hazard rate function plots of the $D\text{-}UEHL(\theta,\lambda)$ distribution for various values of the parameters θ and λ

3. Some Analytical Characteristics of the $D\text{-}UEHL(\theta,\lambda)$ Distribution

This section discusses some statistical characteristics, including the moments, quantile function, stress-strength reliability, likelihood ratio ordering, and MLE of the $D\text{-}UEHL(\theta,\lambda)$ distribution.

3.1. Moments

The moments are useful for understanding the various characteristics of a statistical distribution. In this section, we consider the moments of a $D\text{-}UEHL(\theta,\lambda)$ random variable. Let X be a $D\text{-}UEHL(\theta,\lambda)$ random variable with PDF given by Equation (5). Then, for $r \in \{1,2,3,\cdots\}$, the r-th raw moment of X is

$$E(X^r) = \frac{2\lambda e}{e-1}\int_0^1 x^r \left(\theta x^{\theta-1}\frac{(1-x^\theta)^{\lambda-1}}{(1+x^\theta)^{\lambda+1}}\right) e^{1-\left(\frac{1-x^\theta}{1+x^\theta}\right)^\lambda} dx$$

By substituting $v = x^\theta$, the expectation is written as

$$E(X^r) = \frac{2\lambda e}{e-1}\int_0^1 v^{r/\theta}(1-v)^{\lambda-1}(1+v)^{-\lambda-1} e^{-\left(\frac{1-v}{1+v}\right)^\lambda} dv$$

By using the Taylor expansion $e^x = \sum_{j=0}^\infty \frac{x^j}{j!}$,

$$E(X^r) = \frac{2\lambda e}{e-1}\sum_{i=0}^{\infty}\left[\frac{(-1)^i}{i!}\int_0^1 \left(v^{r/\theta}(1-v)^{\lambda(1+i)-1}(1+v)^{-\lambda(1+i)-1}\right)dv\right]$$

Using Equation 3.197.3 provided in [21], i.e., $\int_0^1 x^{\lambda-1}(1-x)^{\mu-1}(1-\beta x)^{-\nu}dx = B(\lambda,\mu)\ {}_2F_1(\nu,\lambda;\lambda+\mu;\beta)$ such that Re $\lambda > 0$, Re $\mu > 0$, and $|\beta| < 1$, the r-th raw moment is obtained as

$$E(X^r) = \frac{2\lambda e}{e-1}\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}B\left(1+\frac{r}{\theta},\lambda(1+i)\right){}_2F_1\left(\lambda(1+i)+1,1+\frac{r}{\theta};1+\frac{r}{\theta}+\lambda(1+i);-1\right)$$

where

$$B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}\,dt \quad \text{and} \quad {}_2F_1(a,b;c;z) = \sum_{\mathrm{n}=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!}$$

are the beta and Gauss hypergeometric functions, respectively, and $(a)_n = a(a+1)\cdots(a+n+1)$. Table 1 shows the mean, variance, skewness, and kurtosis of $D\text{-}UEHL(\theta,\lambda)$ distribution for selected pairs of (θ,λ). The results indicate that the $D\text{-}UEHL(\theta,\lambda)$ has various shapes depending on the values of the distribution parameters.

Table 1. Mean, variance, skewness, and kurtosis of the $D\text{-}UEHL(\theta,\lambda)$ distribution

θ	λ	Mean	Variance	Skewness	Kurtosis
0.7	0.7	0.4893	0.1057	0.0478	1.6247
	1.5	0.2600	0.0557	0.9766	3.0931
	3	0.1179	0.0168	1.8289	6.8839
1.5	0.7	0.6574	0.0725	-0.5416	2.1708
	1.5	0.4688	0.0573	0.1228	2.1052
	3	0.3153	0.0322	0.5336	2.7771
3	0.7	0.7874	0.0374	-1.0363	3.4153
	1.5	0.6571	0.0369	-0.4670	2.5985
	3	0.5355	0.0285	-0.1805	2.5863

3.2. Quantile Function

The quantile function of the $D\text{-}UEHL(\theta,\lambda)$ distribution is given by

$$Q_{D-UEHL}(u;\theta,\lambda) = F^{-1}(u;\theta,\lambda) = \left[2\left(1+\left(1-\log\left(1+u(e-1)\right)\right)^{1/\lambda}\right)^{-1}-1\right]^{1/\theta} \tag{9}$$

Based on Equation (9), the median of the $D\text{-}UEHL(\theta,\lambda)$ distribution is given by the following function, a function of the parameters θ and λ:

$$Q_{D-UEHL}(0.5;\theta,\lambda) = \left[2\left(1+(1-\log 2-\log{(1+e)})\right)^{1/\lambda}\right)^{-1}-1\right]^{1/\theta} \tag{10}$$

The median values of the $D\text{-}UEHL(\theta,\lambda)$ distribution for selected parameter values are provided in Table 2.

Table 2. Median of the $D\text{-}UEHL(\theta,\lambda)$ distribution

	θ	0.1	0.5	1	2.5	5	10
	0.1	0.99874892	0.05466397	0.00033596	0.00000007	$< 10^{-8}$	$< 10^{-8}$
	0.5	0.99974966	0.55916532	0.20195643	0.03655548	0.00930982	0.00233835
λ	1	0.99987482	0.74777357	0.44939563	0.19119487	0.09648743	0.04835653
	2.5	0.99994993	0.89024193	0.72619213	0.51592966	0.39245358	0.29770219
	5	0.99997496	0.94352633	0.85216907	0.71828244	0.62646116	0.54562093
	10	0.99998748	0.97135283	0.92313004	0.84751545	0.79149299	0.73866158

3.3. Stress-Strength Reliability

Given the stress and strength random variables, Y and X, the stress-strength reliability is defined as $R = P(Y < X)$. In this section, we compute the stress-strength reliability for the model $D\text{-}UEHL(\theta, \lambda)$.

Proposition 1. Let Y and X be independent stress and strength random variables following $D\text{-}UEHL$ distribution with parameters (θ, λ_1) and (θ, λ_2), respectively. Then, the stress-strength reliability is

$$R = \frac{e^2}{(e-1)^2} \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} \gamma\left(\frac{\lambda_1}{\lambda_2} j + 1,1\right) - \frac{1}{e-1} \tag{11}$$

where $\lambda_1 > 0$, $\lambda_2 > 0$, and $\gamma(a, x) = \int_a^x t^{a-1} e^{-t}\, dt$ is the incomplete gamma function.

Proof.

By the definition, the stress-strength reliability can be written as

$$\begin{aligned} R &= \int_0^1 P(Y < X \mid X = x) f_x(x) dx \\ &= \int_0^1 \frac{1}{e-1}\left(e^{1-\left(\frac{1-x^\theta}{1+x^\theta}\right)^{\lambda_1}} - 1\right)\left(\frac{1}{e-1} 2\lambda_2 \theta x^{\theta-1} \frac{\left(1-x^\theta\right)^{\lambda_2 - 1}}{(1+x^\theta)^{\lambda_2+1}} e^{1-\left(\frac{1-x^\theta}{1+x^\theta}\right)^{\lambda_2}}\right) dx \end{aligned}$$

By substituting $t = \left(\frac{1-x^\theta}{1+x^\theta}\right)^{\lambda_1}$ and applying the Taylor expansion of e^x,

$$\begin{aligned} R &= \frac{e}{(e-1)^2} \int_0^1 \left(e^{-t^{\lambda_1/\lambda_2}+1} - 1\right) e^{-t}\; dt \\ &= \frac{e^2}{(e-1)^2} \int_0^1 \left(\sum_{j=0}^{\infty} \frac{(-1)^j t^{\lambda_1 j/\lambda_2}}{j!} - \frac{1}{e}\right) e^{-t}\; dt \\ &= \frac{e^2}{(e-1)^2} \sum_{j=0}^{\infty} \left\{\frac{(-1)^j}{j!}\left(\int_0^1 t^{\lambda_1 j/\lambda_2} e^{-t}\; dt\right)\right\} - \frac{1}{e-1} \\ &= \frac{e^2}{(e-1)^2} \sum_{j=0}^{\infty} \left\{\frac{(-1)^j}{j!} \gamma\left(\frac{\lambda_1}{\lambda_2} j + 1,1\right)\right\} - \frac{1}{e-1} \end{aligned}$$

□

3.4. Likelihood Ratio Ordering

Let X and Y be random variables with PDFs $f_X(\cdot)$ and $f_Y(\cdot)$, respectively. If $\frac{f_X(x)}{f_Y(x)}$ is non-decreasing in x, then it is said to be the random variable X is smaller than Y in likelihood ratio ordering and denoted by $X \leq_{lr} Y$. Proposition 2 gives the property of likelihood ratio ordering for the $D\text{-}UEHL$ distribution.

Proposition 2. Let $X \sim D\text{-}UEHL(\theta_1, \lambda_1)$ and $Y \sim D\text{-}UEHL(\theta_2, \lambda_2)$. If $\theta_1 = \theta_2$, then $X \leq_{lr} Y$.

PROOF.

Let $\theta_1 = \theta_2 = \theta$. The ratio of the PDFs of X and Y is given by

$$g(x) = \frac{2\lambda_1 \theta x^{\theta-1} \frac{\left(1-x^{\theta}\right)^{\lambda_1-1}}{(1+x^{\theta})^{\lambda_1+1}}}{2\lambda_2 \theta x^{\theta-1} \frac{(1-x^{\theta})^{\lambda_2-1}}{(1+x^{\theta})^{\lambda_2+1}}}$$

After simplifications, we obtain $g'(x) < 0$, for $0 < x < 1$ and $\theta > 0$. Hence, $g(\cdot)$ is a non-decreasing function of x, which completes the proof.

3.5. Maximum Likelihood Estimation

Let $X_1, X_2, \cdots, X_n$ be an identically independent distributed sample from $D\text{-}UEHL(\theta, \lambda)$. Then, the likelihood and log-likelihood functions are written as

$$L(\theta, \lambda) = \prod_{i=1}^{n} \left(\frac{1}{e-1} 2\lambda\theta x_i^{\theta-1} \frac{\left(1-x_i^{\theta}\right)^{\lambda-1}}{\left(1+x_i^{\theta}\right)^{\lambda+1}} e^{1-\left(\frac{1-x_i^{\theta}}{1+x_i^{\theta}}\right)^{\lambda}} \right) \tag{12}$$

and

$$\begin{aligned} \ell(\theta, \lambda) = &-n\log(e-1) + n(1+\log 2) + n\log\lambda + n\log\theta + (\theta-1)\sum_{i=1}^{n}\log(x_i) \\ &+(\lambda-1)\sum_{i=1}^{n}\log\left(1-x_i^{\theta}\right) - (\lambda+1)\sum_{i=1}^{n}\log\left(1+x_i^{\theta}\right) - \sum_{i=1}^{n}\left(\frac{1-x_i^{\theta}}{1+x_i^{\theta}}\right)^{\lambda} \end{aligned} \tag{13}$$

respectively. Differentiating the log-likelihood function with respect to the parameters, we get the log-likelihood equations as

$$\frac{\partial \ell(\theta,\lambda)}{\partial\theta} = \frac{n}{\theta} + \sum_{i=1}^{n}\log(x_i) - (\lambda-1)\sum_{i=1}^{n}\frac{x_i^{\theta}\log x_i}{1-x_i^{\theta}} + (\lambda+1)\sum_{i=1}^{n}\frac{x_i^{\theta}\log x_i}{1+x_i^{\theta}} + 2\theta\sum_{i=1}^{n}\frac{x_i^{\theta-1}}{\left(1+x_i^{\theta}\right)^2} = 0 \tag{14}$$

and

$$\frac{\partial \ell(\theta,\lambda)}{\partial\lambda} = \frac{n}{\lambda} + \sum_{i=1}^{n}\log\left(\frac{1-x_i^{\theta}}{1+x_i^{\theta}}\right) - \sum_{i=1}^{n}\left(\frac{1-x_i^{\theta}}{1+x_i^{\theta}}\right)^{\lambda}\log\left(\frac{1-x_i^{\theta}}{1+x_i^{\theta}}\right) = 0 \tag{15}$$

Equations (14) and (15) do not have a closed-form solution. To solve these equations, some iterative methods can be employed. We apply the optim procedure in R to obtain the solution of the equation system.

4. Simulation Study

In this section, a simulation study is carried out to examine the properties of the MLE, which we discussed in detail in Section 3. Table 3 shows the biases and mean squared errors (MSEs) of the parameter estimates with 5000 replications for various values of the parameters (n, θ, λ). According to the results in Table 3, the MLEs are asymptotically unbiased. Moreover, the bias and MSE values of the MLEs decrease to zero as the sample size increases as desired.

Table 3. Bias and MSEs of MLE estimators for selected parameter values

			Bias		MSE	
θ	λ	n	$\hat{\theta}$	$\hat{\lambda}$	$\hat{\theta}$	$\hat{\lambda}$
2	2	50	0.06108	0.10841	0.09290	0.19997
		100	0.02521	0.04550	0.04196	0.08155
		200	0.01127	0.02281	0.01972	0.03706
		300	0.00690	0.01386	0.01311	0.02379
		500	0.00304	0.00752	0.00779	0.01394
3	1.5	50	0.09989	0.06955	0.24153	0.09187
		100	0.04119	0.02920	0.10856	0.03866
		200	0.01851	0.01493	0.05089	0.01787
		300	0.01147	0.00909	0.03392	0.01148
		500	0.00511	0.00496	0.02015	0.00676
1.5	0.8	50	0.06884	0.02793	0.09443	0.01843
		100	0.02873	0.01179	0.04116	0.00817
		200	0.01308	0.00629	0.01895	0.00389
		300	0.00834	0.00381	0.01267	0.00250
		500	0.00396	0.00216	0.00753	0.00148

5. Real Data Application

In this section, we investigate the flexibility performance of the $D\text{-}UEHL(\theta, \lambda)$ distribution with a real data application. For this purpose, we consider the reservoir data set obtained from the monthly water capacity of the Shasta Reservoir in California [22]. We compare the $D\text{-}UEHL(\theta, \lambda)$ distribution with well-known Weibull, Beta, Kumaraswamy, and UEHL distributions. The PDFs of the distributions used for comparison are given as follows:

Weibull distribution

$$f_W(x; \theta, \lambda) = \frac{\theta}{\lambda}\left(\frac{x}{\lambda}\right)^{\theta-1} e^{-(x/\lambda)^{\theta}}, \quad \theta, \lambda > 0$$

Beta distribution

$$f_B(x; \theta, \lambda) = \frac{1}{B(\theta, \lambda)} x^{\theta-1}(1-x)^{\lambda-1}, \quad \theta, \lambda > 0$$

Kumaraswamy distribution

$$f_{Kw}(x; \theta, \lambda) = \theta\lambda x^{\theta-1}\left(1-x^{\theta}\right)^{\lambda-1}, \quad \theta, \lambda > 0$$

We use the maximum likelihood method to estimate the unknown parameters of the considered distributions. We test the goodness of fits of the models with Kolmogorov-Smirnov test statistic (K-S (stat)) and associate p-value (K-S (p-value)). We report the Akaike information criterion (AIC) and Bayesian information criterion (BIC) to compare the proposed model with the other models. AIC and BIC are computed as

$$\text{AIC} = 2k - 2\ell(\theta, \lambda) \quad \text{and} \quad \text{BIC} = k \log n - 2\ell(\theta, \lambda)$$

where k is the number of parameters, n is the number of observations, and ℓ is the maximum value of the likelihood function for the underlying distribution.

Table 4. Bias and MSEs of MLE estimators for selected parameter values

	θ	λ	AIC	BIC	-2ℓ	K-S (stat)	K-S (p-value)
Weibull	7.2987	0.7748	-20.5347	-18.5432	-24.5347	0.2220	0.2396
Beta	7.3157	2.9099	-21.1238	-19.1324	-25.1238	0.2359	0.1834
Kumaraswamy	6.3476	4.4894	-22.9494	-20.9580	-26.9494	0.2209	0.2447
UEHL	6.9010	2.8883	-21.2699	-19.2784	-25.2699	0.2254	0.2248
D-UEHL	6.4316	3.2682	-23.0543	-21.0629	-27.0543	0.2046	0.3267

Table 4 shows the parameter estimates, AIC values, BIC values, and -2ℓ values together with K-S (stat) and K-S (p-value) for all models fitted to the reservoir data set. The K-S test results indicate that the compared distributions are appropriate to fit the underlying data. From Table 4, it can be observed that the $D\text{-}UEHL(\theta,\lambda)$ distribution yields the lowest AIC and BIC values, followed by the Kumaraswamy distribution. Hence, we conclude that the $D\text{-}UEHL(\theta,\lambda)$ model has the best fit for the reservoir data set among the compared distributions in the study. Moreover, Figure 3 manifests the empirical and fitted curves based on the reservoir data set for illustrative purposes.

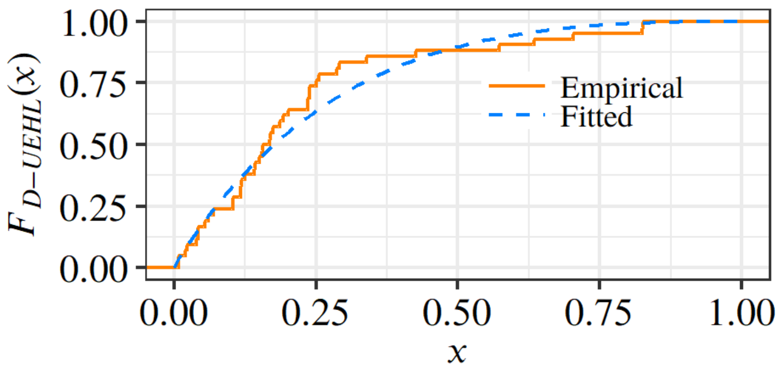


Figure 3: Empirical and fitted distribution functions based on the reservoir data set

6. Conclusion

In this study, we introduce $D\text{-}UEHL$ distribution for record data based on DUS transformation on unit exponentiated half-logistic distribution CDF. We obtain various analytical characteristics, including moments, quantile function, stress-strength reliability, and likelihood ratio ordering of the proposed distribution. We perform a simulation study that illustrates the properties of the maximum likelihood estimates of the $D\text{-}UEHL$ distribution. The real data analysis on a record data shows that the $D\text{-}UEHL$ model performs better on the data than the other well-known models in terms of AIC and BIC criteria.

The findings of this article can be extended by applying the DUS transform to the omega distribution. In this context, the effect of the extra support parameter of the omega distribution with the DUS transformation is a topic for further research.

Author Contributions

All the authors equally contributed to this work. They all read and approved the final version of the paper.

Conflict of Interest

All the authors declare no conflict of interest.

References

[1] D. P. Murthy, M. Xie, R. Jiang, Weibull Models, Wiley, New Jersey, 2004.

[2] J. I. McCool, Using the Weibull Distribution: Reliability, Modeling, and Inference, Wiley, New Jersey, 2012.

[3] M. Aslam, M. Azam, S. Balamurali, C. H. Jun, *An Economic Design of a Group Sampling Plan for a Weibull Distribution Using a Bayesian Approach*, Journal of Testing and Evaluation 43 (6) (2015) 1497–1503.

[4] S. J. Almalki, S. Nadarajah, *Modifications of the Weibull Distribution: A Review*, Reliability Engineering & System Safety 124 (1) (2014) 32–55.

[5] J. M. Carrasco, E. M. Ortega, G. M. Cordeiro, *A Generalized Modified Weibull Distribution for Lifetime Modeling*, Computational Statistics & Data Analysis 53 (2) (2008) 450–462.

[6] H. Pham, C. D. Lai, *On Recent Generalizations of the Weibull Distribution*, IEEE Transactions on Reliability 56 (2007) (3) 454–458.

[7] C. D. Lai, Generalized Weibull Distributions, Springer, Berlin, Heidelberg, 2013.

[8] J. Dombi, T. Jonas, Z. E. Toth, G. Arva, *The Omega Probability Distribution and Its Applications in Reliability Theory*, Quality and Reliability Engineering International 35 (2) (2019) 600–626.

[9] Ö. Özbilen, A. İ. Genç, *A Bivariate Extension of the Omega Distribution for Two-Dimensional Proportional Data*, Mathematica Slovaca 72 (6) (2022) 1605–1622.

[10] J. I. Seo, S. B. Kang, *Notes on the Exponentiated Half Logistic Distribution*, Applied Mathematical Modelling 39 (21) (2015) 6491–6500.

[11] W. Gui, *Exponentiated Half Logistic Distribution: Different Estimation Methods and Joint Confidence Regions*, Communications in Statistics - Simulation and Computation 46 (6) (2017) 4600–4617.

[12] R. C. Gupta, P. L. Gupta, R. D. Gupta, *Modeling Failure Time Data by Lehman Alternatives*, Communications in Statistics-Theory and Methods 27 (4) (1998) 887–904.

[13] G. M. Cordeiro, M. Castro. *A New Family of Generalized Distributions*, Journal of Statistical Computation and Simulation 81 (7) (2011) 883–898.

[14] D. Kumar, U. Singh, S. K. Singh, *A Method of Proposing New Distribution and Its Application to Bladder Cancer Patients Data*, Journal of Statistics Applications & Probability Letters 2 (3) (2015) 235–245.

[15] K. S. Deepthi, V. M. Chacko, *An Upside-Down Bathtub-Shaped Failure Rate Model Using a DUS Transformation of Lomax Distribution*, in: L. Cui, I. B. Frenkel, A. Lisnianski (Eds.), Stochastic Models in Reliability Engineering, CRC Press, Boca Raton, 2020, Ch. 6, pp. 81–100.

[16] P. Kavya, M. Manoharan, *On a Generalized Lifetime Model Using DUS Transformation*, in: V. C. Joshua, S. R. S. Varadhan, V. M. Vishnevsky (Eds.), Applied Probability and Stochastic Processes, Springer, Singapore, 2020, pp. 281–291.

[17] S. Maurya, A. Kaushik, S. Singh, U. Singh, *A New Class of Distribution Having Decreasing, Increasing, and Bathtub-Shaped Failure Rate*, Communications in Statistics-Theory and Methods 46 (20) (2017) 10359–10372.

[18] K. Karakaya, İ. Kınacı, K. Coşkun, Y. Akdoğan, *On the DUS-Kumaraswamy Distribution*, Istatistik Journal of the Turkish Statistical Association 13 (1) (2021) 29–38.

[19] S. B. Kang, J. I. Seo, *Estimation in an Exponentiated Half Logistic Distribution under Progressively Type-2 Censoring*, Communications for Statistical Applications and Methods 18 (5) (2011) 657–666.

[20] M. K. Rastogi, Y. M. Tripathi, *Parameter and Reliability Estimation for an Exponentiated Half-Logistic Distribution Under Progressive Type-II Censoring*, Journal of Statistical Computation and Simulation 84 (8) (2014) 1711–1727.

[21] I. S. Gradshteyn, I. M. Ryzhik, Table of Integrals, Series, and Products, 7th edition, Academic Press, San Diego, 2007.

[22] M. Nadar, A. Papadopoulos, F. Kızılaslan, *Statistical Analysis for Kumaraswamy's Distribution Based on Record Data*, Statistical Papers 54 (2013) 355–369.