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Research Article

Introduction in third-order fuzzy differential subordination

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Abstract

In light of the well-established and widely-used theory of differential subordination, recent works incorporating fuzzy elements into Geometric Function Theory have given rise to the concept of fuzzy differential subordination. Second-order fuzzy differential subordinations were taken into consideration for studies up until this point. The research described in this paper aims to expand the concept of fuzzy differential subordination to third-order fuzzy differential subordination, building on an idea first put forth in 2011 by José A. Antonino and Sanford S. Miller and still being investigated by scholars today. The key concepts and preliminary findings required for the development of this branch of fuzzy differential subordination are introduced. The class of admissible functions is specified, the fundamental theorems are established and the fundamental concepts of the third-order fuzzy subordination approach are presented. Several examples constructed as applications of the new results demonstrate the applicability of the new findings.

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1. Introduction

In order to define the idea of fuzzy subordination [24] using the notion of fuzzy set first introduced by L.A. Zadeh [39] connected with the theory of univalent functions viewed from the perspective of geometric function theory, the study for introducing fuzzy differential subordination theory was started in 2011. As a result of further research employing this idea, the concept of fuzzy differential subordination, which takes as its starting point the classical theory of differential subordination as shown in the monograph [17] which synthesizes it, appeared in 2012 [25]. The key concepts of the differential subordination theory were slowly modified. Prior to discussing Briot-Bouquet fuzzy differential subordination as a special example of fuzzy differential subordination [27], methods for determining dominants and the optimal dominant for fuzzy differential subordinations were first presented

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[26]. Moreover, new operators [4,36,37] and an adaptation of the dual notion of differential superordination embedding the fuzzy set concept [10] were contributed to the study.

In 2011, the first presentation of third-order differential subordination by J.A. Antonino and S.S. Miller [7] emerged. The generalized Bessel functions [33, 35], generalized Mittag-Leffler functions [29], Liu-Srivastava operator [34], a fractional operator [15], and Srivastava-Attiya operator [32] are all connected to third-order differential subordination in fascinating new ways. Third-order differential subordination research is still active, with recent publications using the Hohlov operator [18], Generalized Hurwitz-Lerch Zeta Functions [6], and many additional operators [8, 9, 12]. Third-order differential subordination research is continuing to add to the knowledge regarding second-order differential subordination [28].

As may be seen from recently published papers [22,23], the research on fuzzy differential subordinations is still producing fresh findings. Fractional calculus is used [1], a linear operator is added to the investigations [14], a hypergeometric operator is also taken into consideration [21], a Mittag-Leffler type Borel distribution is investigated in relation to fuzzy differential subordination [31], and various other operators are linked to the research on fuzzy differential subordination theory [5, 38].

Having in mind the latest developments listed above, the present paper aims to introduce the concept of third-order fuzzy differential subordination extending the known knowledge related to second-order fuzzy differential subordination. In the next section some preliminary notions are listed which are used for obtaining the new outcome of the study contained in section 3 of the paper. The notion of solution of the third-order fuzzy differential subordination is defined, followed by the introduction of the concept of fuzzy dominant of the solutions of the third-order fuzzy differential subordination and fuzzy best dominant. Basic theorems needed in future studies for developing the idea of third-order fuzzy differential subordination are next proved. Section 4 of the paper presents several examples of how the results can be used in applications.

2. Preliminaries

The study begins with a summary of the general concepts and classes utilized for the investigation in geometric function theory.

The complex plane's unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ for which we use the notations

$$\mathbb{U}_{r_0}(z_0) = \{z \in \mathbb{C} : |z - z_0| < r_0\}, \overline{\mathbb{U}}_{r_0}(z_0) = \{z \in \mathbb{C} : |z - z_0| \le r_0\}.$$

is the environment where the class of holomorphic functions is denoted by $\mathcal{H}(\mathbb{U})$ and the class of univalent functions is described by $\mathcal{H}(\mathbb{U})$. Important subclasses of $\mathcal{H}(\mathbb{U})$ used for the study are:

$$\mathcal{A}_n = \{f \in \mathcal{H}(\mathbb{U}) : f(z) = z + a_{n+1}z^{n+1} + \cdots, \, z \in \mathbb{U}\},$$

with $\mathcal{A}_1 = \mathcal{A}$, and

$$\mathcal{H}[a, n] = \{f \in \mathcal{H}(\mathbb{U}) : f(z) = a + a_n z^n + a_{n+1}z^{n+1} + \cdots, \, z \in \mathbb{U}\},$$

with $\mathcal{H}_0 = \mathcal{H}[0, 1]$, when $a \in \mathbb{C}$, $n \in \mathbb{N}^* = \{0, 1, 2, 3, ...\}$.

Next, we review the definitions for third-order differential subordinations that were provided in [7].

Definition 2.1 ([7])**.** Let Q denote the set of functions q that are analytic and univalent on the set $\bar{\mathbb{U}} \setminus E(q)$ where $\bar{\mathbb{U}} = \mathbb{U} \cup \partial\mathbb{U}$ and

$$E(q) = \{\xi \in \partial\mathbb{U} : \lim_{z\to\xi} q(z) = \infty\},$$

and are such that $min|q'(\xi)| = \rho > 0$ for $\xi \in \partial\mathbb{U} \setminus E(q)$. The subclass of Q for which $q(0) = a$ is denoted by $Q(a)$.

Class Q is not empty. The function $q(z) = \frac{1+z}{1-z}$ is an illustration of a function found in this class. For this function, $E(q) = \{1\}$, knowing that $min|q'(\xi)| = \frac{1}{2} > 0$ for $\xi \in \mathbb{N} \setminus \{1\}$.

Definition 2.2 ([7])**.** Let Ω be a set in $\mathbb{C}$, $q \in Q$ and $n \in \mathbb{N} \setminus \{1\}$. The class $\Psi_n[\Omega, q]$ of admissible operators consists of those functions $\psi : \mathbb{C}^4 \times \mathbb{U} \to \mathbb{C}$ that satisfy the following admissibility conditions:

$$\psi(r, s, t, u; z) \notin \Omega \tag{2.1}$$

when

$$r = q(\xi),\ s = n\xi q'(\xi),\ Re\left(\frac{t}{s} + 1\right) \geq n\left[Re\left(\frac{\xi q''(\xi)}{q'(\xi)} + 1\right)\right],$$

and

$$Re\left(\frac{u}{s}\right) \geq n^2 Re\left(\frac{\xi^2 q''(\xi)}{q'(\xi)}\right),$$

where $z \in \mathbb{U}, \xi \in \partial\mathbb{U} \setminus E(q)$.

When this definition applies, there is a specific sub-case that can be described. When $\psi : \mathbb{C}^3 \times \mathbb{U} \to \mathbb{C}$, the admissibility condition (2.1) becomes $\psi(r, s, t; z) \notin \Omega$, with

$$r = q(\xi),\ s = n\xi q'(\xi) \text{ and } Re\left(\frac{t}{s} + 1\right) \geq n\left[Re\left(\frac{\xi q''(\xi)}{q'(\xi)} + 1\right)\right],$$

for $\xi \in \partial\mathbb{U} \setminus E(q)$.

Lemma 2.3 ([7])**.** *Let $p(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots$ be analytic in $\mathbb{U}$, with $p(z) \neq a$ and $n \geq 2$, and let $q \in Q(a)$. If there exists points $z_0 = r_0 e^{i\theta_0} \in \mathbb{U}$ and $\xi_0 \in \partial\mathbb{U} \setminus E(q)$ such that $p(z_0) = q(\xi_0)$, $p\left(\overline{\mathbb{U}}_{r_0}\right) \subset q\left(\mathbb{U}\right)$,*

$$Re\frac{\xi_0 q''(\xi_0)}{q'(\xi_0)} \geq 0 \tag{2.2}$$

and

$$\left|\frac{zp'(z)}{q'(\xi)}\right| \leq n \tag{2.3}$$

when $z \in \overline{\mathbb{U}}_{r_0}$ and $\xi \in \partial\mathbb{U} \setminus E(q)$, then there exists a real constant $k \geq n \geq 2$ such that

$$z_0 p'(z_0) = n\xi_0 q'(\xi_0) \tag{2.4}$$

$$Re\frac{z_0 p''(z_0)}{p'(z_0)} + 1 = nRe\frac{\xi_0 q''(\xi_0)}{q'(\xi_0)} + k \geq n\left[Re\frac{\xi_0 q''(\xi_0)}{q'(\xi_0)} + 1\right] \tag{2.5}$$

and

$$Re\frac{z_0^2 p'''(z_0)}{p'(z_0)} + 1 \geq n^2 Re\frac{\xi_0^2 q'''(\xi_0)}{q'(\xi_0)} + (k^2 - 3k + 3) \geq n^2\left[Re\frac{\xi_0^2 q'''(\xi_0)}{q'(\xi_0)} + 1\right] \tag{2.6}$$

or

$$Re\frac{z_0^2 p'''(z_0)}{p'(z_0)} \geq n^2 Re\frac{\xi_0^2 q'''(\xi_0)}{q'(\xi_0)}. \tag{2.7}$$

Next, it is important to remember the fundamental ideas of fuzzy differential subordination as shown in [25, 26].

Definition 2.4 ([25])**.** Let X be nonempty set. An application $F : X \to [0, 1]$ is called fuzzy set.

According to [26], this definition can be used in the more precise form given as:

Definition 2.5 ([26])**.** A pair (A, F_A), where $F_A : X \to [0, 1]$ and $A = \{x \in X : 0 < F_A(x) < 1\}$, is called fuzzy subset of X. The set A is called the support of the fuzzy set (A, F_A) and F_A is called the membership function of the fuzzy set (A, F_A). One can also denote $A = supp(A, F_A)$.

Definition 2.6 ([26]). Let $D \subset \mathbb{C}$ and let $z_0 \in D$ be a fixed point and let the functions f, g be holomorphic in $\mathbb{U}$. The function f is said to be fuzzy subordinate to function g and write $f \prec_F g$ or $f(z) \prec_F g(z)$ if there exists a function $G : \mathbb{C} \to [0,1]$ such that

$$\begin{aligned} &(1)\ f(z_0) = g(z_0), \\ &(2)\ G(f(z)) \le G(g(z)),\ z \in D. \end{aligned} \tag{2.8}$$

The research in [26] provides additional information on examples of functions that can be viewed as function G in the previous definition.

Now that the overall context for the study has been properly discussed, the following section will highlight the key findings of our work into third-order fuzzy differential subordination.

3. Main Results

The following is an introduction to the general aspects of third-order fuzzy differential subordination:

Let $\Omega = (\Omega, F_\Omega) = supp(\Omega, F_\Omega) = \{z \in \mathbb{C} : 0 < F_\Omega(z) \le 1\}$, $\Delta = (\Delta, F_\Delta) = supp(\Delta, F_\Delta) = \{z \in \mathbb{C} : 0 < F_\Delta(z) \le 1\}$, $p(\mathbb{U}) = supp(p(\mathbb{U}), F_{p(\mathbb{U})}) = \{p(z) : 0 < F_{p(\mathbb{U})}(p(z)) \le 1\}$, $z \in \mathbb{U}$ and

$$\begin{aligned} \psi(\mathbb{C}^4 \times \mathbb{U}) &= supp(\mathbb{C}^4 \times \mathbb{U}, F_{\psi(\mathbb{C}^4\times\mathbb{U})}) \\ &= \{\psi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z) : 0 < F_{\psi(\mathbb{C}^4\times\mathbb{U})}(\psi(p(z), zp'(z), z^2p''(z), \\ &\quad z^3p'''(z))) \le 1\}. \end{aligned}$$

The class of admissible functions is first defined.

Definition 3.1. Consider $\Omega \subset \mathbb{C}$, a function $q \in Q$ and $m \ge 2$. Functions $\psi : \mathbb{C}^4 \times \mathbb{U} \to \mathbb{C}$ are called admissible functions and belong to the so called class of admissible functions denoted by $\Psi_{\mathbb{U}}[\Omega, q]$ if the admissibility condition

$$\psi(r, s, t, u; z) = 0, \tag{3.1}$$

is satisfied considering

$$r = q(\xi),\ \ s = m\xi q'(\xi),$$

$$\mathrm{Re}\left\{\frac{t}{s} + 1\right\} \ge m\mathrm{Re}\left\{\frac{\xi q''(\xi)}{q'(\xi)} + 1\right\},$$

and

$$\mathrm{Re}\left\{\frac{u}{s}\right\} \ge m^2 Re\left\{\frac{\xi^2 q'''(\xi)}{q'(\xi)}\right\}$$

with $z \in \mathbb{U}, \xi \in \partial\mathbb{U} \setminus E(q)$ and $m \in \mathbb{N} \setminus \{1\}$.

The third-order fuzzy differential subordination's general form is as follows:

Consider $\Omega = (\Omega, F_\Omega) \subset \mathbb{C}$, $\Delta = (\Delta, F_\Delta) \subset \mathbb{C}$ and let $p \in \mathcal{H}(\mathbb{U})$ satisfying $p(0) = a$, $a \in \mathbb{C}$. For a function $\phi(r, s, t, u; z) : \mathbb{C}^4 \times \mathbb{U} \to \mathbb{C}$, the following implication is studied as proposed problem:

$$F_{\psi(\mathbb{C}^4\times\mathbb{U})}(\psi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z)) \le F_\Omega(z) \ \text{ implies } \ F_{p(\mathbb{U})}(p(z)) \le F_\Delta(z). \tag{3.2}$$

If $\Delta = (\Delta, F_\Delta) \ne \mathbb{C}$ is a simply connected domain with $a \in \Delta$, there exists a function q which maps conformly the unit disc satisfying the condition $q(0) = a$. Then the implication (3.2) can be written as:

$$F_{\psi(\mathbb{C}^4\times\mathbb{U})}(\psi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z)) \le F_\Omega(z) \ \text{ implies } \ F_{p(\mathbb{U})}(p(z)) \le F_{q(\mathbb{U})}(q(z)) \tag{3.3}$$

i.e

$$p(z) \prec_F q(z).$$

If $\Omega = (\Omega, F_\Omega) \neq \mathbb{C}$ is also a simply connected domain, there exists a function h which maps conformly the unit disc satisfying the condition $h(0) = \psi(a, 0, 0, 0; 0)$. If we also have that $\psi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z) \in \mathcal{H}(\mathbb{U})$ then the implication (3.2) can be written as:

$$\psi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z) \prec_F h(z) \,\text{and}\, F_{\psi(\mathbb{U})}(z) \leq F_{h(\mathbb{U})}(z) \tag{3.4}$$

implies

$$p(z) \prec_F q(z) \text{ and } F_{p(\mathbb{U})}(p(z)) \leq F_{q(\mathbb{U})}(q(z)).$$

The solution of a third-order fuzzy differential subordination and the fuzzy best dominant are defined as follows:

Definition 3.2. Consider $\psi : \mathbb{C}^4 \times \mathbb{U} \to \mathbb{C}$, $\psi \in \Psi_n[\Omega, q]$ and let $h \in \mathcal{H}_u(\mathbb{U})$. A function $p \in \mathcal{H}(\mathbb{U})$ is called solution for a third-order fuzzy differential subordination if it satisfies:

$$F_{\psi(\mathbb{C}^4\times\mathbb{U})}(\psi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z)) \leq F_{h(\mathbb{U})}h(z)$$

i.e.

$$\psi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z) \prec_F h(z). \tag{3.5}$$

A function $q \in \mathcal{H}_u(\mathbb{U})$ is said to be a dominant of the solutions of a third-order fuzzy differential subordination if it satisfies the fuzzy subordination $p \prec_F q$ for all solutions p of the third-order fuzzy differential subordination (3.5). The fuzzy best dominant is a dominant $\tilde{q}$ that satisfies the fuzzy subordination $\tilde{q} \prec_F q$ or all dominants q of (3.5). The fuzzy best dominant is known to be unique up to a rotation of $\mathbb{U}$.

Remark 3.3. The implication (3.5) can be replaced by a more general containment of the form:

$$\left\{\psi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z) : z \in \mathbb{U}\right\} \subset \Omega.$$

where $\Omega = (\Omega, F_\Omega)$. In this situation, the containment is also seen as a third-order fuzzy differential subordination and the definitions given above remain valid.

The next theorems are fundamental for the progress of third-order fuzzy differential subordination theory.

Theorem 3.4. *Taking a function $p \in \mathcal{H}[a, n]$, when $n \geq 2$, consider $F : \mathbb{C} \to [0, 1]$ and a function $q \in Q(a)$ for which:*

$$Re\frac{\xi q''(\xi)}{q'(\xi)} \geq 0 \;\; and \;\; \left|\frac{zp'(z)}{q'(\xi)}\right| \leq n \tag{3.6}$$

when $z \in \mathbb{U}$ and $\xi \in \partial\mathbb{U} \setminus E(q)$. If $(\Omega, F_\Omega) \subset \mathbb{C}$ and $\psi \in \Psi_n[\Omega, q]$ satisfying

$$\left\{\psi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z) : z \in \mathbb{U}\right\} \subset \Omega \tag{3.7}$$

then

$$p(z) \prec_F q(z) \text{ or } F_{p(\mathbb{U})}(p(z)) \leq F_{q(\mathbb{U})}(q(z)).$$

Proof. For $z = z_0$ using (3.7),

$$\psi(p(z_0), z_0p'(z_0), z_0^2p''(z_0), z_0^3p'''(z_0); z_0) \in \Omega. \tag{3.8}$$

Assuming that p isnt fuzzy subordinated to function q, by applying Lemma 2.3, we get that there exist points $z_0 = r_0e^{i\theta_0}$ and $\xi_0 \in \partial\mathbb{U} \setminus E(q)$ satisfying $p(z_0) = q(z_0)$ and $p(\overline{\mathbb{U}}_{r_0}) \subset q(\mathbb{U})$, $z_0p'(z_0) = n\xi_0q'(\xi_0)$ and

$$\text{Re}\frac{z_0p''(z_0)}{p'(z_0)} + 1 \geq n\text{Re}\left(\frac{\xi_0q''(\xi_0)}{q'(\xi_0)} + 1\right),$$

$$\mathrm{Re}\frac{z_0^2p'''(z_0)}{p'(z_0)}+1\geq n^2\mathrm{Re}\frac{\xi_0^2q'''(\xi_0)}{q'(\xi_0)}.$$

Since the conditions needed for Lemma 2.3 are satisifed

$$r=p(z_0),\ s=z_0p'(z_0),\ t=z_0^2p''(z_0),\ u=z_0^2p'''(z_0),$$

the admissibility condition given in (3.1) becomes

$$\psi(p(z_0),z_0p'(z_0),z_0^2p''(z_0),z_0^3p'''(z_0);z_0)=0 \tag{3.9}$$

from which we deduce

$$\psi(p(z_0),z_0p'(z_0),z_0^2p''(z_0),z_0^3p'''(z_0);z_0)\notin\Omega. \tag{3.10}$$

Since (3.10) contradicts (3.8) the assumption made in false and we have

$$p(z)\prec_F q(z),\ z\in\mathbb{U}.$$

□

If the behavior of function q is not known on the closed unit disc, then we can use $q^2(z)=q(z^2)$ which satisfies the conditions for applying Lemma 2.3. In this case, the following result can be stated.

Theorem 3.5. *Take a function $q\in\mathcal{H}_u(\mathbb{U})$ with $q(0)=a$ and $\rho\in(0,1)$. Consider $F:\mathbb{C}\to[0,1]$ and $q_\rho(z)\equiv q(\rho z)$,$z\in\mathbb{U}$. Taking a function $p\in\mathcal{H}[a,n]$, when $n\geq 2$, let p and q_ρ satisfy:*

$$Re\frac{\xi q_\rho''(\xi)}{q_\rho'(\xi)}\geq 0\ \ and\ \ \left|\frac{zp'(z)}{q_\rho'(\xi)}\right|\leq n$$

when $z\in\mathbb{U}$ and $\xi\in\partial\mathbb{U}\setminus E(q_\rho)$.

If $\Omega\subset\mathbb{C}$ and $\psi\in\Psi_n[\Omega,q_\rho]$ then

$$\left\{\psi(p(z),zp'(z),z^2p''(z),z^3p'''(z);z):z\in\mathbb{U}\right\}\subset\Omega \tag{3.11}$$

hence

$$p(z)\prec_F q(z),\ \ z\in\mathbb{U}.$$

Proof. We must prove that $q_\rho\in\mathcal{H}_u(\mathbb{U})$. This is given by the following:

$$q_\rho(z_1)=q_\rho(z_2),\ q_\rho(z_1)=q(\rho z_1),\ q_\rho(z_2)=q(\rho z_2)\Rightarrow q(\rho z_1)=q(\rho z_2).$$

Since q is univalent we have

$$\rho z_1=\rho z_2\Rightarrow z_1=z_2,$$

from which we deduce that q_ρ is univalent in $\mathbb{U}$. Since it is univalent on $\overline{\mathbb{U}}$, we have $E(q_\rho)=\emptyset$ and $q_\rho(0)=q(\rho,0)=q(0)=a$, which gives $q_\rho\in Q(a)$ and $\Psi_n[\Omega,q_\rho]$ is the class of admissible functions. Since $\psi\in\Psi_n[\Omega,q_\rho]$, using (3.9) we obtain

$$p(z)\prec_F q(z),\ \ z\in\mathbb{U},\ \ or\ F_{p(\mathbb{U})}(p(z))\leq F_{q_\rho(\mathbb{U})}(q_\rho(z)). \tag{3.12}$$

But $q_\rho(z)=q_(\rho z)=q(z)$ and we get

$$F_{q_\rho(\mathbb{U})}(q_\rho(z))\leq F_{q(\mathbb{U})}(q(z)). \tag{3.13}$$

Using relations (3.12) (3.13), we obtain

$$F_{p(\mathbb{U})}(p(z))\leq F_{q_{\rho(\mathbb{U})}}(q_\rho(z))\leq F_{q(\mathbb{U})}(q(z)),\ or\ p(z)\prec_F q(z),z\in\mathbb{U}, \tag{3.14}$$

hence

$$F_{p(\mathbb{U})}(p(z))\leq F_{q(\mathbb{U})}(q(z)),\ or\ p(z)\prec_F q(z),z\in\mathbb{U}.$$

□

Considering the situation of $\Omega \neq \mathbb{C}$ being a simply connected domain and knowing that function h maps conformly $\mathbb{U}$ into Ω, the class of admissible functions given in Definition 3.1 is denoted by $\psi \in \Psi_n[h, g]$. Using this notation, Theorem 3.4 and 3.5 can be written as follows:

Theorem 3.6. *Taking a function $p \in \mathcal{H}[a, n]$ when $n \geq 2$, consider a function $q \in Q(a)$ satisfying the conditions:*

$$Re\frac{\xi q''(\xi)}{q'(\xi)} \geq 0 \text{ and } \left|\frac{zp'(z)}{q'(\xi)}\right| \leq n \tag{3.15}$$

when $z \in \mathbb{U}$ and $\xi \in \partial\mathbb{U} \setminus E(q)$. Also consider $F : \mathbb{C} \to [0, 1]$. If $\psi \in \Psi_n[h, q]$ and $\psi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z) \in \mathcal{H}(\mathbb{U})$, then we have

$$F_{\Psi(\mathbb{C}^4\times\mathbb{U})}(\psi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z)) \leq F_{h(\mathbb{U})}(h(z)) \tag{3.16}$$

or

$$\psi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z) \prec_F h(z) \tag{3.17}$$

implies

$$F_{p(\mathbb{U})}(p(z)) \leq F_{q(\mathbb{U})}(q(z)) \text{ or } p(z) \prec_F q(z),\ z \in \mathbb{U}. \tag{3.18}$$

Considering the definition of $Q(a)$, we can see that by hypothesis of Theorem 3.6 is restrictive regarding the function q considered. When the behaviour of function q on the closed unit disc $\bar{\mathbb{U}}$ is not known, it is still possible to state a similar result using function $q_\rho(z) \equiv q(\rho z),\ \rho \in (0, 1), z \in \mathbb{U}$, which satisfies the conditions required by Lemma 2.3.

Theorem 3.7. *Let $q \in \mathcal{H}_u(\mathbb{U})$ satisfying*

$$Re\frac{\xi q''(\xi)}{q'(\xi)} \geq 0 \text{ and } \left|\frac{zp'(z)}{q'(\xi)}\right| \leq n \tag{3.19}$$

when $z \in \mathbb{U}$ and $\xi \in \partial\mathbb{U} \setminus E(q_\rho)$. Also consider $F : \mathbb{C} \to [0, 1]$. If $\psi \in \Psi_n[h, q]$ and $\psi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z) \in \mathcal{H}(\mathbb{U})$ then we have

$$F_{\Psi(\mathbb{C}^4\times\mathbb{U})}(\psi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z)) \leq F_{h(\mathbb{U})}(h(z)) \tag{3.20}$$

or

$$\psi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z) \prec_F h(z) \tag{3.21}$$

implies

$$F_{p(\mathbb{U})}(p(z)) \leq F_{q(\mathbb{U})}(q(z)) \text{ or } p(z) \prec_F q(z),\ z \in \mathbb{U}. \tag{3.22}$$

The following theorem provides means for obtaining the third-order best fuzzy dominant.

Theorem 3.8. *Let $p \in \mathcal{H}[a, n]$ when $n \geq 2$, consider $\Psi : \mathbb{C}^4 \times \mathbb{U} \to \mathbb{C}$ and assume that $\psi(q(z), zq'(z), z^2q''(z), z^3q'''(z); z) \in \mathcal{H}(\mathbb{U})$. Consider also the functions $F : \mathbb{C} \to [0, 1]$ and $h \in \mathcal{H}_u(\mathbb{U})$. Suppose that function $q \in Q(a)$ is a solution for*

$$\psi(q(z), zq'(z), z^2q''(z), z^3q'''(z); z) = h(z),\ z \in \mathbb{U}, \tag{3.23}$$

and that q satisfies

$$Re\frac{\xi q''(\xi)}{q'(\xi)} \geq 0 \text{ and } \left|\frac{zp'(z)}{q'(\xi)}\right| \leq n, \tag{3.24}$$

when $z \in \mathbb{U}$ and $\xi \in \partial\mathbb{U} \setminus E(q)$. If $\psi \in \Psi_n[h, q]$ with $n \geq 2$.
Then

$$F_{\Psi(\mathbb{C}^4\times\mathbb{U})}(\psi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z)) \leq F_{h(\mathbb{U})}(h(z))$$

or

$$\psi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z) \prec_F h(z) \tag{3.25}$$

implies

$$F_{p(\mathbb{U})}(p(z)) \leq F_{q(\mathbb{U})}(q(z)) \text{ or } p(z) \prec_F q(z),\ z \in \mathbb{U}. \tag{3.26}$$

Function q satisfying (3.26) *is the third-order fuzzy best dominant.*

Proof. Using the results proved in Theorem 3.4 and (3.25), we obtain the fuzzy differential subordination

$$F_{p(\mathbb{U})}(p(z)) \leq F_{q(\mathbb{U})}(q(z)),\ \ z \in \mathbb{U},$$

written equivalently as

$$p(z) \prec_F q(z),\ \ z \in \mathbb{U}.$$

This means that function q is a dominant of the third-order fuzzy differential subordination (3.25). We also know that $q \in \mathcal{H}_u(\mathbb{U})$ is a solution for (3.24), hence q is the third-order fuzzy best dominant. □

Remark 3.9. In the case when the behavior of function q isnt known on $\partial\mathbb{U}$, the result proved in Theorem 3.8 can be extended using the function $q_\rho(z) = q(\rho z)$, $z \in \mathbb{U}$, $\rho \in (0,1)$ as it can be seen in the next theorem.

Theorem 3.10. *Take $h, q \in \mathcal{H}_u(\mathbb{U})$ with $q(0) = a$ and let $\rho \in (0,1)$. Consider the functions $F : \mathbb{C} \to [0,1]$ and $\Psi : \mathbb{C}^4 \times \mathbb{U} \to \mathbb{C}$, with Ψ satisfying one of the following:*
(i) $\psi \in \Psi_n[h_\rho, q\rho]$ for certain $\rho \in (0,1)$ or,
(ii) $\exists \rho_0 \in (0,1)$ such that $\psi \in \Psi_n[h_\rho, q\rho]$,$\forall \rho \in (0,1)$.

Taking a function $p \in \mathcal{H}[a,1]$ and considering that the function $\psi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z)$ is analytical in U with

$$\psi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z) \prec_F h(z),$$

or

$$F_{\Psi(\mathbb{C}^4\times\mathbb{U})}(\psi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z)) \leq F_{h(\mathbb{U})}(h(z)),$$

then

$$F_{p(\mathbb{U})}(p(z)) \leq F_{q(\mathbb{U})}(q(z)) \ \text{or}\ p(z) \prec_F q(z),\ \ z \in \mathbb{U}.$$

Proof. **Case (i):** We have that $q_\rho(z) = q(\rho z)$, hence $q_\rho(0) = q(\rho 0) = q(0) = a$. We also know that $q_\rho \in \mathcal{H}_u(\mathbb{U}))$(proved in Theorem 3.5), and since $q_\rho(0) = a$, we conclude that $q_\rho \in Q(a)$. This gives that $\Psi_n[\Omega, q_\rho]$ is the class of admissible functions.
Since $\psi \in \Psi_n[\Omega, q_\rho]$, by using Theorem 3.5 relation (3.11), we get that

$$p(z) \prec_F q_\rho(z),$$

or

$$F_{p(\mathbb{U})}(p(z)) \leq F_{q_\rho(\mathbb{U})}(q_\rho(z)).$$

Since $q_\rho(z) \prec_F q(z)$ or $F_{q_\rho(\mathbb{U})}(q_\rho(z)) \leq F_{q(\mathbb{U})}(q(z))$, we have

$$p(z) \prec_F q_\rho(z) \prec_F q(z),$$

or

$$F_{p(\mathbb{U})}(p(z)) \leq F_{q_\rho(\mathbb{U})}(q_\rho(z)) \leq F_{q(\mathbb{U})}(q(z)),$$

implies

$$p(z) \prec_F q(z),$$

or

$$F_{p(\mathbb{U})}(p(z)) \leq F_{q(\mathbb{U})}(q(z)), z \in \mathbb{U}.$$

Case (ii): We have that $h_\rho(z) = h(\rho z)$. Hence, we can write

$$\psi(p_\rho(z), z{p_\rho}'(z), z^2{p_\rho}''(z), z^3{p_\rho}'''(z); z) \in h_\rho(\mathbb{U}).$$

Using the results proved in Theorem 3.5, we get that $p_\rho(z) \prec_F q_\rho(z)$ for all $\rho \in (0,1)$. If we make $\rho \to 1$, we obtain the fuzzy differential subordination $p(z) \prec_F q(z)$ subordination equivalently written as $F_{p(\mathbb{U})}(p(z)) \leq F_{q(\mathbb{U})}(q(z))$, $z \in \mathbb{U}$. □

The next theorem provides sufficient conditions for finding the third-order fuzzy best dominant in case when n is any natural number satisfying $n \geq 2$.

Theorem 3.11. *Take $h \in \mathcal{H}_u(\mathbb{U})$, $F : \mathbb{C} \to [0,1]$, $p \in \mathcal{H}[a,n]$ when $n \geq 2$, and consider $\Psi : \mathbb{C}^4 \times \mathbb{U} \to \mathbb{C}$. Let function q, with $q(0) = a$, be a solution for:*

$$\psi(q(z), nzq'(z), n(n-1)zq'(z)+n^2z^2q''(z), n(n-1)(n-2)zq'(z)+3n^2(n-1)z^2q''(z)+n^3z^3q'''(z)) = h(z), \quad (3.27)$$

such that q satisfies

$$Re\frac{\xi q''(\xi)}{q'(\xi)} \geq 0 \text{ and } \left|\frac{zp'(z)}{q'(\xi)}\right| \leq n,$$

when $z \in \mathbb{U}$ and $\xi \in \partial\mathbb{U} \setminus E(q)$. If $\psi \in \Psi_n[h,q]$ with $n \geq 2$.

Assume that q satisfies one of the following assertions:
(i) $q \in Q$ and $\psi \in \Psi_n[h,q]$,
(ii) $q \in \mathcal{H}_u(\mathbb{U})$, and $\psi \in \Psi_n[h,q_\rho]$ for certain $\rho \in (0,1)$,
(iii) $q \in \mathcal{H}_u(\mathbb{U})$ and there exists $\exists\rho_0 \in (0,1)$ such that $\psi \in \Psi_n[h_\rho, q_\rho]$, $\forall\rho \in (\rho_0, 1)$.

Taking a function $p \in \mathcal{H}[a,n]$, if $\psi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z) \in \mathcal{H}(\mathbb{U})$ and function p satisfies

$$F_{\Psi(\mathbb{C}^4\times\mathbb{U})}(\psi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z)) \leq F_{h(\mathbb{U})}(h(z)),$$

or

$$\psi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z) \prec_F h(z), \quad (3.28)$$

then

$$F_{p(\mathbb{U})}(p(z)) \leq F_{q(\mathbb{U})}(q(z)) \text{ or } p(z) \prec_F q(z),\ z \in \mathbb{U},$$

where q is said to be the third-order fuzzy best dominant.

Proof. Using the results proved in Theorem 3.6 and Theorem 3.7, we obtain the fuzzy differential subordination

$$F_{p(\mathbb{U})}(p(z)) \leq F_{q(\mathbb{U})}(q(z)),\ z \in \mathbb{U},$$

written equivalently as $p(z) \prec_F q(z),\ z \in \mathbb{U}$.
In order to show that q is the third-order fuzzy best dominant we must prove that q is a solution for (3.27). If we let $p(z) = q(z^n)$, $p(\mathbb{U}) = q(\mathbb{U})$ then

$$\begin{aligned} zp'(z) &= nz^nq'(z^n) \\ z^2p''(z) &= n(n-1)z^nq'(z^n) + n^2z^{2n}q''(z) \\ z^3p'''(z) &= n(n-1)(n-2)z^nq'(z^n) + 3n^2(n-1)z^{2n}q''(z^n) + n^3z^{3n}q'''(z^n). \end{aligned}$$

By applying $z = z^n$ in (3.27), we get

$$\begin{aligned} &\psi(q(z^n), nz^nq'(z^n), n(n-1)z^nq'(z^n) + n^2z^{2n}q''(z^n), \\ &\quad n(n-1)(n-2)z^nq'(z^n) + 3n^2(n-1)z^{2n}q''(z^n) + n^3z^{3n}q'''(z^n)) = h(z^n) \\ &\quad = \psi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z). \end{aligned} \quad (3.29)$$

Considering relation (3.29) in (3.28) we conclude

$$\begin{aligned} &\psi(q(z^n), nz^nq'(z^n), n(n-1)z^nq'(z^n) + n^2z^{2n}q''(z^n), \\ &\quad n(n-1)(n-2)z^nq'(z^n) + 3n^2(n-1)z^{2n}q''(z^n) + n^3z^{3n}q'''(z^n)) \prec_F h(z^n). \end{aligned}$$

We now have that q is a solution for (3.27), hence q is the third-order fuzzy best dominant. □

4. Examples

In this section, some numerical examples constructed as applications of the results obtained in the theorems are presented.

Example 4.1. Consider $p(z) = 1 + z^2$, hence $p \in \mathcal{H}[1, 2]$. Compute $p'(z) = 2z$, and further obtain $p''(z) = 2$ and $p'''(z) = 0$. Let

$$\Psi : \mathbb{C}^4 \times \mathbb{U} \to \mathbb{C}, \Psi(r, s, t, u; z) = r + \frac{1}{3}s + \frac{1}{2}t + \frac{1}{4}u.$$

Taking $r = p(z) = 1 + z^2$, $s = zp'(z) = 2z^2$, $t = z^2p''(z) = 2z^2$, and $u = z^3p'''(z) = 0$, we get

$$\psi(1 + z^2, 2z^2, 2z^2, 0; z) = 1 + \frac{1}{3}2z^2 + \frac{1}{2}2z^2 + \frac{1}{4}0 = 1 + \frac{5}{3}z^2.$$

Using Theorem 3.6 we have the following. Let $p(z) = 1 + z^2$, $p \in \mathcal{H}[1, 2]$ and $q \in Q(1)$, $q(z) = 1 + z$, an univalent function in $\mathbb{U}$ which satisfies

$$\mathrm{Re}\frac{\xi q''(\xi)}{q'(\xi)} \geq 0 \ \ and \ \ \left|\frac{zp'(z)}{q'(\xi)}\right| \leq 2. \tag{4.1}$$

Function $h(z) = \frac{1-z}{1+z}$ maps conformally $\mathbb{U}$ into the half plane $\{z \in \mathbb{C} : Rez > 0\}$. Let $F : \mathbb{C} \to [0, 1]$. If $\psi \in \Psi_n[h, q]$,

$$\psi(1 + z^2, 2z^2, 2z^2, 0; z) = 1 + \frac{5}{3}z^2$$

and the following third-order fuzzy differential subordination holds

$$1 + \frac{5}{3}z^2 \prec_F \frac{1 - z}{1 + z}, \ z \in \mathbb{U},$$

or

$$F_{\Psi(\mathbb{C}^4\times\mathbb{U})}(\psi(1 + z^2, 2z^2, 2z^2, 0; z)) \leq F_{h(\mathbb{U})}(h(z)), \tag{4.2}$$

then

$$p(z) = 1 + z^2 \prec_F q(z) = 1 + z, \ z \in \mathbb{U},$$

or

$$F_{p(\mathbb{U})}(p(z)) \leq F_{q(\mathbb{U})}(q(z)), \ z \in \mathbb{U}.$$

Indeed, we first prove that $q(z) = 1 + z$ satisfies the conditions given in (4.1):

$$\mathrm{Re}\frac{\xi q''(\xi)}{q'(\xi)} = Re\frac{\xi \cdot 0}{1} \geq 0 \ \ and \ \ \left|\frac{zp'(z)}{q'(\xi)}\right| = \left|\frac{2z^2}{1}\right| = 2|z|^2 \leq 2.$$

The function $q(z) = 1 + z$ being convex in $\mathbb{U}$, relation (4.2) can be written as:

$$Re(1 + z^2) \geq Re(1 + z), \ z \in \mathbb{U}.$$

$Rep(z) = Re(1 + z^2) = Re(1 + \cos^2\theta + i \sin\theta\cos\theta - \sin^2\theta) = 1 + \cos^2\theta - \sin^2\theta = 2\cos^2\theta$ which gives that

$$p(\mathbb{U}) = \{z \in \mathbb{C} : Rez > 2\cos^2\theta\} = \{z \in \mathbb{C} : Rez \geq 0\},$$

and

$$Req(z) = Re(1 + z) = Re(1 + \cos\theta + i \sin\theta) = 1 + \cos\theta = 2\cos^2\frac{\theta}{2} \geq 0$$

which gives that

$$q(\mathbb{U}) = \{z \in \mathbb{C} : Rez \geq 0\}.$$

We know that $p(0) = q(0) = 1$ and $p(\mathbb{U}) \subseteq q(\mathbb{U})$, and hence we conclude that

$$1 + z^2 \prec_F 1 + z, \ z \in \mathbb{U}.$$

The third-order best fuzzy dominant is given using the outcome of Theorem 3.8 for the case when n=2.

Example 4.2. Consider $p(z) = 1 + \frac{z^2}{2} + \frac{z^3}{3}$, hence $p \in \mathcal{H}[1,2]$. Let

$$\Psi : \mathbb{C}^4 \times \mathbb{U} \to \mathbb{C}, \Psi(r,s,t,u;z) = r + s + \frac{t}{2} + \frac{u}{2}.$$

Consider also the functions $F : \mathbb{C} \to [0,1]$, $F(z) = \frac{1}{|z|}$ and $q(z) = 1 + \frac{z^3}{3}$, $q(0) = p(0) = 1$, $q'(z) = z^2, q''(z) = 2z, q'''(z) = 2$.

For the function q considered, we determine using (3.23):

$$h(z) = \Psi(1 + \frac{z^3}{3}, z \cdot z^2, z^2 \cdot 2, z^3 \cdot 2) = 1 + \frac{z^3}{3} + z^3 + \frac{2z^3}{2} + \frac{2z^3}{2} = 1 + \frac{10}{3}z^3.$$

Taking $r = p(z) = 1 + \frac{z^2}{2} + \frac{z^3}{3}$, $s = zp'(z) = z(z+z^2)$, $t = z^2p''(z) = z^2(1+2z)$, and $u = z^3p'''(z) = z^3 \cdot 2$, we get

$$\psi(1+\frac{z^2}{2}+\frac{z^3}{3}, z(z+z^2), z^2(1+2z), z^3\cdot 2; z) = 1+\frac{z^2}{2}+\frac{z^3}{3}+z^2+z^3\frac{z^2+2z^3}{2}+\frac{2z^3}{2} = 1+2z^2+\frac{10z^3}{3}.$$

Applying Theorem 3.8 we have the following.

Let $p(z) = 1 + \frac{z^2}{2} + \frac{z^3}{3}$, $p \in \mathcal{H}[1,2]$ and $q \in Q(1)$, $q(z) = 1 + \frac{z^3}{3}$, an univalent function in $\mathbb{U}$ which satisfies

$$\mathrm{Re}\frac{\xi q''(\xi)}{q'(\xi)} \geq 0 \ \ and \ \ \left|\frac{zp'(z)}{q'(\xi)}\right| \leq 2$$

and the third-order fuzzy differential subordination

$$F_{\Psi(\mathbb{C}^4\times\mathbb{U})}(\psi(1 + \frac{z^2}{2} + \frac{z^3}{3}, z(z+z^2), z^2(1+2z), z^3 \cdot 2; z)) \leq F_{h(\mathbb{U})}(h(z)),$$

equivalently written

$$1 + 2z^2 + \frac{10z^3}{3} \prec_F 1 + \frac{10}{3}z^3, \ z \in \mathbb{U},$$

or

$$\frac{1}{|1 + 2z^2 + \frac{10z^3}{3}|} \leq \frac{1}{|1 + \frac{10}{3}z^3|}$$

holds, then

$$p(z) = 1 + \frac{z^2}{2} + \frac{z^3}{3} \prec_F q(z) = 1 + \frac{z^3}{3}, \ z \in \mathbb{U},$$

or

$$\frac{1}{|1 + \frac{z^2}{2} + \frac{z^3}{3}|} \leq \frac{1}{|1 + \frac{z^3}{3}|}$$

equivalently written

$$F_{p(\mathbb{U})}(p(z)) \leq F_{q(\mathbb{U})}(q(z)), \ z \in \mathbb{U},$$

and q is the third-order fuzzy best dominant.

Indeed, we first prove that $q(z) = 1 + \frac{z^3}{3}$ satisfies the conditions given in (3.24) when n=2:

$$\mathrm{Re}\frac{\xi q''(\xi)}{q'(\xi)} = Re\frac{2\xi^2}{\xi^2} = 2 \geq 0 \ \ and$$

$$\left|\frac{zp'(z)}{q'(\xi)}\right| = \left|\frac{z(z+z^2)}{\xi^2}\right| = \frac{|z^3||1+z|}{|\xi|^2} = |1+z| = \sqrt{2(1+\cos\theta)} \leq 2.$$

The function $q(z) = 1 + \frac{z^3}{3}$ being the solution of the equation

$$h(z) = \Psi(1 + \frac{z^3}{3}, z \cdot z^2, z^2 \cdot z, z^3 \cdot 2; z) = 1 + \frac{10}{3}z^3,$$

we conclude that it is the third-order fuzzy best dominant.

The third-order best fuzzy dominant is given using the outcome of Theorem 3.11 for the case when $n \geq 2$.

Example 4.3. Consider $p(z) = 1 + \frac{z^4}{4} + \frac{z^5}{5}$, hence $p \in \mathcal{H}[1, 4], 4 > 2$. Let

$$\Psi : \mathbb{C}^4 \times \mathbb{U} \to \mathbb{C}, \Psi(r, s, t, u; z) = r + s + \frac{t}{2} + \frac{u}{2}.$$

Consider also the functions $F : \mathbb{C} \to [0, 1]$, $F(z) = \frac{1}{|z|}$ and $q(z) = 1+\frac{z^4}{4}$, $q(0) = p(0) = 1$, $q'(z) = z^3, q''(z) = 3z^2, q'''(z) = 6z$.

For the function q considered, we determine using (3.27):

$$h(z) = \Psi(1 + \frac{z^4}{4}, 4z \cdot z^3, 4 \cdot 3 \cdot z \cdot z^3 + 16z^2 \cdot 3z^2, 4 \cdot 3 \cdot 2 \cdot z \cdot z^3 + 3 \cdot 4^2 \cdot 3z^2 \cdot 3z^2 + 64z^3 \cdot 6z; z)$$

$$= 1 + \frac{z^4}{4} + 12z^4 + \frac{12z^4 + 48z^4}{2} + \frac{24z^4 + 472z^4 + 384z^4}{2}$$

$$= 1 + \frac{z^4}{4} + 12z^4 + 30z^4 + 440z^4 = 1 + \frac{1929z^4}{4}.$$

Taking $r = p(z) = 1 + \frac{z^4}{4} + \frac{z^5}{5}$, $s = zp'(z) = z(z^3 + z^4)$, $t = z^2p''(z) = z^2(3z^2 + 4z^3)$, and $u = z^3p'''(z) = z^3(6z + 12z^2)$, we get:

$$\psi(1 + \frac{z^4}{4} + \frac{z^5}{5}, z(z^3 + z^4), z^2(3z^2 + 4z^3), z^3(6z + 12z^2); z)$$

$$= 1 + \frac{z^4}{4} + \frac{z^5}{5} + z^4 + z^5 + \frac{3z^4 + 4z^5}{2} + \frac{6z^4 + 12z^5}{2}$$

$$= 1 + \frac{23z^4}{4} + \frac{48z^5}{5}.$$

Applying Theorem 3.11 we obtain the following.

Let $p(z) = 1 + \frac{z^4}{4} + \frac{z^5}{5}$, $p \in \mathcal{H}[1, 4]$ and $q \in Q(1)$, $q(z) = 1 + \frac{z^4}{4}$, an univalent function in $\mathbb{U}$ which satisfies

$$\mathrm{Re}\frac{\xi q''(\xi)}{q'(\xi)} \geq 0 \ \ and \ \ \left|\frac{zp'(z)}{q'(\xi)}\right| \leq n = 4$$

and the third-order fuzzy differential subordination

$$F_{\Psi(\mathbb{C}^4\times\mathbb{U})}(\psi(1 + \frac{z^4}{4} + \frac{z^5}{5}, z(z^3 + z^4), z^2(3z^2 + 4z^3), z^3(6z + 12z^2); z)) \leq F_{h(\mathbb{U})}(h(z)),$$

equivalently written

$$1 + \frac{23z^4}{4} + \frac{46z^5}{5} \prec_F 1 + \frac{1929z^4}{4}, \ z \in \mathbb{U},$$

or

$$\frac{1}{|1 + \frac{23z^4}{4} + \frac{46z^5}{5}|} \leq \frac{1}{|1 + \frac{1929z^4}{4}|}$$

holds, then

$$p(z) = 1 + \frac{z^4}{4} + \frac{z^5}{5} \prec_F q(z) = 1 + \frac{z^4}{4}, \ z \in \mathbb{U},$$

or

$$\frac{1}{|1 + \frac{z^4}{4} + \frac{z^5}{5}|} \leq \frac{1}{|1 + \frac{z^4}{4}|}$$

equivalently written

$$F_{p(\mathbb{U})}(p(z)) \leq F_{q(\mathbb{U})}(q(z)), \ z \in \mathbb{U},$$

and q is the third-order fuzzy best dominant.

Indeed, we first prove that $q(z) = 1 + \frac{z^4}{4}$ satisfies the conditions given in (3.24) when $n \geq 2$:

$$\mathrm{Re}\frac{\xi q''(\xi)}{q'(\xi)} = Re\frac{2\xi^3}{\xi^3} = 2 \geq 0 \ \ and$$

$$\left|\frac{zp'(z)}{q'(\xi)}\right| = \left|\frac{z(z^3+z^4)}{\xi^3}\right| = \frac{|z^4||1+z|}{|\xi|^3} = |1+z| = \sqrt{2(1+\cos\theta)} \leq 4.$$

The function $q(z) = 1 + \frac{z^4}{4}$ being the solution of the equation

$$h(z) = \Psi(1 + \frac{z^4}{4}, 12z^4, 60z^4, 880z^4) = 1 + \frac{1929z^4}{4},$$

we conclude that it is the fuzzy best dominant.

5. Conclusion

This paper lays the groundwork for the third-order fuzzy differential subordination as part of fuzzy differential subordination theory. The fundamental principles of the novel theory of third-order fuzzy differential subordination are derived from the third-order subordination concepts contained in [7] combined with the general theory of fuzzy differential subordination presented in [25]. The Introduction in Section 1 contains the fundamental ideas required for the study described in this paper as well as the justification for the topic's investigation. The preliminary known results used for the research are reported in Section 2 and the main findings are presented in Section 3. The new third-order fuzzy differential subordination theory's essential theorems are established and proven, and the examples given illustrate the application of both the new concepts and the findings of the research established in the theorems. The novel findings presented in this publication are meant to serve as the foundation for a new line of research. Therefore, it is anticipated that they will be put to use in establishing new third-order fuzzy differential subordinations for particular classes of univalent functions, as seen in [13, 19, 30], various operators, as shown in [2, 3, 20], or fractional calculus aspects, as seen in [11] and [16].

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