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On the Continuous Composition of Integrable Functions

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Abstract. We prove if α be a function of bounded variation on $[a,b]$, $[m_i, M_i] \subset \mathbb{R}$ be a closed interval for $1 \leq i \leq n$, $f_i : [a,b] \to [m_i, M_i]$ be Riemann-Stieltjes integrable with respect to α, and $G : \Pi_{i=1}^{i=n}[m_i, M_i] \to \mathbb{R}$ be continuous, then $H = G \circ (f_1, \ldots, f_n)$ is Riemann-Stieltjes integrable with respect to α. Some other consequences, applications and counterexamples are also provided.

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1. Introduction and Preliminaries

The definition of Riemann-Stieltjes integral (simply R-S integral) has yet remained as the subject of many current mathematical researches in the last four decades. Some of them, which are in its own theory, are [6, 11, 13–15].

Some of the recent findings in this area such as [2, 7] are about to approximation theory, or refer to the bounds of functionals. In [5], the authors gave lower and upper bounds of the Čebyšev functional for the R-S integral, and provided some applications to the three-point quadrature rules of some special functions.

Many papers have been especially assigned to investigate the analogies of the R-S integration known for different types of differential equations and their applications to analysis of their solutions. In [10], Rezounenko investigated parabolic partial differential equations with delays presented by Stieltjes integral, and proved the existence of a compact global attractor. Finally, some early results, related to the R-S integration, is interpreted in [8], where the author provided some applications of R-S integration in complex analysis and probability theory.

The present paper is also a survey on R-S integrability. Note that, if f, g are two R-S integrable functions with respect to an increasing function α on $[a,b]$, then, in general, $g \circ f$ is not R-S integrable with respect to same integrator. But if f be a R-S integrable function with respect to an increasing function α on $[a,b]$, $m \leq f \leq M$ for $m, M \in \mathbb{R}$, and g be a continuous function on $[m, M]$, then by a theorem in mathematics, $g \circ f$ is R-S integrable with respect to α on $[a,b]$ [12]. Here, we provide a generalization of this theorem, and some related consequences. The essential concepts and symboles which we use in this study, such as " partition of the interval $[a,b]$", "a function of bounded variation (simply B-V)", and "total variation of a function of B-V", are defined as [1, 3, 4]. "A partition of $[a,b]$", and "the set of all partitions of $[a,b]$" are denoted by P, and $\mathbb{P}[a,b]$ respectively. Finally, "the set of all R-S integrable functions $f : [a,b] \to \mathbb{R}$ with respect to $h : [a,b] \to \mathbb{R}$" is denoted by $\mathfrak{R}(h)[a,b]$.

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2. Main Results

We intend to give an extension of a celebrated theorem in mathematical analysis. The structutre of a part of the proof, is by an argument which generalizes the applied one in [12].

Theorem 2.1. *Let the integrator α be a function of B-V on $[a,b]$, $m_i, M_i \in \mathbb{R}, m_i < M_i$, $f_i : [a,b] \to [m_i, M_i]$ and $f_i \in \Re(\alpha)[a,b]$ for $1 \le i \le n$. If G be a continuous function on $C = \Pi_{i=1}^{i=n}[m_i, M_i]$ and $H = G \circ (f_1, \dots, f_n)$, then $H \in \Re(\alpha)[a,b]$.*

Proof. We devide the proof into two steps.
step 1. α is increasing:
Let the integrator α be a bounded increasing function on $[a,b]$ and $\epsilon > 0$. Since C is compact, then G is uniformly continuous on C, and there exists $\delta > 0$ less than ϵ such that, $|G(t) - G(s)| < \epsilon$ if $\| t - s \| < \delta$ for all $s, t \in C$. Riemann's criterion of R-S integrability for f_i implies that there exists a partition P_i of $[a,b]$ such that $U(P_i, f_i, \alpha) - L(P_i, f_i, \alpha) < \delta^2$ for $1 \le i \le n$ [12]. Let

$$P = \cup_{i=1}^{i=n} P_i = \{a = x_0, x_1, \dots, x_{r-1}, x_r = b\},$$

so, $P_i \subseteq P$, and $U(P, f_i, \alpha) - L(P, f_i, \alpha) < \delta^2$ for $1 \le i \le n$. For $1 \le j \le r$, let

$$\begin{aligned}
M_{ij} &= sup\{f_i(x) | x_{j-1} \le x \le x_j\},\\
m_{ij} &= inf\{f_i(x) | x_{j-1} \le x \le x_j\},\\
M_j^* &= sup\{H(x) | x_{j-1} \le x \le x_j\},\\
m_j^* &= inf\{H(x) | x_{j-1} \le x \le x_j\},\\
m_j &= (m_{1j}, \dots, m_{nj}),\\
M_j &= (M_{1j}, \dots, M_{nj}),\\
\|M_j - m_j\| &= \sqrt{(M_{1j} - m_{1j})^2 + \cdots + (M_{nj} - m_{nj})^2},\\
A &= \{j | 1 \le j \le r, \|M_j - m_j\| < \delta\},\\
B &= \{j | 1 \le j \le r, \|M_j - m_j\| \ge \delta\}.
\end{aligned}$$

Then, $A \cap B = \emptyset$, $A \cup B = \{1, \dots, r\}$.

(1) If $j \in A$, $x_{j-1} \le y \le x_j$, $x_{j-1} \le z \le x_j$, $Y_j = (f_1(y), \dots, f_n(y))$ and $Z_j = (f_1(z), \dots, f_n(z))$, then $m_{ij} \le f_i(y) \le M_{ij}$, $m_{ij} \le f_i(z) \le M_{ij}$ for $1 \le i \le n$, and $\|Y_j - Z_j\| \le \|M_j - m_j\| < \delta$. Therefore, $|G(Y_j) - G(Z_j)| < \epsilon$ and $M_j^* - m_j^* \le \epsilon$.

(2) If $j \in B$, then clearly there exists a natural number $k = k(j) \in \mathbb{N}$, such that $1 \le k \le n$ and $M_{kj} - m_{kj} \ge \frac{\delta}{\sqrt{n}}$. Therefore,

$$\begin{aligned}
\frac{\delta}{\sqrt{n}} \Sigma_{j \in B} \Delta\alpha_j &\le \Sigma_{j \in B}(M_{kj} - m_{kj})\Delta\alpha_j\\
&\le \Sigma_k (U(P_k, f_k, \alpha) - L(P_k, f_k, \alpha)) < n\delta^2
\end{aligned}$$

and so $\Sigma_{j \in B} \Delta\alpha_j \le n\sqrt{n}\delta$.

Let $M = sup\{H(x) | x \in [a,b]\}$, then $M_j^* - m_j^* \le 2M$ for $1 \le j \le r$. Now, we have

$$\begin{aligned}
U(P, H, \alpha) - L(P, H, \alpha) &= \Sigma_{j=1}^{j=r}(M_j^* - m_j^*)\Delta\alpha_j\\
&= \Sigma_{j \in A}(M_j^* - m_j^*)\Delta\alpha_j + \Sigma_{j \in B}(M_j^* - m_j^*)\Delta\alpha_j\\
&\le \epsilon(\alpha(b) - \alpha(a)) + 2Mn\sqrt{n}\delta < (\alpha(b) - \alpha(a) + 2Mn\sqrt{n})\epsilon.
\end{aligned}$$

Therefore, $H \in \Re(\alpha)[a,b]$.
step 2. α is of B-V:
Let V be the total variation of α. Since $f_i \in \Re(\alpha)[a,b]$ for $(1 \le i \le n)$, then $f_i \in \Re(V)[a,b]$ and $f_i \in \Re(V - \alpha)[a,b]$ [1]. Therefore, $H \in \Re(V)[a,b]$ and $H \in \Re(V - \alpha)[a,b]$ by step *1*. Thus $H \in \Re(\alpha)[a,b]$. □

Proving the Corollary 2.3, is up to the following Theorem.

Theorem 2.2 ([9]). *Let the bounded integrator α is not of B-V on $[a,b]$, then there exists a continuous function on $[a,b]$ doesn't belong to $\Re(\alpha)[a,b]$.*

Corollary 2.3. *Let the integrator α be a bounded function on $[a,b]$ such that $H = G \circ (f_1, \ldots, f_n) \in \Re(\alpha)[a,b]$ for all continuous functions $f_i : [a,b] \to [m_i, M_i]$, $(1 \le i \le n)$, and all continuous functions G on $C = \Pi_{i=1}^{i=n}[m_i, M_i]$. Then, α is of B-V on $[a,b]$.*

Corollary 2.4. *If the integrator α be of B-V on $[a,b]$, $f_i \in \Re(\alpha)[a,b]$ for $1 \le i \le n$, then,*

(1) $\Sigma_{i=1}^{i=n} c_i f_i \in \Re(\alpha)[a,b]$ *for* $c_1, \ldots, c_n \in \mathbb{R}$.
(2) $\Sigma_{i=1}^{i=n} |f_i| \in \Re(\alpha)[a,b]$,
(3) $\Pi_{i=1}^{i=n} f_i \in \Re(\alpha)[a,b]$,
(4) $min\{f_1, \ldots, f_n\} \in \Re(\alpha)[a,b]$,
(5) $max\{f_1, \ldots, f_n\} \in \Re(\alpha)[a,b]$.

2.1. Applications and Counterexamples.

Corollary 2.5. *Let $p_i, P_i \in \mathbb{R}$, $p_i < P_i$ for $1 \le i \le n$ and $G : \Pi_{i=1}^{i=n}[p_i, P_i] \to \mathbb{R}$ be continuous. Let $f : [a,b] \to \mathbb{R}$ be of B-V, $\alpha_i : [a,b] \to [p_i, P_i]$ be such that $f \in \Re(\alpha_i)[a,b]$ for $1 \le i \le n$, and $K = G \circ (\alpha_1, \ldots, \alpha_n)$, then $f \in \Re(K)[a,b]$.*

Corollary 2.6. *Suppose that f is of B-V on $[a,b]$, $f \in \Re(\alpha_i)[a,b]$ for $1 \le i \le n$. Then,*

(1) $f \in \Re(\Sigma_{i=1}^{i=n} c_i \alpha_i)[a,b]$, *for* $c_1, \ldots, c_n \in \mathbb{R}$,
(2) $f \in \Re(\Sigma_{i=1}^{i=n} |\alpha_i|)[a,b]$,
(3) $f \in \Re(\Pi_{i=1}^{i=n} \alpha_i)[a,b]$,
(4) $f \in \Re(min\{\alpha_1, \ldots, \alpha_n\})[a,b]$,
(5) $f \in \Re(max\{\alpha_1, \ldots, \alpha_n\})[a,b]$.

The following example shows that without the continuty of G, Theorem 2.1 does not hold.

Example 2.7. Suppose that $f(x) = \alpha(x) = x$ on $[0,1]$, and

$$G(x) = \begin{cases} 1, & if \quad x \in [0,1] \cap \mathbb{Q}, \\ 0, & if \quad x \in [0,1] \cap (\mathbb{R} - \mathbb{Q}). \end{cases}$$

Then, α is of B-V on $[0,1]$, $f \in \Re(\alpha)[0,1]$, but $(G \circ f) \notin \Re(\alpha)[0,1]$ [12].

A similar form of Theorem 2.1 does not hold for the functions of B-V instead of R-S integrable ones.

Example 2.8. Let

$$G(x) = \begin{cases} x\cos(\frac{\pi}{2x}), & if \quad x \in [0,1], \\ 0, & if \quad x = 0 \end{cases}$$

and $\alpha = Id_{[0,1]}$. Then, G is continuous on $[0,1]$, α is a function of B-V on $[0,1]$, but $G \circ \alpha$ is not of B-V on $[0,1]$ [1].

Despite the property that α and G be of B-V, in general, if G is not continuous, then the Corollary 2.5 does not hold.

Example 2.9. Let

$$G(x) = \begin{cases} 1, & if \quad x \ne 0, \\ 0, & if \quad x = 0, \end{cases} \quad , \quad \alpha(x) = \begin{cases} \frac{1}{q^3}, & if \quad x = \frac{p}{q}, p \ne 0, q > 0, gcd(p,q) = 1, \\ 0, & if \quad x \in (\mathbb{R} - \mathbb{Q}) \cup \{0\}. \end{cases}$$

Then,

$$(G \circ \alpha)(x) = \begin{cases} 1, & if \quad x \in \mathbb{Q} - \{0\}, \\ 0, & if \quad x \in (\mathbb{R} - \mathbb{Q}) \cup \{0\}. \end{cases}$$

Obviously, the functions G, α and $f = Id_{[0,1]}$ are of B-V on $[0,1]$, and G is not continuous on $[0,1]$. Now, Lebesgue's integrability criterion for Riemann's integral [1, 12], besides the theorem of integration by parts imply that $f \in \Re(G)[0,1] \cap \Re(\alpha)[0,1]$ separately, but $f \notin \Re(G \circ \alpha)[0,1]$.

The continuous composition of two functions of B-V, in general, is not of B-V.

Example 2.10. The function $G : (0,1] \times (0,1] \to \mathbb{R}$, $G(x,y) = \frac{x}{y}$ is continuous, and $f, g : [0,1] \to \mathbb{R}$, defined by $f(x) = 1$ and

$$g(x) = \begin{cases} 1, & if \quad x = 0, \\ x, & if \quad x \in [0,1] \end{cases}$$

are of B-V on $[0, 1]$, but $H = G \circ (f, g) : [0, 1] \to \mathbb{R}$, defined by

$$H(x) = \begin{cases} 1, & if \quad x = 0, \\ \frac{1}{x}, & if \quad x \in]0, 1] \end{cases}$$

is not of B-V on $[0, 1]$.

In the next Example, as an extension of the results in [1,3,4], some continuous composition of the functions of B-V, which again are of B-V, are provided.

Example 2.11. Suppose that the functions $\alpha_i (1 \leq i \leq n)$ are of B-V on $[a, b]$, and $c_1, \ldots, c_n \in \mathbb{R}$. Then, $\Sigma_{i=1}^{i=n} c_i \alpha_i$, $\Sigma_{i=1}^{i=n} |\alpha_i|$, $min\{\alpha_1, \ldots, \alpha_n\}$, $max\{\alpha_1, \ldots, \alpha_n\}$, are of B-V on $[a, b]$.

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this article.

Authors Contribution Statement

The author has read and agreed the published version of the manuscript.

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