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SECOND-ORDER HANKEL DETERMINANT FOR A SUBCLASS OF ANALYTIC FUNCTIONS SATISFYING SUBORDINATION CONDITION CONNECTED WITH MODIFIED q-OPOOLA DERIVATIVE OPERATOR

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Abstract. This paper introduces a new subclass of analytic functions employing the operator that was recently defined by the authors. The coefficients estimate $|a_s|$ $(s = 2, 3)$ of the Taylor-Maclaurin series in this new class, as well as the Fekete-Szegö functional problems, have been derived. Furthermore, we obtained the sharp upper bound for the functional $|\mathfrak{a}_2\mathfrak{a}_4 - \mathfrak{a}_3^2|$ for functions belonging to this new subclass.

1. Introduction

By $\mathcal{A}$, we express the functions class f of the form

$$f(\mathbf{z}) = \mathbf{z} + \sum_{s=2}^{\infty} \mathfrak{a}_s \mathbf{z}^s, \tag{1}$$

which are considered analytic with respect to the symmetric open unit disk $\mathbb{U} = \{\mathbf{z} \in \mathbb{C} : |\mathbf{z}| < 1\}$, with the normalization conditions given by $f(0) = f^{'}(0) - 1 = 0$. Furthermore, we denote $\mathcal{S}$ as the subclass of $\mathcal{A}$, which are univalent in $\mathbb{U}$.

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Let $\mathfrak{P}$ be the family of all analytic functions p having positive real parts, given by

$$p(\mathbf{z}) = 1 + \sum_{s=1}^{\infty} \mathfrak{d}_s \mathbf{z}^s \quad , (\Re e\{\mathfrak{d}_s\} > 0, \, \mathbf{z} \in \mathbb{U}) \,. \tag{2}$$

A number of subclasses with respect to normalized analytic functions is studied as part of Geometric Function Theory.

The concept of quantum calculus, also known as q-calculus, has played a significant role in the advancement of Geometric Function Theory (GFT) and its extensive application in diverse fields, including mathematical science and quantum physics. For analyzing a variety of subclasses, q-calculus technique are essential. In Geometric Function Theory, the fundamental q-hypergeometric functions were initially applied by Srivastava and Owa (1989), who also provided a clear foundation for employing calculus inside this theory.

Additionally, using q-calculus theory, it is possible to express univalent function theory. More recently, the use of a fractional q-derivative operator has been observed in the creation of numerous families of analytic functions (for example, in Alsoboh and Darus [8], Elhaddad and Darus [10, 11], Mahmood and Darus [23]). For instance, Purohit and Raina [28] investigated the usage of q-fractional operators with respect to defining several analytic function classes for $\mathbb{U}$ as an open unit disc. Meanwhile, Mohammed and Darus [23] assessed properties q-analogue operator with respect to approximation and geometry concerning specific families of analytic function within the compact disc. A rather comprehensive analysis of applied q-analysis in the theory of operators can be discovered in Aral et al. [9] and Exton [12], also see ([7], [14], [15], [16], [17], [29], [31], [34]) for further studies.
The k^{th} Hankel determinant was explored by Noonan and Thomas [24] in 1976, which is expressed as

$$H_k(s) = \begin{vmatrix} \mathfrak{a}_s & \mathfrak{a}_{s+1} & \mathfrak{a}_{s+2} & \cdots & \mathfrak{a}_{s+k+1} \\ \mathfrak{a}_{s+1} & \mathfrak{a}_{s+2} & \mathfrak{a}_{s+3} & \cdots & \mathfrak{a}_{s+k+2} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \mathfrak{a}_{s+k-1} & \mathfrak{a}_{s+k} & \mathfrak{a}_{s+k+1} & \cdots & \mathfrak{a}_{s+2k-2} \end{vmatrix}, \qquad (s, k \in \mathbb{N}).$$

This determinant has garnered significant attention from several researchers. The rate of growth of $H_k(s)$ as s tends to ∞ was determined by Noor [25] with bounded boundary. For $k = 2$ and $s = 1$, we have $H_2(1) = |\mathfrak{a}_3 - \mathfrak{a}_2^2|$, which is well-known by Fekete–Szegö functional, and this may be generalized to $|\mathfrak{a}_3 - \mu\mathfrak{a}_2^2|$ for $(\mu \in \mathbb{C})$ (see, for example, [8,10]). For $k = 2$ and $s = 2$, we obtain the second Hankel determinant $H_2(2) = |\mathfrak{a}_2\mathfrak{a}_4 - \mathfrak{a}_3^2|$.

Determining the upper bounds for $H_2(2)$ attracts the attention of many authors who have determined several families of analytic functions. In 1967, Pommerenke [27] estimated the sharp upper bounds for the class $\mathcal{A}$. Some recent applications are studied by Abubaker and Darus [1], Ullah et al. [33], and Elhaddad and Darus [11].

Several authors have investigated it before, which may be referred to ([2,19,22,32, 34,35,37]).

More recently, Alatawi and Darus [5,6] provided the new q-derivative operator $\mathsf{D}_q^n(\mu,\beta,\eta,t)f(\mathbf{z}) : \mathcal{A} \to \mathcal{A}$, which is a modified Opoola operator as follows:

$$\mathsf{D}_q^n(\mu,\beta,\eta,t)f(\mathbf{z}) = \mathbf{z} + \sum_{s=2}^{\infty} \Omega_s^n(\eta,\beta,\mu,t)\mathfrak{a}_s\mathbf{z}^s, \tag{3}$$

where

$$\Omega_s^n(\eta,\beta,\mu,t) = \Big[\eta + \Big([s]_q + \beta - \mu - \eta\Big)t\Big]^n,$$

where $n \in \mathbb{N}_0$, $t \geq 0$ and $1 \leq \mu + \eta \leq \beta$.

Remark 1. *Some special operators are listed here:*

(1) *When $q \to 1^-$ and $\eta = 1$, then $\mathsf{D}_q^n(\mu,\beta,\eta,t)f(\mathbf{z})$ becomes the Opoola differential operator [26].*
(2) *When $q \to 1^-$, $t = 1$ and $\mu = \beta$, then $\mathsf{D}_q^n(\mu,\beta,\eta,t)f(\mathbf{z})$ becomes the Sãlãgean differential operator [30].*
(3) *When $t = 1$ and $\mu = \beta$, then $\mathsf{D}_q^n(\mu,\beta,\eta,t)f(\mathbf{z})$ becomes the q- Sãlãgean differential operator [13].*
(4) *When $q \to 1^-$, $\mu = \beta$ and $\eta = 1$, then $\mathsf{D}_q^n(\mu,\beta,\eta,t)f(\mathbf{z})$ becomes the Al-Oboudi differential operator [4].*

Definition 1. *Let f be given by (1). Hence, $f \in \mathcal{L}_{q,b}^n(\mu,\beta,\eta,t)$ if it complies with the inequality condition given below*

$$\mathfrak{Re}\Big\{\partial_q \mathsf{D}_q^n(\mu,\beta,\eta,t)f(\mathbf{z})\Big\} > 0, \quad (\mathbf{z} \in \mathbb{U}). \tag{4}$$

If $q \to 1-$ and $n = 0$, then the subclass $\mathcal{L}_{q,b}^n(\mu,\beta,\eta,t)$ is reduced to the class of positive real parts, denoted by $\mathcal{R}$, which was created by MacGregor [21] then studied by Janteng et al. [18].

To demonstrate our main findings, we require the lemmas as expressed below:

Lemma 1. *[20] Let $p \in \mathfrak{P}$ as is in (2), then $|\mathfrak{d}_2 - \nu\mathfrak{d}_1^2| \leq 2\max\{1, |2\nu - 1|\}$ and the sharpness result of the functions given by*

$$p(\mathbf{z}) = \frac{1+\mathbf{z}^2}{1-\mathbf{z}^2}, \quad p(\mathbf{z}) = \frac{1+\mathbf{z}}{1-\mathbf{z}}.$$

.

Lemma 2. *[27] Suppose $p \in \mathfrak{P}$ given by (2), therefore $|\mathfrak{d}_m| \leq 2$ for all $m \geq 1$.*

Lemma 3. *[19] Let $p \in \mathfrak{P}$ as in (2), then*

$$2\mathfrak{d}_2 = \mathfrak{d}_1^2 + x(4 - \mathfrak{d}_1^2), \quad |x| < 1 \tag{5}$$

and

$$4\mathfrak{d}_3 = \mathfrak{d}_1^3 + 2x(4-\mathfrak{d}_1^2)\mathfrak{d}_1 - \mathfrak{d}_1(4-\mathfrak{d}_1^2)x^2 + 2(4-\mathfrak{d}_1^2)(1-|x|^2)\mathbf{z}, \quad |\mathbf{z}| < 1. \tag{6}$$

In this current work, we determine the sharp upper limits with respect to $H_2(2)$ for the class of analytic functions $\mathcal{L}_{q,b}^n(\mu,\beta,\eta,t)$ as follows.

2. Main Results

In our first theorem, motivated by the result of Zaprawa [36], we determine the coefficients estimate $|a_s|$ $(s=2,3)$ of the Taylor-Maclaurin series in this new class, as well as the Fekete-Szegö functional problems for functions in $\mathcal{L}_{q,b}^n(\mu,\beta,\eta,t)$.

Theorem 1. *If $f\in\mathcal{L}_{q,b}^n(\mu,\beta,\eta,t)$, then*

$$\left|\mathfrak{a}_2\right|\leq\frac{2}{[2]_q\left(\eta+\left([2]_q+\beta-\mu-\eta\right)t\right)^n},$$

$$\left|\mathfrak{a}_3\right|\leq\frac{2}{[3]_q\left(\eta+\left([3]_q+\beta-\mu-\eta\right)t\right)^n},$$

and

$$|a_3-\aleph a_2^2|\leq\frac{2}{[3]_q\left(\eta+\left([3]_q+\beta-\mu-\eta\right)t\right)^n}\max\left\{1;\frac{2\aleph[3]_q\left(\eta+\left([3]_q+\beta-\mu-\eta\right)t\right)^n}{[2]_q^2\left(\eta+\left([2]_q+\beta-\mu-\eta\right)t\right)^{2n}}-1\right\}.$$

The best possible result is achieved by Köebe function.

Proof. Since $f\in\mathcal{L}_{q,b}^n(\mu,\beta,\eta,t)$. From (3), we have

$$1+\sum_{s=2}^{\infty}[s]_q\left[\eta+\Big([s]_q+\beta-\mu-\eta\Big)t\right]^n\mathfrak{a}_s\mathbf{z}^{s-1}=1+\sum_{s=1}^{\infty}\mathfrak{d}_s\mathbf{z}^s. \tag{7}$$

By equating the coefficients on both sides of (7) yields

$$\mathfrak{a}_2=\frac{\mathfrak{d}_1}{[2]_q\left(\eta+\left([2]_q+\beta-\mu-\eta\right)t\right)^n}, \tag{8}$$

$$\mathfrak{a}_3=\frac{\mathfrak{d}_2}{[3]_q\left(\eta+\left([3]_q+\beta-\mu-\eta\right)t\right)^n}, \tag{9}$$

$$\mathfrak{a}_4=\frac{\mathfrak{d}_3}{[4]_q\left(\eta+\left([4]_q+\beta-\mu-\eta\right)t\right)^n}. \tag{10}$$

From (8), (9) and using Lemma 2, yields

$$\left|\mathfrak{a}_2\right|\leq\frac{2}{[2]_q\left(\eta+\left([2]_q+\beta-\mu-\eta\right)t\right)^n},$$

and

$$\left|\mathfrak{a}_3\right|\leq\frac{2}{[3]_q\left(\eta+\left([3]_q+\beta-\mu-\eta\right)t\right)^n}.$$

Now,

$$
\begin{aligned}
a_3 - \aleph a_2^2 &= \frac{\mathfrak{d}_2}{[3]_q\left(\eta + \left([3]_q + \beta - \mu - \eta\right)t\right)^n} - \frac{\aleph \mathfrak{d}_1^2}{[2]_q^2\left(\eta + \left([2]_q + \beta - \mu - \eta\right)t\right)^{2n}} \\
&= \frac{1}{[3]_q\left(\eta + \left([3]_q + \beta - \mu - \eta\right)t\right)^n}\left(\mathfrak{d}_2 - \frac{\aleph[3]_q\left(\eta + \left([3]_q + \beta - \mu - \eta\right)t\right)^n}{[2]_q^2\left(\eta + \left([2]_q + \beta - \mu - \eta\right)t\right)^{2n}}\mathfrak{d}_1^2\right).
\end{aligned}
$$

Using Lemma 1, we have $|\mathfrak{d}_2 - \nu\mathfrak{d}_1^2| \leq 2\max\{1, |2\nu - 1|\}$

$$
|a_3 - \aleph a_2^2| \leq \frac{2}{[3]_q\left(\eta + \left([3]_q + \beta - \mu - \eta\right)t\right)^n}\max\left\{1; \frac{2\aleph[3]_q\left(\eta + \left([3]_q + \beta - \mu - \eta\right)t\right)^n}{[2]_q^2\left(\eta + \left([2]_q + \beta - \mu - \eta\right)t\right)^{2n}} - 1\right\}.
$$

Using the techniques employed by Abubaker and Darus [1], Libera and Zlotkiewicz [19], and Janteng et al. [18], we prove the theorem given below.

Theorem 2. *If $f \in \mathcal{L}_{q,b}^n(\mu, \beta, \eta, t)$, then*

$$
\left|\mathfrak{a}_2\mathfrak{a}_4 - \mathfrak{a}_3^2\right| \leq \frac{4}{[3]_q^2\left(\Omega_3^n(\eta, \beta, \mu, t)\right)^2}.
$$

The best possible result is achieved by Köebe function.

Proof. Since $f \in \mathcal{L}_{q,b}^n(\mu, \beta, \eta, t)$. from (8), (9) and (10), we observe the following

$$
\begin{aligned}
\left|\mathfrak{a}_2\mathfrak{a}_4 - \mathfrak{a}_3^2\right| = \Bigg| &\frac{\mathfrak{d}_1\mathfrak{d}_3}{[2]_q[4]_q\left(\eta + \left([2]_q + \beta - \mu - \eta\right)t\right)^n\left(\eta + \left([4]_q + \beta - \mu - \eta\right)t\right)^n} \\
&- \frac{\mathfrak{d}_2^2}{[3]_q^2\left(\eta + \left([3]_q + \beta - \mu - \eta\right)t\right)^{2n}}\Bigg|.
\end{aligned}
$$

Since the function $p(\mathbf{z}) \in \mathfrak{P}$, we assume without loss of generality that $\mathfrak{a}_1 > 0$, and for the sake of notation's accessibility, we let $\mathfrak{a}_1 = \mathbf{z}$, $(0 \leq \mathbf{z} \leq 2)$. By substituting the values of $\mathfrak{a}_2$ and $\mathfrak{a}_3$ from the system of equations (9), we have

$$\begin{aligned}\left|\mathfrak{a}_2\mathfrak{a}_4-\mathfrak{a}_3^2\right| &= \frac{1}{4}\left|\frac{\left(\mathbf{z}^4+2x(4-\mathbf{z}^2)\mathbf{z}^2-\mathbf{z}^2(4-\mathbf{z}^2)x^2+2\mathbf{z}(4-\mathbf{z}^2)(1-|x|^2)\right)}{[2]_q[4]_q\Omega_2^n(\eta,\beta,\mu,t)\Omega_4^n(\eta,\beta,\mu,t)}\right.\\
&\quad\left.-\frac{\mathbf{z}^4+2\mathbf{z}^2(4-\mathbf{z}^2)x+x^2(4-\mathbf{z}^2)^2}{[3]_q^2\left(\Omega_3^n(\eta,\beta,\mu,t)\right)^2}\right|\\
&=\frac{1}{4}\left|\left(\frac{1}{[2]_q[4]_q\Omega_2^n(\eta,\beta,\mu,t)\Omega_4^n(\eta,\beta,\mu,t)}-\frac{1}{[3]_q^2\left(\Omega_3^n(\eta,\beta,\mu,t)\right)^2}\right)\mathbf{z}^4\right.\\
&\quad+\left(\frac{1}{[2]_q[4]_q\Omega_2^n(\eta,\beta,\mu,t)\Omega_4^n(\eta,\beta,\mu,t)}-\frac{1}{[3]_q^2\left(\Omega_3^n(\eta,\beta,\mu,t)\right)^2}\right)2x(4-\mathbf{z}^2)\mathbf{z}^2\\
&\quad-\left(\frac{\mathbf{z}^2}{[2]_q[4]_q\Omega_2^n(\eta,\beta,\mu,t)\Omega_4^n(\eta,\beta,\mu,t)}-\frac{4-\mathbf{z}^2}{[3]_q^2\left(\Omega_3^n(\eta,\beta,\mu,t)\right)^2}\right)x^2(4-\mathbf{z}^2)\\
&\quad\left.+\frac{2\mathbf{z}(4-\mathbf{z}^2)(1-|x|^2)}{[2]_q[4]_q\Omega_2^n(\eta,\beta,\mu,t)\Omega_4^n(\eta,\beta,\mu,t)}\right|.\end{aligned}$$

Employing the triangle inequality, $|\mathbf{z}|\leq 1$ and replacing $|x|$ by ν, we obtain

$$\begin{aligned}\left|a_2a_4-a_3^2\right| &= \left|\left(\frac{1}{[2]_q[4]_q\Omega_2^n(\eta,\beta,\mu,t)\Omega_4^n(\eta,\beta,\mu,t)}-\frac{1}{[3]_q^2\left(\Omega_3^n(\eta,\beta,\mu,t)\right)^2}\right)\mathbf{z}^4\right.\\
&\quad+\left(\frac{1}{[2]_q[4]_q\Omega_2^n(\eta,\beta,\mu,t)\Omega_4^n(\eta,\beta,\mu,t)}-\frac{1}{[3]_q^2\left(\Omega_3^n(\eta,\beta,\mu,t)\right)^2}\right)2\nu(4-\mathbf{z}^2)\mathbf{z}^2\\
&\quad-\left(\frac{\mathbf{z}^2}{[2]_q[4]_q\Omega_2^n(\eta,\beta,\mu,t)\Omega_4^n(\eta,\beta,\mu,t)}-\frac{4-\mathbf{z}^2}{[3]_q^2\left(\Omega_3^n(\eta,\beta,\mu,t)\right)^2}\right)\nu^2(4-\mathbf{z}^2)\\
&\quad\left.+\frac{2\mathbf{z}(4-\mathbf{z}^2)(1-\nu^2)}{[2]_q[4]_q\Omega_2^n(\eta,\beta,\mu,t)\Omega_4^n(\eta,\beta,\mu,t)}\right|.\end{aligned}$$

$$\begin{aligned}\left|\mathfrak{a}_2\mathfrak{a}_4-\mathfrak{a}_3^2\right| &= \frac{1}{4}\left\{\left(\frac{1}{[2]_q[4]_q\Omega_2^n(\eta,\beta,\mu,t)\Omega_4^n(\eta,\beta,\mu,t)}-\frac{1}{[3]_q^2\left(\Omega_3^n(\eta,\beta,\mu,t)\right)^2}\right)\mathbf{z}^4\right.\\
&\quad+\left(\frac{1}{[2]_q[4]_q\Omega_2^n(\eta,\beta,\mu,t)\Omega_4^n(\eta,\beta,\mu,t)}-\frac{1}{[3]_q^2\left(\Omega_3^n(\eta,\beta,\mu,t)\right)^2}\right)2\nu(4-\mathbf{z}^2)\mathbf{z}^2\\
&\quad-\left(\frac{\mathbf{z}(\mathbf{z}-2)}{[2]_q[4]_q\Omega_2^n(\eta,\beta,\mu,t)\Omega_4^n(\eta,\beta,\mu,t)}-\frac{4-\mathbf{z}^2}{[3]_q^2\left(\Omega_3^n(\eta,\beta,\mu,t)\right)^2}\right)\nu^2(4-\mathbf{z}^2)\\
&\quad\left.+\frac{2\mathbf{z}(4-\mathbf{z}^2)}{[2]_q[4]_q\Omega_2^n(\eta,\beta,\mu,t)\Omega_4^n(\eta,\beta,\mu,t)}\right\}=\mathcal{H}(\nu,\mathbf{z}),\end{aligned} \tag{11}$$

where $\mathbf{z} \in [0,2]$ and $|x| = \nu \leq 1$.

Subsequently, we maximize the function $\mathfrak{H}(\nu, \mathbf{z})$ on the closed square $[0,1]\times[0,2]$. We now partially differentiate $\mathfrak{H}(\nu, \mathbf{z})$ given in (11) with respect to ν, which yields

$$\begin{aligned}\frac{\partial \mathfrak{H}_q(\nu, \mathbf{z})}{\partial \nu} &= \frac{1}{2}\left(\frac{1}{[2]_q[4]_q\Omega_2^n(\eta,\beta,\mu,t)\Omega_4^n(\eta,\beta,\mu,t)} - \frac{1}{[3]_q^2\left(\Omega_3^n(\eta,\beta,\mu,t)\right)^2}\right)(4-\mathbf{z}^2)\mathbf{z}^2\\ &\quad - \frac{1}{2}\left(\frac{\mathbf{z}(\mathbf{z}-2)}{[2]_q[4]_q\Omega_2^n(\eta,\beta,\mu,t)\Omega_4^n(\eta,\beta,\mu,t)} - \frac{4-\mathbf{z}^2}{[3]_q^2\left(\Omega_3^n(\eta,\beta,\mu,t)\right)^2}\right)\nu(4-\mathbf{z}^2),\end{aligned}$$

implying that $\mathfrak{H}(\nu, \mathbf{z})$ increases with respect to $\mathbf{z}$. This suggests that $\mathfrak{H}(\nu, \mathbf{z})$ may not possess a maximum value in the closed square $[0,1] \times [0,2]$. Apart from that, by fixing $\mathbf{z} \in [0,2]$ we obtain

$$\max_{\nu \in [0,1]} \mathfrak{H}(\nu, \mathbf{z}) = \mathfrak{H}(1, \mathbf{z}) = \mathfrak{K}(\mathbf{z}).$$

$$\begin{aligned}\mathfrak{K}(\mathbf{z}) = \frac{1}{4}\Bigg\{&\left(\frac{1}{[2]_q[4]_q\Omega_2^n(\eta,\beta,\mu,t)\Omega_4^n(\eta,\beta,\mu,t)} - \frac{1}{[3]_q^2\left(\Omega_3^n(\eta,\beta,\mu,t)\right)^2}\right)\mathbf{z}^4\\ &+\left(\frac{1}{[2]_q[4]_q\Omega_2^n(\eta,\beta,\mu,t)\Omega_4^n(\eta,\beta,\mu,t)} - \frac{1}{[3]_q^2\left(\Omega_3^n(\eta,\beta,\mu,t)\right)^2}\right)2(4-\mathbf{z}^2)\mathbf{z}^2\\ &-\left(\frac{\mathbf{z}(\mathbf{z}-2)}{[2]_q[4]_q\Omega_2^n(\eta,\beta,\mu,t)\Omega_4^n(\eta,\beta,\mu,t)} - \frac{4-\mathbf{z}^2}{[3]_q^2\left(\Omega_3^n(\eta,\beta,\mu,t)\right)^2}\right)(4-\mathbf{z}^2)\\ &+\frac{2\mathbf{z}(4-\mathbf{z}^2)}{[2]_q[4]_q\Omega_2^n(\eta,\beta,\mu,t)\Omega_4^n(\eta,\beta,\mu,t)}\Bigg\}.\end{aligned}$$

Then

$$\mathfrak{K}'(\mathbf{z}) = \frac{2\mathbf{z}(4-\mathbf{z}^2)}{[2]_q[4]_q\Omega_2^n(\eta,\beta,\mu,t)\Omega_4^n(\eta,\beta,\mu,t)} - \frac{2\mathbf{z}(4-\mathbf{z}^2)}{[3]_q^2\left(\Omega_3^n(\eta,\beta,\mu,t)\right)^2}.$$

It is now clear that $\mathfrak{K}'(\mathbf{z}) < 0$ for $0 < \mathbf{z} < 2$ and $\mathfrak{K}(\mathbf{z})$ possess real critical points at $\mathbf{z} = 0$, implying the upper bound with respect to (11) corresponding to $\mathbf{z} = 0$ and $\nu = 1$. Here,

$$\left|\mathfrak{a}_2\mathfrak{a}_4 - \mathfrak{a}_3^2\right| \leq \frac{4}{[3]_q^2\left(\Omega_3^n(\eta,\beta,\mu,t)\right)^2}.$$

Setting $n = 0$ and $q \to 1-$, we obtain the following results.

Corollary 1. *[18] If $f \in \mathcal{R}$, then*

$$\left|\mathfrak{a}_2\mathfrak{a}_4 - \mathfrak{a}_3^2\right| \leq \frac{4}{9}.$$

3. Conclusions

The q-calculus gained great importance among many researchers due to its numerous various applications in geometric function theory, especially in analytic function theory. This article primarily aims to estimates of the Taylor-Maclaurin coefficients $|a_s|$ $(s = 2, 3)$ for functions in this new class, as well as solve the Fekete-Szegö functional problems. Additionally, we aim to derive the second Hankel determinants for functions within the new subclass $\mathcal{L}^n_{q,b}(\mu, \beta, \eta, t)$ of analytic functions in the open unit disk $\mathbb{U}$. This subclass is attained by using a differential Operator Involving q-Opoola Operator. Using the results obtained in this article, we can generalize and enhance some recently published articles.

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