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Advancing modal analysis of mechanical linkages via differential algebraic equations

Diferansiyel denklemler yoluyla mekanik bağlantıların modal analizinin geliştirilmesi

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Advancing Modal Analysis of Mechanical Linkages via Differential Algebraic Equations

Highlights

- *Modal Analysis*
- *Mechanism Design*
- *Differential-Algebraic Equations*
- *Embedding Technique*

Graphical Abstract

In this study, a new method was proposed to make modal analysis of mechanical linkages at arbitrary configurations in space.

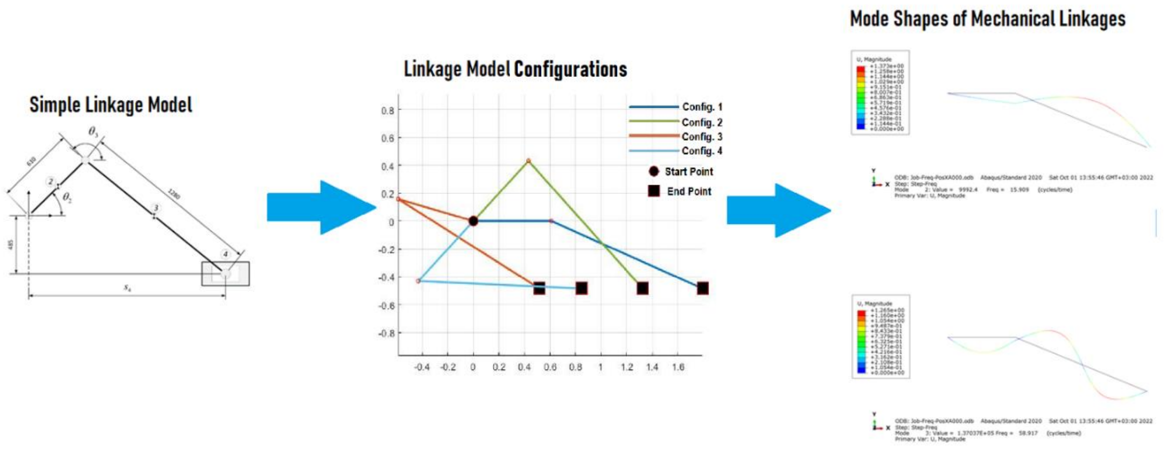

Figure. *Flowchart of mechanical linkage mode shape analysis for arbitrary configurations*

Aim

Aim of this study is to show that modal analysis of mechanical linkages depend on the configuration of the mechanism in space. Both mode shapes and spectrum of natural frequencies are a function of the boundary condition and configuration of the linkages.

Design & Methodology

In this study, differential algebraic equations and embedding techniques were used to develop a novel modal analysis method for mechanical linkages.

Originality

Differential algebraic equations were used to make a modal analysis of mechanical linkages. An analytical proof was made to show that modal analysis characteristics of mechanism is a function of mechanism configuration.

Findings

Analytical expressions were provided to show that mode shapes and spectrum of mechanical linkages are a function of mechanism configuration in space.,

Conclusion

Modal analysis of mechanical linkages must be repeated for every new configuration in space. Differential algebraic equations can be used to accelerate this repeated modal analysis.

Declaration of Ethical Standards

The author(s) of this article declare that the materials and methods used in this study do not require ethical committee permission and/or legal-special permission.

Advancing Modal Analysis of Mechanical Linkages via Differential Algebraic Equations

Araştırma Makalesi / Research Article

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ABSTRACT

In this study, the modal analysis of mechanical linkages were investigated across arbitrary configurations. Rigid body motion was successfully decoupled from the flexible motion associated with pseudo-dynamic equilibrium configurations by leveraging linear elasto-dynamic modeling. These equilibrium configurations were determined through the solution of nonlinear differential equations. The analytical investigation reveals that the modal analysis of a mechanical linkage at arbitrary configurations is intricately tied to the position parameters of the linkage. Mode shapes and the spectrum of natural frequencies were dynamically evolved as the mechanical linkage traversed space. To validate the findings, numerical results were presented for a slider-crank mechanism, which were further corroborated through analytical simulations.

Keywords: Coordinate partitioning; Embedding technique; Kinematic chain; Modal analysis

Diferansiyel Denklemler Yoluyla Mekanik Bağlantıların Modal Analizinin Geliştirilmesi

ÖZ

Bu çalışmada, farklı konfigürasyonlara sahip mekanik bağlantıların modal analizi derinlemesine incelenmiştir. Doğrusal elasto-dinamik modellemeden yararlanarak, katı cisim hareketi, dinamik denge konfigürasyonlarıyla ilişkili esnek hareketten ayrıştırılmıştır. Bu denge konfigürasyonları doğrusal olmayan diferansiyel denklemlerin çözümü yoluyla belirlenmiştir. Analitik yöntemlerle farklı konfigürasyonlardaki mekanik bağlantının modal analizinin, bağlantının konum parametrelerine karmaşık bir şekilde bağlı olduğu ortaya konmuştur. Buna göre mod şekilleri ve doğal frekansların spektrumu, mekanik bağlantıların farklı konfigürasyonlarında dinamik olarak değişmektedir. Sonuçların doğrulanması için, simülasyonlarla desteklenen krank - biyel mekanizmasının sayısal sonuçları sunulmuştur.

Anahtar Kelimeler: Koordinat bölümleme; Gömme tekniği; Kinematik zincir;Modal analiz

1. INTRODUCTION

The structural parts which have mechanical linkages are exposed to vibrational forces. One of the efficient way to understand those dynamic forces of corresponding part is to apply modal analysis [1]. Modal analysis guides the investigation of the dynamic characteristics of parts and the method is used for many industrial applications such as analyzing vibrations, fatigue life, flexible multi-body systems etc. [2,3]. Spectrum of natural frequencies and mode shapes can be extracted. Besides, effect of distinct dynamic forces on the part can be visualized through mode shapes [4].

On the other hand, structural parts can be formed of versatile flexible linkage configurations [5-7]. Basic structures such as four-bar, slider-crank, and robot linkage mechanisms were affected by dynamic forces according to their using operations [8,9]. Dynamic responses of the structures can be refined by changing configuration of linkage connections [10]. Analytical solutions of mechanical linkages can be acquired with different disciplines such as linear elasto-dynamic analysis [11-13]. In the nature, materials show some degree of nonlinear elastoplastic response. Most of the time, material complexity is in high levels and hard to analyse. Therefore, dynamic analysis is evaluated in the linear region that assumes part's material properties are homogeneous and isotropic. Then, this simplifies the fact that material's behavior doesn't change in different directions of space [14-15].

In the literature, different mechanical linkage configurations such as four-bar, slider-crank, internal combustion engine mechanisms were studied in the way of dynamic responses to vibrations [16-20]. Wang et al studied robot mechanism theory and dynamic control of linkage dependent mechanisms [21]. Mehta et al. investigated four-bar linkage dynamic analysis in space [22]. Palmieri et al. studied configuration dependent modal analysis of a robot [23]. Wang et al. and Aannaque et al studied elasto-dynamic analysis of robots and four-bar mechanisms, respectively [24-25]. Midha et al. studied different positional configuration of linkage mechanisms and extracted configure-dependent natural frequencies and mode shapes which are presented as functions of the crank position [26]. Eberhard et al. used Timoshenko beam model to study both the axial and

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lateral system vibrations of a manipulator linkage configurations [27].

In the case of mechanism geometric dimensions, there are different aspects which varies according to application field. In this study, slider crank mechanisms, show that the crank and connecting rod lengths in a relation to each other. Besides, even the dimensions used in industrial type mechanisms are versatile, they have analytical and numerical solutions in terms of the applicability of the method. Golebiewski et al investigated slider crank mechanism with both analytical and experimental methods in terms of various parameters such as crank length, slider offset etc. [28]. Matekar et al. and Khemili et al. studied slider-crank mechanisms with changing clearance dimensions. Both study claimed that joint clearance and slider offset have an impact on modelling the mechanism [29,30]. Chai et al. investigated modal analysis of a specific slider-crank mechanism which is mainly used as crank shaft and piston in internal combustion chambers. Mode shapes and natural frequencies for crank shaft and piston parts were analyses through simulation and experiments [31]. Bonisoli et al. improved a detection method which reveals undesired dynamics in structures such as motorcycle piston-crank mechanisms. The method showed the critical components in the assembly and their mode-shapes requiring modifications [32].

In the light of those studies, an analytical study was made on the modal analysis of mechanical linkages that moves in space continuously in this article. With this goal in mind, several open problems like change in mode shapes and natural frequencies as a function of mechanical linkage configuration were addressed and conditions for a successful modal analysis of mechanical linkages were sought.

The rest of the paper is organized as follows: in section two, a review of related works is made and the contribution of this paper is discussed. In section three, the problem statement is defined. Section four is devoted to theory where an analytical proof of the main claim of this paper is made. Section Five presents the numerical results of the proposed modal analysis approach. The main claim of this paper is that the mode shapes and spectrum of the natural frequencies change as a function of the mechanical linkage position parameters, and Abaqus software simulations validate these results. Finally, section six concludes the paper with conclusive statements.

1.1. Related Works

Basically, dynamic analysis of multi-body systems can be made using either differential-algebraic equations or ordinary differential equations. Vibration analysis of multi-body systems can be made assuming small motions around some equilibrium configuration [33,34]. Generally, differential algebraic formulation can be presented as in augmented formulation where Lagrange multipliers are used to augment the constraint equations to differential equations that are written as a function of redundant generalized coordinates or they can be transformed using the embedding technique so differential algebraic equations can be written in terms of a set of minimal differential equations. The second method, embedding technique can be visualized as a generalization of constraint elimination technique [35]. As it was mentioned above, vibration analysis of a multi body system can be formulated as a perturbation in the generalized coordinate set around some equilibrium point [36-37].

The main claim of this paper is that when the vibration characteristic of any mechanical linkage changes as a function of its configuration parameters, it can be accounted for easily as proposed here. As an example case; A slider crank mechanism was studied and it has been shown that both spectrum of natural frequencies and the set of mode shapes change in great amounts when configuration of the mechanism changes as a function of crank angle. The authors of this paper used Matlab simulations to calculate these results. Abaqus software simulations were used to validate these numerical results. All simulation results have verified that both the spectrum of natural frequencies and the set of mode shapes are a function of mechanical linkage's configuration parameters. This conclusion was explained using the fact that boundary condition of each linkage in the mechanism change as the mechanical linkage moves in space; boundary conditions are a function of the mechanical linkage configuration parameters.

2. PROBLEM STATEMENT

For body-i in the multi-body system, Lagrange's equation takes the form

$$\frac{d}{dt}\left(\frac{\partial T^i}{\partial \dot{\mathbf{q}}^i}\right) - \left(\frac{\partial T^i}{\partial \mathbf{q}^i}\right) + \mathbf{C}_{q^i}^T \boldsymbol{\lambda} = \mathbf{Q}^i \tag{1}$$

$\mathbf{q}^i$ is the redundant set of generalized coordinates of body-i. T^i is the kinetic energy of the body-i. $\mathbf{C}_{q^i}$ is the Jacobian matrix of the constraint equation associated with the body-i. $\boldsymbol{\lambda}$ is the vector of Lagrange multipliers. In general, $\mathbf{C}_{q^i}^T \boldsymbol{\lambda}$ represents the generalized joint forces acting on body-i. $\mathbf{Q}^i$ is the vector of generalized forces and it is defined as

$$\mathbf{Q}^i = -\mathbf{K}^i \mathbf{q}^i + \mathbf{Q}_v^i + \mathbf{Q}_e^i \tag{2}$$

$\mathbf{K}^i$ is the stiffness matrix of body-i. Let us define $\mathbf{Q}_v^i$ to be

$$\mathbf{Q}_v^i = -\dot{\mathbf{M}}^i \dot{\mathbf{q}}^i + \left[\frac{\partial}{\partial \mathbf{q}^i}\left(\frac{1}{2}\dot{\mathbf{q}}^{iT}\mathbf{M}^i\dot{\mathbf{q}}^i\right)\right]^T \tag{3}$$

$\mathbf{M}^i$ is the mass matrix of body-i. $\mathbf{Q}_v^i$ is called the quadratic velocity vector. Using Lagrange multipliers, equations of motion are formulated in term of a set of redundant coordinates including the independent and dependent coordinate sets:

$$\mathbf{M}^i\ddot{\mathbf{q}}^i + \mathbf{K}^i\mathbf{q}^i + \mathbf{C}_{q^i}^T\boldsymbol{\lambda} = \mathbf{Q}_e^i + \mathbf{Q}_v^i; \quad \forall i \in n_b \tag{4}$$

Where

$$\mathbf{M}^i = \begin{bmatrix} \mathbf{m}_{RR}^i & \mathbf{m}_{R\theta}^i & \mathbf{m}_{Rf}^i \\ & \mathbf{m}_{\theta\theta}^i & \mathbf{m}_{\theta f}^i \\ sym & & \mathbf{m}_{ff}^i \end{bmatrix} \quad \mathbf{q}^i = \begin{bmatrix} \mathbf{R}^i \\ \theta_2^i \\ \mathbf{q}_f^i \end{bmatrix} \tag{5}$$

$$\mathbf{K}^i = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \mathbf{K}_{ff}^i \end{bmatrix} \quad \mathbf{C}_{q^i} = \begin{bmatrix} \mathbf{C}_{R^i}^T \\ \mathbf{C}_{\theta^i}^T \\ \mathbf{C}_{q_f^i}^T \end{bmatrix} \tag{6}$$

$$\mathbf{Q}_e^i = \begin{bmatrix} \left(\mathbf{Q}_e^i\right)_R \\ \left(\mathbf{Q}_e^i\right)_\theta \\ \left(\mathbf{Q}_e^i\right)_f \end{bmatrix} \quad \mathbf{Q}_v^i = \begin{bmatrix} \left(\mathbf{Q}_v^i\right)_R \\ \left(\mathbf{Q}_v^i\right)_\theta \\ \left(\mathbf{Q}_v^i\right)_f \end{bmatrix} \tag{7}$$

The solution of Equation 4 has to satisfy the constraint equations given below:

$$\mathbf{C}(\mathbf{q}, t) = 0 \tag{8}$$

The equation of motion of the multi-body system can be obtained by stacking the equation of motion of each body in the system, see Equation 4-7; they are represented as follows:

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{K}\mathbf{q} + \mathbf{C}_q^T\boldsymbol{\lambda} = \mathbf{Q}_e + \mathbf{Q}_v \tag{9}$$

In this paper, it is aimed to develop a general approach to setup an eigenvalue problem for the modal analysis of open/close loop mechanical linkages which take arbitrary configuration in space continuously. To this end, linear-elasto dynamic model of the open/close loop mechanical linkage was formulated as described in the next section. Then, embedding technique was used to remove the redundant differential equations from the set of equations as represented in Equation 9. This paper shows that the minimal set of nonlinear differential equations are a function of the open/close loop mechanical linkage's configuration parameters. Hence, modal analysis has to be repeated for every successive configuration taken by the mechanical linkage.

3. THEORY

3.1 Linear Elasto-Dynamic Modeling for Modal Analysis

The general form of dynamic equation is given in Equation 4-9. Using the theory of linear elasto-dynamics, the differential equations are separated into two sets: one is for the rigid body motion and the second one is for the flexible motion.

Changes in the mass matrix caused by the elastic deformation are not considered. Equation 4 can be split up into two row-wise equations as follows:

$$\begin{bmatrix} \mathbf{m}_{R^iR^i}^i & \mathbf{m}_{R^i\theta^i}^i \\ & \mathbf{m}_{\theta^i\theta^i}^i \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{R}}^i \\ \ddot{\theta}_2^i \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{R}^i \\ \theta_2^i \end{bmatrix} + \begin{bmatrix} \mathbf{C}_{R^i}^T \\ \mathbf{C}_{\theta^i}^T \end{bmatrix} \boldsymbol{\lambda}$$
$$= \begin{bmatrix} \left(\mathbf{Q}_e^i\right)_{R^i} \\ \left(\mathbf{Q}_e^i\right)_{\theta^i} \end{bmatrix} + \begin{bmatrix} \left(\mathbf{Q}_v^i\right)_{R^i} \\ \left(\mathbf{Q}_v^i\right)_{\theta^i} \end{bmatrix} - \begin{bmatrix} \mathbf{m}_{R_f}^i \\ \mathbf{m}_{\theta_f}^i \end{bmatrix} \left[\ddot{\mathbf{q}}_f^i\right] \quad i \in n_b \tag{10}$$

$$\left[\mathbf{m}_{q_f^iq_f^i}^i\right]\left[\ddot{\mathbf{q}}_f^i\right] + \left[\mathbf{K}_{q_f^iq_f^i}^i\right]\left[\mathbf{q}_f^i\right] = \left[\left(\mathbf{Q}_e^i\right)_{q_f^i}\right]$$
$$+ \left[\left(\mathbf{Q}_v^i\right)_{q_f^i}\right] - \left[\mathbf{C}_{q_f^i}^T\right]\boldsymbol{\lambda} - \begin{bmatrix} \mathbf{m}_{Rf}^i \\ \mathbf{m}_{\theta f}^i \end{bmatrix}^T \begin{bmatrix} \ddot{\mathbf{R}}^i \\ \ddot{\theta}_2^i \end{bmatrix} \quad i \in n_b \tag{11}$$

Equation 10 represents a set of nonlinear equations defining the dynamics of the mechanical linkage. This equation can be used to represent the pseudo-dynamic equilibrium position. Equation 11 is a system of second-order differential equations defining the flexible motion field about this pseudo-dynamic equilibrium position. Constraint equations are expressed as a vector:

$$\mathbf{C}(\mathbf{q}^d, \mathbf{q}^e, t_p) = 0 \tag{12}$$

Here, $\mathbf{q}^d$, $\mathbf{q}^e$ are the independent/dependent rigid/flexible coordinates, respectively. Note that constraint equations are dependent on both rigid body coordinates and coordinates associated with flexible coordinates. Using the generalized coordinate partitioning of $\mathbf{q} = \left[\mathbf{q}^{dT} \quad \mathbf{q}^{eT}\right]^T$ let us take the derivative of constraint equation given in Equation 12, twice. This equation leads to an expression where the acceleration term $\ddot{\mathbf{q}}$ appears explicitly. Define some new constants:

$$\mathbf{C}_q^{de} = -\mathbf{C}_q^{-e}\mathbf{C}_q^d \tag{13}$$

$$\mathbf{C}_\oplus = -\mathbf{C}_q^{-e}\left[\left(\mathbf{C}_q\dot{\mathbf{q}}\right)_q\dot{\mathbf{q}} + 2\mathbf{C}_{qt}\dot{\mathbf{q}} + [\mathbf{C}_{tt}]\right] \tag{14}$$

Let us define $\mathbf{B}_q^{de}$ and $\boldsymbol{\gamma}^{de}$ as

$$\mathbf{B}_q^{de} = \begin{bmatrix} \mathbf{I} \\ \mathbf{C}_q^{de} \end{bmatrix} \quad \boldsymbol{\gamma}^{de} = \begin{bmatrix} \mathbf{0} \\ \mathbf{C}_\oplus \end{bmatrix} \tag{15}$$

One can write an expression for $\ddot{\mathbf{q}}$ where it is expressed in terms of the acceleration of independent coordinates i.e. $\ddot{\mathbf{q}}^d$.

$$\ddot{\mathbf{q}} = \mathbf{B}_q^{de}\ddot{\mathbf{q}}^d + \boldsymbol{\gamma}^{de} \tag{16}$$

When the flexible mechanism is fixed at a particular configuration to make the modal analysis of mechanism links, it is assumed that all velocity and acceleration terms associated with the rigid body motion take zero values. Since, in the application of linear elasto-dynamic modeling to the modal analysis of a mechanism, it is assumed that the mechanism is fixed in space or the rigid body motion is removed from the subsequent vibration analysis, it is claimed that the velocity/acceleration terms related to rigid body motion are all go to zero. Hence, Equation 10-11 can be simplified further using this assumption.

$$\begin{bmatrix} \mathbf{C}^T_{R^i} \\ \mathbf{C}^T_{\theta^i} \end{bmatrix} \boldsymbol{\lambda} = \begin{bmatrix} \left(\mathbf{Q}^i_e\right)_{R^i} \\ \left(\mathbf{Q}^i_e\right)_{\theta^i} \end{bmatrix} + \begin{bmatrix} \left(\mathbf{Q}^i_v\right)_{R^i} \\ \left(\mathbf{Q}^i_v\right)_{\theta^i} \end{bmatrix} - \begin{bmatrix} \mathbf{m}^i_{R_f} \\ \mathbf{m}^i_{\theta_f} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{q}}^i_f \end{bmatrix} \tag{17}$$

$$\begin{bmatrix} \mathbf{m}^i_{q^i_f q^i_f} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{q}}^i_f \end{bmatrix} + \begin{bmatrix} \mathbf{K}^i_{q^i_f q^i_f} \end{bmatrix} \begin{bmatrix} \mathbf{q}^i_f \end{bmatrix} = \begin{bmatrix} \left(\mathbf{Q}^i_e\right)_{q^i_f} \end{bmatrix} - \begin{bmatrix} \mathbf{C}^T_{q^i_f} \end{bmatrix} \boldsymbol{\lambda} \tag{18}$$

Note that in structural dynamic analysis of a mechanism at an arbitrary position, the acceleration vector can take high values, however, the displacement vector is defined as an infinitesimal deformation about the static equilibrium position i.e. elastic deformations $\mathbf{q}_f$. Therefore, displacements are expected to take small values. One can use the first-order Taylor series expansion of the constraint equations using constraint Jacobian matrix and express the dependent displacements in terms of independent displacement. This process follows the same approach that is employed for the virtual displacement vector. $\mathbf{C}(\mathbf{q})$ represents the component due to displacements in coordinates. Now, let us decompose the vector of elastic coordinates into dependent and independent coordinates. If $\mathbf{q}^e_f$ are selected such that $\mathbf{C}^e_{q_f}$ has full rank then $\mathbf{q}^e_f$ can be expressed in terms of $\mathbf{q}^d_f$:

$$\mathbf{q}^e_f = \mathbf{C}^{de}_{q_f}\mathbf{q}^d_f + \mathbf{C}^{-e}_{q_f}\mathbf{C}(\mathbf{q}) \tag{19}$$

$$\mathbf{C}^{de}_{q_f} = -\mathbf{C}^{-e}_{q_f}\mathbf{C}^d_{q_f} \tag{20}$$

Finally, $\mathbf{q}_f$ can be written in terms of $\mathbf{q}^d_f$

$$\mathbf{q}_f = \begin{bmatrix} \mathbf{I} \\ \mathbf{C}^{de}_{q_f} \end{bmatrix} \mathbf{q}^d_f + \begin{bmatrix} \mathbf{0} \\ \mathbf{C}^{-e}_{q_f}\mathbf{C}(\mathbf{q}) \end{bmatrix} \tag{21}$$

In structural dynamic analysis of a mechanism that starts from standstill; if all initial conditions are zero then displacement, velocity and acceleration vectors are zero. Then one can assume that $\mathbf{C}(\mathbf{q}) = 0$. Furthermore, when the flexible mechanism is fixed at a particular configuration, all displacement related to rigid body motion goes to zero as well so Equation 21 is simplified. Then, one can write

$$\mathbf{q}_f = \mathbf{B}^{de}_{q_f}\mathbf{q}^d_f \Rightarrow \ddot{\mathbf{q}}_f = \mathbf{B}^{de}_{q_f}\ddot{\mathbf{q}}^d_f \tag{22}$$

Equation 22 can be substituted into Equation 18. This equation is pre-multiplied by $\mathbf{B}^{deT}_{q_f}$. One can show that $\mathbf{B}^{deT}_{q_f}\mathbf{C}^T_{q_f}$ is a null matrix. Then, one can write the following equation:

$$\mathbf{M}^{dd}_{ff}\ddot{\mathbf{q}}^d_f + \mathbf{K}^{dd}_{ff}\mathbf{q}^d_f = \mathbf{Q}^{dd}_{ff} \tag{23}$$

$$\mathbf{M}^{dd}_{ff} = \mathbf{B}^{deT}_{q_f}\mathbf{M}_{ff}\mathbf{B}^{de}_{q_f} \tag{24}$$

$$\mathbf{K}^{dd}_{ff} = \mathbf{B}^{deT}_{q_f}\mathbf{K}_{ff}\mathbf{B}^{de}_{q_f} \tag{25}$$

$$\mathbf{Q}^{dd}_{ff} = \mathbf{B}^{deT}_{q_f}\mathbf{Q}_{fe} \tag{26}$$

This is a classical equation of motion for the modal analysis of a linear system. Note that both $\mathbf{K}^{dd}_{ff}$ and $\mathbf{M}^{dd}_{ff}$ are functions of $\mathbf{q}^d$ since $\mathbf{B}^{de}_{q_f}$ is a function of $\mathbf{q}^d$ i.e. $\mathbf{B}^{de}_{q_f}(\mathbf{q}^d)$. It is obvious that natural frequency spectrum is a function of $\mathbf{q}^d$.

3.2 LU Decomposition of Constraint Jacobian Matrix

LU decomposition can be applied to the Jacobian matrix of the constraint equations i.e. $\mathbf{C}_q$ to find the set of independent and dependent coordinates. LU decomposition of the $\mathbf{C}_q$ is represented as follows:

$$\mathbf{C}^T_q = \mathbf{P}^T(\mathbf{LU}) \Rightarrow \mathbf{C}_q = [\mathbf{LU}]^T\mathbf{P} \tag{27}$$

$$\mathbf{C}_q\boldsymbol{\delta}\mathbf{q} = [\mathbf{LU}]^T\boldsymbol{\delta}\mathbf{q}^p = \mathbf{0} \cup \boldsymbol{\delta}\mathbf{q}^p = \mathbf{P}\boldsymbol{\delta}\mathbf{q} \tag{28}$$

The first n_c columns of $[\mathbf{LU}]^T$ gives an $n_c \times n_c$ matrix; this matrix can be used as $\mathbf{C}^e_q$. It can be shown that $\mathbf{C}^e_q$ has full rank, so it is inverse exists. Hence, one can show that

$$\boldsymbol{\delta}\mathbf{q}^{ep} = -\mathbf{C}^{-e}_q\mathbf{C}^d_q\boldsymbol{\delta}\mathbf{q}^{dp} \tag{29}$$

where

$$\mathbf{C}^e_q = -\left[\mathbf{U}^T\mathbf{L}^T\right]_e \quad \mathbf{C}^d_q = \left[\mathbf{U}^T\mathbf{L}^T\right]_d \tag{30}$$

The original coordinates can always be recovered by the following transformation:

$$\boldsymbol{\delta}\mathbf{q} = \mathbf{B}^{dep}\boldsymbol{\delta}\mathbf{q}^{dp} \cup \mathbf{B}^{dep} = \mathbf{P}^{-1}\begin{bmatrix} \mathbf{C}^{-e}_q\mathbf{C}^d_q \\ \mathbf{I} \end{bmatrix} \tag{31}$$

Since, matrix $\mathbf{P}$ is a permutation matrix, its inverse always exists. Note that $\boldsymbol{\delta}\mathbf{q}^{dp}$ is a permutation of the original set of redundant coordinates.

4. MODAL ANALYSIS OF A SLIDER-CRANK MECHANISM AT ARBITRARY CONFIGURATION IN SPACE

In this section, modal analysis of a slider-crank mechanism was made. The dimensions of the mechanism are given in Figure 1.

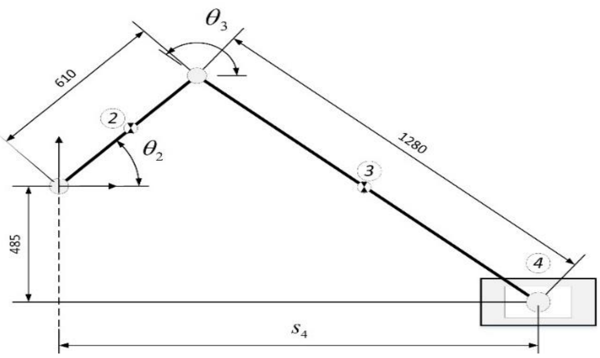

Figure 1. Slider-crank mechanism. Dimensions are given in mm.

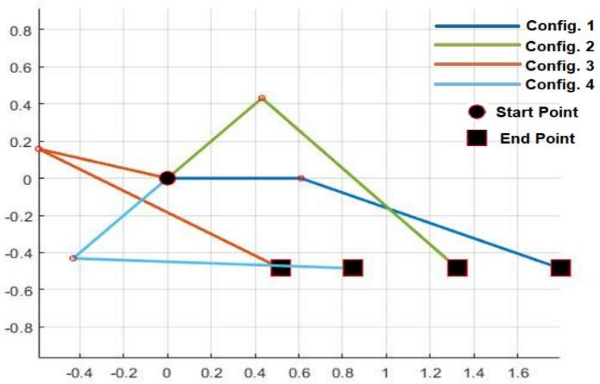

Figure 2. Four-different configurations of slider-crank mechanism.

It is assumed that the cross-section of the links are $2\times 2mm^2$. It was assumed that the mechanism is made of steel material. The slider mass is assumed to be $0.1kg$ .Modal analysis of this mechanism was made at four different crank angle: i) $\theta_2 = 0^o$ ii) $\theta_2 = 45^o$ iii) $\theta_2 = 165^o$ iv) $\theta_2 = 225^o$. Mainly, four different configurations of slider-crank mechanism were generated and shown in Figure 2 with Cartesian coordinate system. A finite element model was constructed in Abaqus software at four different configurations and natural frequency and mode shapes were extracted and compared with the Matlab software analytical results. Abaqus software results have verified that the method proposed here is very accurate. Natural frequencies calculated by the method proposed here is compared with the Abaqus software results in Table 1-2. Modes are not shown here due to space limitations but numerical comparisons have shown that mode shapes are almost identical. The function of the modal scale factor (MSF) provides a qualitative way of comparing two modal vector sets. In this approach, scaled modal vectors are used for correlation. When two modal vectors are scaled similarly, elements of each vector can be averaged, differentiated, or sorted to provide a best estimate of the modal vector [38]. According to that, Table 3-6 show MSF numbers which shows comparison of mode shapes for four configurations with crank and coupler members and calculated by the method developed here and run on Matlab program and mode shapes calculated by Abaqus software. To investigate the modal characteristics of the mechanism, configuration one was examined in detail. First five mode shapes and natural frequencies calculated by Abaqus for configuration one are plotted in Figure 3-7.

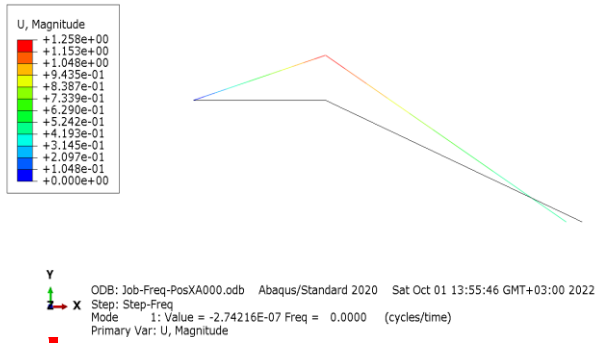

Figure 3. Modal analysis of slider-crank mechanism with Abaqus software for configuration-one: mode shape-01.

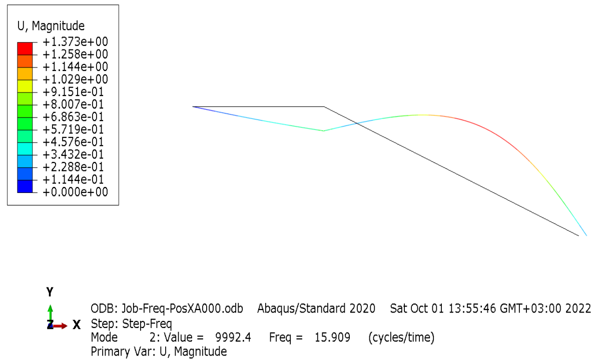

Figure 4. Modal analysis of slider-crank mechanism with Abaqus software for configuration-one: mode shape-02.

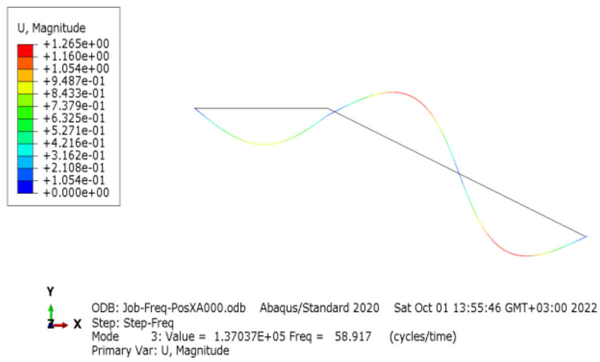

Figure 5. Modal analysis of slider-crank mechanism with Abaqus software for configuration-one: mode shape-03.

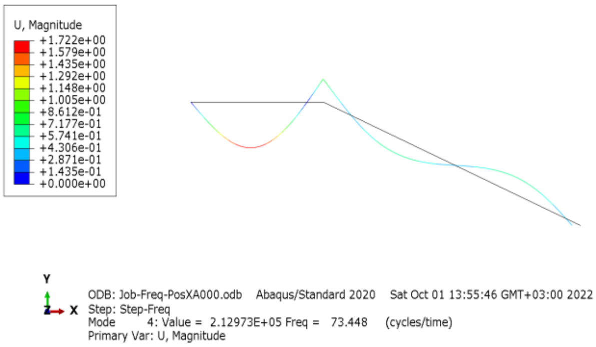

Figure 6. Modal analysis of slider-crank mechanism with Abaqus software for configuration-one: mode shape-04.

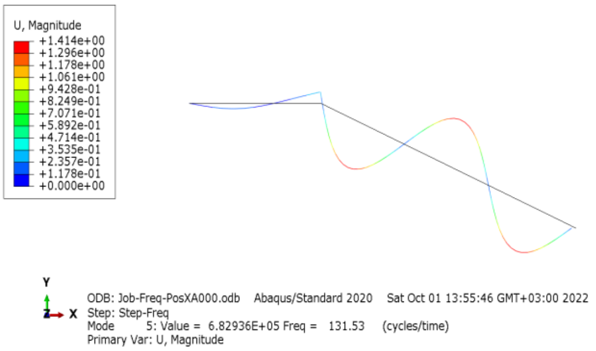

Figure 7. Modal analysis of slider-crank mechanism with Abaqus software for configuration-one: mode shape-05.

These mode shapes show that modal characteristics of the mechanism depends on the boundary condition of individual components and the joint kinematics that constrains the relative motion among neighboring components. Modal analysis of the slider-crank mechanism shows that mode shapes of the crank and coupler link almost identical to each other at different configurations. This observation is explained by simple beam model of crank and coupler links, boundary conditions of both links change very little so modal analysis results do not change considerably.

Table 1. Natural frequency of the slider-crank mechanism at four different configurations: Matlab analytical results are compared with Abaqus software results for configuration one and two.

	Natural Frequency (rad/sec)			
Mode	**Matlab-01**	**%Err-01**	**Matlab-02**	**%Err-02**
Mode-02	15.912	0.0188	15.007	0.013
Mode-03	58.932	0.0265	57.932	0.021
Mode-04	73.507	0.0799	64.6971	0.025
Mode-05	131.60	0.0617	129.8657	0.036
Mode-06	233.07	0.0827	230.1752	0.059
Mode-07	262.08	0.2543	253.9726	0.141
Mode-08	359.33	0.1511	359.2785	0.086
Mode-09	518.05	0.2242	516.7660	0.122
Mode-10	565.45	0.6078	568.5836	0.334

Table 2. Natural frequency of the slider-crank mechanism at four different configurations: Matlab analytical results are compared with Abaqus software results for configuration three and four.

	Natural Frequency (rad/sec)			
Mode	**Matlab-01**	**%Err-01**	**Matlab-02**	**%Err-02**
Mode-02	16.4753	0.0157	14.9218	-0.0039
Mode-03	59.1462	0.0186	57.8785	0.0054
Mode-04	77.9673	0.1150	66.6812	0.0394
Mode-05	133.299	0.1115	129.7825	0.0304
Mode-06	234.494	0.0908	230.1621	0.0579
Mode-07	268.291	0.5287	255.7737	0.1852
Mode-08	359.830	0.4361	358.9230	0.1018
Mode-09	518.912	0.9028	516.2853	0.1525
Mode-10	557.894	1.8261	569.0805	0.4341

Table 3. MSF number of mode shapes for configuration-one: Matlab analytical results are compared with Abaqus software results.

MSF-01	Crank-x	Crank-y	Coupler-x	Coupler-y
Mode-02	1.0000	1.0000	1.0000	1.0000
Mode-03	1.0000	1.0000	1.0000	1.0000
Mode-04	1.0000	1.0000	1.0000	1.0000
Mode-05	1.0000	1.0000	1.0000	1.0000
Mode-06	1.0000	1.0000	1.0000	1.0000
Mode-07	1.0000	1.0000	1.0000	0.9999
Mode-08	1.0000	1.0000	1.0000	1.0000
Mode-09	1.0000	1.0000	1.0000	0.9999
Mode-10	1.0000	0.9999	1.0000	0.9736

Table 4. MSF number of mode shapes for configuration-two: Matlab analytical results are compared with Abaqus software results.

MSF-02	Crank-x	Crank-y	Coupler-x	Coupler-y
Mode-02	1.0000	1.0000	1.0000	1.0000
Mode-03	1.0000	1.0000	1.0000	1.0000
Mode-04	1.0000	1.0000	1.0000	1.0000
Mode-05	1.0000	1.0000	1.0000	1.0000
Mode-06	1.0000	1.0000	1.0000	1.0000
Mode-07	1.0000	1.0000	1.0000	0.9999
Mode-08	1.0000	1.0000	1.0000	1.0000
Mode-09	1.0000	1.0000	1.0000	1.0000
Mode-10	1.0000	1.0000	1.0000	0.9972

5. CONCLUSIONS

In this article, modal analysis of open/close loop kinematic chains were studied. Equation of motion of a mechanical linkage was derived using the embedding technique and differential-algebraic equations.

Table 5. MSF number of mode shapes for configuration three: Matlab analytical results are compared with Abaqus software results.

MSF-03	Crank-x	Crank-y	Coupler-x	Coupler-y
Mode-02	1.0000	1.0000	1.0000	1.0000
Mode-03	1.0000	1.0000	1.0000	1.0000
Mode-04	1.0000	1.0000	1.0000	1.0000
Mode-05	1.0000	1.0000	1.0000	1.0000
Mode-06	1.0000	1.0000	1.0000	1.0000
Mode-07	1.0000	0.9999	1.0000	0.9994
Mode-08	1.0000	1.0000	1.0000	0.9997
Mode-09	1.0000	0.9998	0.9999	0.9972
Mode-10	1.0000	0.9984	1.0000	0.9272

Table 6. MSF number of mode shapes for configuration-four: Matlab analytical results are compared with Abaqus software results.

MSF-04	Crank-x	Crank-y	Coupler-x	Coupler-y
Mode-02	1.0000	1.0000	1.0000	1.0000
Mode-03	1.0000	1.0000	1.0000	1.0000
Mode-04	1.0000	1.0000	1.0000	1.0000
Mode-05	1.0000	1.0000	1.0000	1.0000
Mode-06	1.0000	1.0000	1.0000	1.0000
Mode-07	1.0000	1.0000	1.0000	1.0000
Mode-08	1.0000	1.0000	1.0000	1.0000
Mode-09	1.0000	1.0000	1.0000	1.0000
Mode-10	1.0000	1.0000	1.0000	0.9854

In this formulation, if any component of the mechanical linkages is replaced then the equation of motion can still be obtained by a local modification of the differential-algebraic equations.

In this study, it has been shown that modal analysis of a mechanical linkage depends on the configuration of the mechanism in space. A slider-crank mechanism was examined as a case study. It has been shown that mode shapes and natural frequency are a function of the configuration. This study has made two contributions to the literature: i) differential-algebraic equation and embedding technique formulation to the modal analysis was proposed and ii) interaction of the mechanism configuration in space and the results of modal analysis was examined.

One can be inferred from this study that, since the length of the crank is shorter than the connecting rod, the mode shapes and natural frequencies of the crankshaft are not identical with the mode shapes and natural frequencies of the connecting rod. Therefore, the dynamic behavior of the assembly and components does not coincide. And yet, the dynamic behavior of the crankshaft and connecting rod behaves approximately like a cantilever beam. The fact that similar results were obtained in the Abaqus program supports this analytical approach.

For the case of industrial applications, slider crank mechanism could be improved in terms of its dynamic response. Specifically, if operating frequency was known, high-frequency vibrations can be detected, natural frequencies of assembly could be modified in order to eliminate resonances and mode shapes which are requiring modifications could be revealed by the presented method.

DECLARATION OF ETHICAL STANDARDS

The author of this article declares that the materials and methods they use in their work do not require ethical committee approval and/or legal-specific permission.

AUTHORS' CONTRIBUTIONS

Can Ulaş DOĞRUER: He introduced the general concept of article, set up the analytical configurations of mechanical linkages and decoupled the rigid body motion model.

Can Barış TOPRAK: He contributed to literature survey of article and supported in configuration of slider-crank mechanisms.

Bora YILDIRIM: He carried out numerical Abaqus simulations and verified numerical results.

CONFLICT OF INTEREST

There is no conflict of interest in this study.

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