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Hermitian self-dual quasi-abelian codes

Research Article

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Abstract: Quasi-abelian codes constitute an important class of linear codes containing theoretically and practically interesting codes such as quasi-cyclic codes, abelian codes, and cyclic codes. In particular, the sub-class consisting of 1-generator quasi-abelian codes contains large families of good codes. Based on the well-known decomposition of quasi-abelian codes, the characterization and enumeration of Hermitian self-dual quasi-abelian codes are given. In the case of 1-generator quasi-abelian codes, we offer necessary and sufficient conditions for such codes to be Hermitian self-dual and give a formula for the number of these codes. In the case where the underlying groups are some p-groups, the actual number of resulting Hermitian self-dual quasi-abelian codes are determined.

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1. Introduction

Quasi-cyclic codes form an important class of linear codes due to their rich algebraic structures, large number of codes with good parameters, and various applications (see [9], [10], [11], [12], [14], [17], and references therein). Let $\mathbb{F}_q$ denote a finite field of order q. It is known that quasi-cyclic codes of length ml and index l over $\mathbb{F}_q$ can be regarded as $\mathbb{F}_q[\mathbb{Z}_m]$-submodules of the $\mathbb{F}_q[\mathbb{Z}_m]$-module $(\mathbb{F}_q[\mathbb{Z}_m])^l$, where $\mathbb{Z}_m$ denotes the cyclic group of order m and $\mathbb{F}_q[\mathbb{Z}_m]$ is the group algebra of $\mathbb{Z}_m$ over $\mathbb{F}_q$ (see [10]).

In a more general setting, quasi-abelian codes are defined by replacing $\mathbb{Z}_m$ with a finite abelian group. Particularly, if G is a finite abelian group and $H \leq G$, then an H-*quasi-abelian code* is defined to be an $\mathbb{F}_q[H]$-submodule of the $\mathbb{F}_q[H]$-module $\mathbb{F}_q[G]$. This class of codes was first introduced in [18] and further studies of their properties have been made in [4, Section 7] and [1]. More recently in [6], via the

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Discrete Fourier Transform, the structural characterization of quasi-abelian codes have been established together with the existence of asymptotically good quasi-abelian codes. Quasi-abelian codes serve as the general case for quasi-cyclic codes (if $H \neq G$ is cyclic), abelian codes (if $H = G$), and cyclic codes (if $H = G$ is cyclic). Since the theory of quasi-abelian codes generalizes that of quasi-cyclic codes, a link can be established between 1-generator quasi-abelian codes and irreducible or minimal cyclic codes which plays a central role in the theory of cyclic codes [2].

Self-dual codes form another fascinating family of codes and are known to be closely related with other objects such as lattices and possess variety of practical applications (see [13]). Moreover, both Euclidean and Hermitian self-dual codes have close connection with quantum stabilizer codes [8]. In [6], the authors presented necessary and sufficient conditions for quasi-abelian codes to be Euclidean self-dual and gave enumeration of those codes based on the classification of q-cyclotomic classes of the underlying group. Moreover, they have shown that some class of binary Euclidean self-dual strictly-quasi-abelian codes are asymptotically good.

To the best of our knowledge, no study has been done yet on Hermitian self-dual quasi-abelian codes. It is therefore of natural interest to investigate such family of codes and compare the result of this study with that of [6]. In this work, considering finite abelian groups $H \leq G$, we offer sufficient and necessary conditions for an H-quasi-abelian code in $\mathbb{F}_q[G]$ to be Hermitian self-dual using similar decomposition given in [6, Section 3] (see Proposition 2.3). Consequently, enumeration of Hermitian self-dual H-quasi-abelian codes is presented (see Corollary 3.1). In similar fashion, the sufficient and necessary conditions for a 1-generator quasi-abelian code to be Hermitian self-dual are obtained (see Corollary 4.3). Enumeration of Hermitian self-dual 1-generator quasi-abelian codes is also given. In the case $H \cong (\mathbb{Z}_{p^k})^s$ is a p-group, p is a prime, $k > 0$ and $s > 0$, we classify completely the q-cyclotomic classes of H (see Propositions 3.6 and 3.10) which lead to the actual number of the resulting Hermitian self-dual H-quasi-abelian codes. The asymptotic goodness of Hermitian self-dual strictly-quasi-abelian codes over $\mathbb{F}_{2^{2s}}$ is guaranteed by [6, Section 7] since every code over $\mathbb{F}_{2^{2s}}$ with generator matrix containing only elements from $\mathbb{F}_2$ is Hermitian self-dual if and only if such a matrix generates a Euclidean self-dual code over $\mathbb{F}_2$.

The paper is organized as follows. In Section 2, we recall notations and definitions which are essential to this work as well as the well-known decomposition of semi-simple group algebras. Enumeration of Hermitian self-dual quasi-abelian codes, where the underlying groups are some p-groups, is established in Section 3. Finally in Section 4, we focus on the characterization and enumeration of Hermitian self-dual 1-generator quasi-abelian codes.

2. Preliminaries

For a prime power q and positive integer n, let $\mathbb{F}_q$ denote a finite field of order q and let G be a finite abelian group of order n, written additively. Denote by $\mathbb{F}_q[G]$ the *group algebra* of G over $\mathbb{F}_q$. The elements in $\mathbb{F}_q[G]$ will be written as $\sum_{g \in G} \alpha_g Y^g$, where $\alpha_g \in \mathbb{F}_q$. The addition and the multiplication in $\mathbb{F}_q[G]$ are given as in the usual polynomial rings over $\mathbb{F}_q$ with the indeterminate Y, where the indices are computed additively in G. As convention, Y^0 is treated as the multiplicative identity of $\mathbb{F}_q[G]$, where 0 is the identity of G.

Let R be a finite commutative ring with unity. A linear code of length n over R is defined to be an R-submodule of R^n. A *(linear) code* C in $\mathbb{F}_q[G]$ refers to an $\mathbb{F}_q$-subspace of $\mathbb{F}_q[G]$. This can be viewed as a linear code of length n over $\mathbb{F}_q$ by indexing the n-tuples by the elements of G. For more details, the reader is referred to [6].

Consider a subgroup H of G, a code C in $\mathbb{F}_q[G]$ is called an *H-quasi-abelian code* (specifically, an *H-quasi-abelian code of index l*, where $l := [G : H]$) if C is an $\mathbb{F}_q[H]$-module, i.e., C is closed under addition and multiplication by the elements in $\mathbb{F}_q[H]$. If H is a non-cyclic subgroup of G, then we say that C is a *strictly-quasi-abelian code*. If it is clear in the context or if H is not specified, such a code will be called simply a *quasi-abelian code*. An H-quasi-abelian code C is said to be of 1-*generator* if C is a cyclic $\mathbb{F}_q[H]$-module.

Let $\{\mathfrak{g}_1, \mathfrak{g}_2, \ldots, \mathfrak{g}_l\}$ be a fixed set of representatives of the cosets of H in G. Let $\mathcal{R} := \mathbb{F}_q[H]$. Define $\Phi : \mathbb{F}_q[G] \to \mathcal{R}^l$ by

$$\Phi\left(\sum_{h\in H}\sum_{i=1}^{l}\alpha_{h+\mathfrak{g}_i}Y^{h+\mathfrak{g}_i}\right) = \left(\boldsymbol{\alpha}_1(Y), \boldsymbol{\alpha}_2(Y), \ldots, \boldsymbol{\alpha}_l(Y)\right),$$

where $\boldsymbol{\alpha}_i(Y) = \sum_{h\in H}\alpha_{h+\mathfrak{g}_i}Y^h \in \mathcal{R}$, for all $i = 1, 2, \ldots, l$. It is well known that Φ is an $\mathcal{R}$-module isomorphism interpreted as follows.

Lemma 2.1. *The map Φ induces a one-to-one correspondence between H-quasi-abelian codes in $\mathbb{F}_q[G]$ and linear codes of length l over $\mathcal{R}$.*

In $\mathbb{F}_q^n$, the *Euclidean inner product* of $\boldsymbol{u} = (u_1, u_2, \ldots, u_n)$ and $\boldsymbol{v} = (v_1, v_2, \ldots, v_n)$ is defined to be $\langle \boldsymbol{u}, \boldsymbol{v}\rangle_{\mathrm{E}} := \sum_{i=1}^{n} u_i v_i$. From this point, we assume $q = q_0^2$, where q_0 is a prime power. Consequently, the *Hermitian inner product* of $\boldsymbol{u}$ and $\boldsymbol{v}$ is defined as $\langle \boldsymbol{u}, \boldsymbol{v}\rangle_{\mathrm{H}} := \sum_{i=1}^{n} u_i\overline{v_i}$, where $^{-}$ is the automorphism on $\mathbb{F}_q$ defined by $\alpha \mapsto \alpha^{q_0}$ for all $\alpha \in \mathbb{F}_q$. For a code C of length n over $\mathbb{F}_q$, let $C^{\perp_{\mathrm{E}}}$ and $C^{\perp_{\mathrm{H}}}$ denote its Euclidean dual and Hermitian dual, respectively. The code C is said to be *Euclidean* (resp., *Hermitian*) *self-dual* if $C^{\perp_{\mathrm{E}}} = C$ (resp., $C^{\perp_{\mathrm{H}}} = C$).

The *Hermitian inner product* in $\mathbb{F}_q[G]$ is defined by

$$\langle \boldsymbol{u}, \boldsymbol{v}\rangle_{\mathrm{H}} := \sum_{g\in G}\alpha_g\overline{\beta_g}$$

for all $\boldsymbol{u} = \sum_{g\in G}\alpha_g Y^g$ and $\boldsymbol{v} = \sum_{g\in G}\beta_g Y^g$ in $\mathbb{F}_q[G]$. The *Hermitian dual* of a code $C \subseteq \mathbb{F}_q[G]$ is given by

$$C^{\perp_{\mathrm{H}}} := \{\boldsymbol{u} \in \mathbb{F}_q[G] \mid \langle \boldsymbol{u}, \boldsymbol{v}\rangle_{\mathrm{H}} = 0 \text{ for all } \boldsymbol{v} \in C\}.$$

Similarly, the code C in $\mathbb{F}_q[G]$ is said to be *Hermitian self-dual* if $C^{\perp_{\mathrm{H}}} = C$. Note that without confusion, we use the symbol $\perp_{\mathrm{H}}$ to indicate both the Hermitian dual of a code over $\mathbb{F}_q$ and the Hermitian dual of a code in $\mathbb{F}_q[G]$. All throughout, the self-duality of quasi-abelian codes is studied with respect to the given Hermitian inner product in $\mathbb{F}_q[G]$.

2.1. Decomposition and Hermitian dual codes

The main tool of this work appears in this subsection. The idea is to have a convenient decomposition of quasi-abelian codes using the well-known decomposition of semi-simple group algebras introduced in [16]. Then, combining this technique with the results of [7, Proposition 2.7] and [6, Proposition 4.1], we obtain characterization of Hermitian self-dual quasi-abelian codes (see Proposition 2.3). This will lead to enumeration of such class of codes.

For completeness, we discuss the concepts of q-cyclotomic classes and primitive idempotents as appeared in [7, Section II.C]. Given coprime positive integers i and j, the *multiplicative order of j modulo i*, denoted by $\mathrm{ord}_i(j)$, is defined to be the smallest positive integer s such that i divides $j^s - 1$. For each $a \in H$, denote by $\mathrm{ord}(a)$ the *additive order* of a in H.

From this point, we assume that $\gcd(|H|, q) = 1$. A *q-cyclotomic class* of H containing $a \in H$, denoted by $S_q(a)$, is defined to be the set

$$S_q(a) := \{q^i \cdot a \mid i = 0, 1, \ldots\} = \{q^i \cdot a \mid 0 \le i < \mathrm{ord}_{\mathrm{ord}(a)}(q)\},$$

where $q^i \cdot a := \sum_{j=1}^{q^i} a$ in H.

For a positive integer r and $a \in H$, denote by $-r \cdot a$ the element $r \cdot (-a) \in H$. A q-cyclotomic class $S_q(a)$ is said to be of *type I* if $S_q(a) = S_q(-q_0 \cdot a)$ and it is of *type II* if $S_q(-q_0 \cdot a) \neq S_q(a)$. Clearly, $S_q(0)$ is a q-cyclotomic class of type I.

An *idempotent* in a ring is a non-zero element e such that $e^2 = e$, and it is called *primitive idempotent* if, for every other idempotent f, either $ef = e$ or $ef = 0$. The primitive idempotents in $\mathcal{R} := \mathbb{F}_q[H]$ are induced by the q-cyclotomic classes of H (see [5, Proposition II.4]).

Assume that H contains t q-cyclotomic classes. Without loss of generality, let $\{a_1 = 0, a_2, \ldots, a_t\}$ be a set of representatives of the q-cyclotomic classes of H such that $\{a_i \mid i = 1, 2, \ldots, r_I\}$ and $\{a_{r_I+j}, a_{r_I+r_{II}+j} = -q_0 \cdot a_{r_I+j} \mid j = 1, 2, \ldots, r_{II}\}$ are sets of representatives of q-cyclotomic classes of types I and II, respectively, where $t = r_I + 2r_{II}$. Let $\{e_1, e_2, \ldots, e_t\}$ be the set of primitive idempotents of $\mathcal{R}$ induced by $\{S_q(a_i) \mid i = 1, 2, \ldots, t\}$, respectively. It is well known that $\mathcal{R}e_i$ is isomorphic to an extension field of $\mathbb{F}_q$ of degree $|S_q(a_i)|$ for each $i = 1, 2, \ldots, t$.

In [16], $\mathcal{R} := \mathbb{F}_q[H]$ is decomposed in terms of e_i's. Later, the components in the decomposition of $\mathcal{R}$ are rearranged in [7] and obtain the following.

$$\mathcal{R} = \bigoplus_{i=1}^{t} \mathcal{R}e_i \cong \left(\prod_{i=1}^{r_I} \mathbb{E}_i\right) \times \left(\prod_{j=1}^{r_{II}} (\mathbb{K}_j \times \mathbb{K}'_j)\right), \tag{1}$$

where $\mathbb{E}_i \cong \mathcal{R}e_i$, $\mathbb{K}_j \cong \mathcal{R}e_{r_I+j}$, and $\mathbb{K}'_j \cong \mathcal{R}e_{r_I+r_{II}+j}$ are finite extension fields of $\mathbb{F}_q$ for all $i = 1, 2, \ldots, r_I$ and $j = 1, 2, \ldots, r_{II}$.

Remark 2.2. *It is known that* $\mathbb{E}_i \cong \mathbb{F}_{q^{s_i}}$, $\mathbb{K}_j \cong \mathbb{F}_{q^{t_j}}$ *and* $\mathbb{K}'_j \cong \mathbb{F}_{q^{t'_j}}$, *where* $s_i := |S_q(a_i)|$, $t_j := |S_q(a_{r_I+j})|$, *and* $t'_j := |S_q(a_{r_I+r_{II}+j})|$ *for* $i = 1, 2, \ldots, r_I$ *and* $j = 1, 2, \ldots, r_{II}$. *Note that* $|S_q(a_{r_I+j})| = |S_q(a_{r_I+r_{II}+j})|$ *for each* $j = 1, 2, \ldots, r_{II}$. *Thus,* $\mathbb{K}_j \cong \mathbb{K}'_j$ *for each* $j = 1, 2, \ldots, r_{II}$.

From (1), we have

$$\mathbb{F}_q[G] \cong \mathcal{R}^l \cong \left(\prod_{i=1}^{r_I} \mathbb{E}_i^l\right) \times \left(\prod_{j=1}^{r_{II}} (\mathbb{K}_j^l \times {\mathbb{K}'_j}^l)\right), \tag{2}$$

where the isomorphisms are $\mathcal{R}$-module isomorphisms. They can be viewed as $\mathbb{F}_q$-linear isomorphisms as well. Consequently, every quasi-abelian code C in $\mathbb{F}_q[G]$ can be viewed as

$$C \cong \left(\prod_{i=1}^{r_I} C_i\right) \times \left(\prod_{j=1}^{r_{II}} (D_j \times D'_j)\right), \tag{3}$$

where C_i, D_j and D'_j are linear codes of length l over $\mathbb{E}_i$, $\mathbb{K}_j$, and $\mathbb{K}'_j$, respectively, for all $i = 1, 2, \ldots, r_I$ and $j = 1, 2, \ldots, r_{II}$.

Using arguments similar to the proofs of [7, Proposition 2.7] and [6, Proposition 4.1], it can be concluded that the Hermitian dual of C is of the form

$$C^{\perp_{\mathrm{H}}} \cong \left(\prod_{i=1}^{r_I} C_i^{\perp_{\mathrm{H}}}\right) \times \left(\prod_{j=1}^{r_{II}} \left((D'_j)^{\perp_{\mathrm{E}}} \times D_j^{\perp_{\mathrm{E}}}\right)\right). \tag{4}$$

From (3) and (4), we have the following necessary and sufficient conditions for quasi-abelian codes to be Hermitian self-dual.

Proposition 2.3. *An* H*-quasi-abelian code* C *in* $\mathbb{F}_q[G]$ *is Hermitian self-dual if and only if, in the decomposition (3),*

i) C_i *is Hermitian self-dual for all* $i = 1, 2, \ldots, r_I$, *and*

ii) $D'_j = D_j^{\perp_{\mathrm{E}}}$ *for all* $j = 1, 2, \ldots, r_{II}$.

3. Enumeration of Hermitian self-dual quasi-abelian codes

In this section, we enumerate Hermitian self-dual quasi-abelian codes by using the decomposition in (3), Proposition 2.3 and the following formulas. Let $N(q,l)$ (resp., $N_{\mathrm{H}}(q,l)$) denote the number of linear codes (resp., Hermitian self-dual codes) of length l over $\mathbb{F}_q$. It is well known (see [15] and [13]) that

$$N(q,l) = \sum_{i=0}^{l} \prod_{j=0}^{i-1} \frac{q^l - q^j}{q^i - q^j}, \tag{5}$$

$$N_{\mathrm{H}}(q,l) = \begin{cases} \prod_{i=0}^{\frac{l}{2}-1} (q^{i+\frac{1}{2}} + 1) & \text{if } l \text{ is even,} \\ 0 & \text{otherwise,} \end{cases} \tag{6}$$

where the empty product is set to be 1.

In general, to count the number of Hermitian self-dual quasi-abelian codes in $\mathbb{F}_q[G]$, in (3), we count the number of Hermitian self-dual codes C_i of length l over $\mathbb{F}_{q^{s_i}}$ for all $i = 1, 2, \ldots, r_I$ and multiply it with the number of all possible linear codes D_j of length l over $\mathbb{F}_{q^{t_j}}$ for all $j = 1, 2, \ldots, r_{II}$. This technique is clear in the following corollary. Hereafter, the numbers s_i, t_j, and t_j' will appear frequently in the succeeding results. If needed, the reader is referred back to Remark 2.2 for the definitions of s_i, t_j, and t_j'.

Corollary 3.1. *Let $H \leq G$ be finite abelian groups such that $\gcd(|H|, q) = 1$ and $l = [G : H]$. Assume that $\mathbb{F}_q[H]$ contains r_I (resp., $2r_{II}$) primitive idempotents of type I (resp., II). Assume further that the primitive idempotents of type I are induced by q-cyclotomic classes of size s_i for each $i = 1, 2, \ldots, r_I$ and the primitive idempotents of type II are induced by q-cyclotomic classes of sizes t_j and t_j', pair-wise, for each $j = 1, 2, \ldots, r_{II}$. Then the number of Hermitian self-dual H-quasi-abelian codes in $\mathbb{F}_q[G]$ is*

$$\prod_{i=1}^{r_I} N_{\mathrm{H}}(q^{s_i}, l) \prod_{j=1}^{r_{II}} N(q^{t_j}, l). \tag{7}$$

We note that $S_q(0)$ is a q-cyclotomic class of H of type I. Then $r_I \geq 1$, and hence, the product $\prod_{i=1}^{r_I} N_{\mathrm{H}}(q^{s_i}, l) = 0$ for all odd positive integers l. Hence, there are no Hermitian self-dual H-quasi-abelian codes if $l = [G : H]$ is odd. Therefore, we have the following result derived from (6) and (7).

Lemma 3.2. *There exists a Hermitian self-dual H-quasi-abelian code in $\mathbb{F}_q[G]$ if and only if the index $l = [G : H]$ is even.*

Remark 3.3. *From Lemma 3.2, it is apparent that given a finite abelian group G and $q = q_0^2$, the existence of Hermitian self-dual H-quasi-abelian codes in $\mathbb{F}_q[G]$ depends only on the choice of H, particularly on index l being even.*

In the theory of quasi-cyclic codes, it is practical to use a relatively small fixed value of the index l mainly for the purpose of efficient decoding [3]. Moreover, this case contains the known case of double circulant codes (see [10, Section VI.A] and [12, Section II.A]). Since the theory of quasi-abelian codes generalizes that of quasi-cyclic codes, we can adopt those concepts. Note that a quasi-cyclic code is cyclic when $l = 1$. Thus $l = 2$ is the smallest index such that a code is quasi-cyclic. Specifically for $l = 2$, one can talk about self-dual 1-generator quasi-abelian codes (see Section 4). Consider the example below for the number of quasi-abelian codes of index 2.

Example 3.4. *Let $H \leq G$ be finite abelian groups such that $\gcd(|H|, q) = 1$ and $l = [G : H] = 2$. Assume that $\mathbb{F}_q[H]$ contains r_I (resp., $2r_{II}$) primitive idempotents of type I (resp., II). Assume further that the primitive idempotents of type I are induced by q-cyclotomic classes of size s_i for each $i = 1, 2, \ldots, r_I$ and the primitive idempotents of type II are induced by q-cyclotomic classes of sizes t_j and t_j', pair-wise, for* *each $j = 1, 2, \dots, r_{II}$. Then the number of Hermitian self-dual H-quasi-abelian codes of index 2 in $\mathbb{F}_q[G]$ is*

$$\prod_{i=1}^{r_I}(q_0^{s_i}+1)\prod_{j=1}^{r_{II}}(q^{t_j}+3).$$

In the next two subsections, we consider the case where the subgroups H of G are some p-groups. It is interesting to see that for this particular case, the cardinality and the number of q-cyclotomic classes of H can be explicitly determined. Hence, one can obtain the actual number of resulting Hermitian self-dual H-quasi-abelian codes. In this regard, we offer sufficient and necessary conditions for a q-cyclotomic class of H to be of type I or type II.

3.1. $H \cong (\mathbb{Z}_{2^k})^s$

The succeeding discussion is instrumental in determining the explicit forms of r_I and r_{II}. Let $H \cong (\mathbb{Z}_{p^k})^s$, where k and s are positive integers, and p is prime such that $\gcd(p, q) = 1$. Define

$$H_{p^i} := \{h \in H |\, \mathrm{ord}(h) = p^i\},$$

for each $0 \le i \le k$. Observe that $H_1, H_p, \dots, H_{p^k}$ are pair-wise disjoint and $H = H_1 \cup H_p \cup \cdots \cup H_{p^k}$, where $H_1 = \{0\}$. For each $1 \le i \le k$, it is not difficult to see that $H_{p^i} = \left(p^{k-i}\mathbb{Z}_{p^k}\right)^s \setminus \left(p^{k-(i-1)}\mathbb{Z}_{p^k}\right)^s$. Consequently, we have $|H_1| = 1$ and, via inclusion-exclusion principle,

$$|H_{p^i}| = p^{is} - p^{(i-1)s},$$

for each $i = 1, 2, \dots, k$. Recall that $q = q_0^2$ where q_0 is a prime power. Hereafter, let $\nu_{p^i} := \mathrm{ord}_{p^i}(q)$ and $\mu_{p^i} := \mathrm{ord}_{p^i}(q_0)$, for $i = 0, 1, \dots, k$. Note that if $h \in H_{p^i}$, $|S_q(h)| = \mathrm{ord}_{\mathrm{ord}(h)}(q) = \nu_{p^i}$.

Now, consider the case where q is odd and $p = 2$, i.e., $H \cong (\mathbb{Z}_{2^k})^s$. Suppose $h \in H_2$. Since $\mathrm{ord}(h) = 2$ for all $h \in H_2$, $q \equiv \pm 1 \pmod{\mathrm{ord}(h)}$ and $q_0 \equiv \pm 1 \pmod{\mathrm{ord}(h)}$, then we have $h = q \cdot h = q_0 \cdot h = q_0 \cdot (-h) = -q_0 \cdot h$. Then $S_q(h) = S_q(-q_0 \cdot h)$ is of type I and having cardinality equal to 1. For the case where $h \in H_{2^i}$, $2 \le i \le k$, we have the same result. Suppose $h \in H_{2^i}$, for a given $2 \le i \le k$, and assume $S_q(h)$ is of type I. Then $|S_q(h)| = \nu_{2^i}$ is odd (see [7, Remark 2.6 (2)]). Moreover, the elements of H_{2^i} are partitioned into q-cyclotomic classes of the same type and size (see [7, Remark 2.5 (ii)]). Thus, ν_{2^i} divides $|H_{2^i}|$. In particular, ν_{2^i} divides $|2^{k-i}\mathbb{Z}_{2^k} \setminus 2^{k-i+1}\mathbb{Z}_{2^k}| = 2^i - 2^{i-1} = 2^{i-1}$. Since ν_{2^i} is odd, it must be 1.

Furthermore, it can be shown that $\mu_{2^i} = 2$ for all $i = 2, 3, \dots, k$. Note that $2^i \mid (q-1)$ since $\nu_{2^i} = 1$ and thus, $2^i \mid (q_0^2 - 1)$. We show that indeed, $\mu_{2^i} = 2$. Suppose contrary, i.e., $\mu_{2^i} = 1 = \nu_{2^i}$. It implies that $q_0 \cdot h = h$ and $-h = -q_0 \cdot h = q \cdot h = h$, since $S_q(h)$ is assumed to be of type I. It implies that $h = 0$ or $\mathrm{ord}(h) = 2$ which contradicts that $h \in H_{2^i}$, $i = 2, 3, \dots, k$. We state these observations in the following lemma.

Lemma 3.5. *Let $h \in H_{2^i}$, for a given $0 \le i \le k$. If $S_q(h)$ is of type I, then $\nu_{2^i} = 1$. Moreover, $\mu_{2^i} = 2$ for all $i = 2, 3, \dots, k$.*

In the next proposition, we give the necessary and sufficient conditions for a q-cyclotomic class of H to be of type I or type II. Since all q-cyclotomic classes in H_{2^i} are of the same type and size, we characterize the q-cyclotomic classes of H through its subsets H_{2^i}, for $0 \le i \le k$, keeping in mind that $S_q(h)$ is always of type I, for all $h \in H_1 \cup H_2$.

Proposition 3.6. *Let $h \in H_{2^i}$, for a given $0 \le i \le k$. Then $S_q(h)$ is of type I if and only if $q_0 \equiv -1 \pmod{2^i}$. Equivalently, $S_q(h)$ is of type II if and only if $q_0 \not\equiv -1 \pmod{2^i}$.*

Proof. Clearly, the proposition holds for the case where $h \in H_1 \cup H_2$. Now, consider $h \in H_{2^i}$, for a given $2 \le i \le k$, and assume $S_q(h)$ is of type I. From Lemma 3.5, $\nu_{2^i} = 1$ and $\mu_{2^i} = 2$. Thus, $q \equiv 1 \pmod{2^i}$ and $q_0 \not\equiv 1 \pmod{2^i}$. Hence, $q_0 \equiv -1 \pmod{2^i}$.

On the other hand, assume $q_0 \equiv -1 \pmod{2^i}$. Thus, for each $h \in H_{2^i}$, $-q_0 \cdot h = h \in S_q(h)$. Hence, $S_q(h)$ is of type I. □

Remark 3.7. *Using Proposition 3.6, we can completely classify the sets H_{2^i}, $0 \le i \le k$, that contain q-cyclotomic classes of type I or type II. Choose the largest integer $0 \le r' \le k$ such that $2^{r'} | (q_0 + 1)$. Hence, by Proposition 3.6 H_{2^i} contains q-cyclotomic classes of type I for all $i = 0, 1, \dots, r'$ and the rest of the sets H_{2^j} contain elements of type II, for $j = r' + 1, \dots, k$. This will lead to a decomposition of $\mathbb{F}_q[H]$.*

Let r' be a positive integer as described in Remark 3.7. Since $\nu_{2^i} = 1$ for all $0 \le i \le r'$, then

$$r_I = \sum_{i=0}^{r'} \frac{|H_{2^i}|}{\nu_{2^i}} = 2^{r's}$$

and

$$r_{II} = \sum_{r=r'+1}^{k} \frac{|H_{2^r}|}{2\nu_{2^r}} = \sum_{r=r'+1}^{k} \frac{2^{rs} - 2^{(r-1)s}}{2\nu_{2^r}}.$$

Thus, from (1), this will give the following decomposition,

$$\mathbb{F}_q[H] \cong \left(\prod_{i=1}^{2^{r's}} \mathbb{F}_q \right) \times \left(\prod_{r=r'+1}^{k} \left(\prod_{j'=1}^{\frac{2^{rs}-2^{(r-1)s}}{2\nu_{2^r}}} (\mathbb{F}_{q^{\nu_{2^r}}} \times \mathbb{F}_{q^{\nu_{2^r}}}) \right) \right).$$

Similar with (3), every H-quasi-abelian code C in $\mathbb{F}_q[G]$ can be written as

$$C \cong \left(\prod_{i=1}^{2^{r's}} C_i \right) \times \left(\prod_{r=r'+1}^{k} \left(\prod_{j'=1}^{\frac{2^{rs}-2^{(r-1)s}}{2\nu_{2^r}}} (D_{r,j'} \times D'_{r,j'}) \right) \right), \tag{8}$$

where C_i, $D_{r,j'}$ and $D'_{r,j'}$ are linear codes of length l over $\mathbb{F}_q$, $\mathbb{F}_{q^{\nu_{2^r}}}$ and $\mathbb{F}_{q^{\nu_{2^r}}}$, respectively, for $i = 1, 2, \dots, 2^{r's}$, $r = r' + 1, \dots, k$, and $j' = 1, 2, \dots, (2^{rs} - 2^{(r-1)s})/2\nu_{2^r}$. Given the decomposition of C in (8), we deduce the next proposition.

Proposition 3.8. *Let $H \le G$ be finite abelian groups such that $H \cong (Z_{2^k})^s$, $\gcd(|H|, q) = 1$ and $l = [G : H]$. Let $0 \le r' \le k$ be the largest integer such that $2^{r'} | (q_0 + 1)$. The number of Hermitian self-dual H-quasi-abelian codes in $\mathbb{F}_q[G]$ is*

$$\begin{cases} \left(\prod_{i=0}^{\frac{l}{2}-1} (q^{i+\frac{1}{2}} + 1)^{2^{r's}} \right) \left(\prod_{r=r'+1}^{k} \left(\sum_{i=0}^{l} \prod_{j=0}^{i-1} \frac{(q^{\nu_{2^r}})^l - (q^{\nu_{2^r}})^j}{(q^{\nu_{2^r}})^i - (q^{\nu_{2^r}})^j} \right)^{\frac{2^{rs}-2^{(r-1)s}}{2\nu_{2^r}}} \right) & \textit{if } l \textit{ is even,} \\ 0 & \textit{if } l \textit{ is odd.} \end{cases}$$

Proof. The result follows from (8) and Proposition 2.3 by counting the number of all possible Hermitian self-dual linear codes C_i over $\mathbb{F}_q$ of length l and linear codes $D_{r,j'}$ over $\mathbb{F}_{q^{\nu_{2^r}}}$ of length l, for $i = 1, 2, \dots, r's$, $r = r' + 1, \dots, k$, and $j' = 1, 2, \dots, (2^{rs} - 2^{(r-1)s})/2\nu_{2^r}$, then apply formulas (5) and (6). □

A specific case of Proposition 3.8 is given in the example below, where $H \cong (\mathbb{Z}_2)^s$ (i.e., $r' = k = 1$) is an elementary 2-group.

Example 3.9. *Let $H \leq G$ be finite abelian groups such that $H \cong (\mathbb{Z}_2)^s$, $\gcd(|H|, q) = 1$ and $l = [G : H]$. The number of Hermitian self-dual H-quasi-abelian codes in $\mathbb{F}_q[G]$ is*

$$\begin{cases} \prod_{i=0}^{\frac{l}{2}-1} (q^{i+\frac{1}{2}} + 1)^{2^s} & \text{if } l \text{ is even,} \\ 0 & \text{if } l \text{ is odd.} \end{cases}$$

Table 3.1 illustrates Proposition 3.8 when $q = 9$, $l = 2$, for $k = 1, 2$ and $s = 1, 2$. Note that in the last column, $A \cdot B$ gives the number of the resulting codes. Moreover, since the value of $k \leq 2$ and $q_0 = 3$, then $r' = k$, for $k = 1, 2$. Hence, the second factor in the formula given by B is empty and set to be 1. In other words, all cyclotomic classes of H is of type I, for $k = 1, 2$. In this case, the numbers in the last column of the table also gives the number of Hermitian self-dual 1-generator H-quasi-abelian codes as presented in Corollary 4.5 (i).

Table 1. Number of Hermitian self-dual H-quasi-abelian codes in $\mathbb{F}_q[G]$, $H \cong (\mathbb{Z}_{2^k})^s$, $l = [G : H] = 2$ and $q = 9$.

s	k	$\|H\|$	$\|G\|$	r'	$A = (q_0 + 1)^{2^{r'}s}$	$B = \prod_{r=r'+1}^{k} (q^{\nu_{2^r}} + 3)^{\|H_{2^r}\|/2\nu_{2^r}}$	$A \cdot B$
1	1	2	4	1	16	1	16
	2	4	8	2	256	1	256
2	1	4	8	1	256	1	256
	2	16	32	2	4^{16}	1	4^{16}

3.2. $H \cong (\mathbb{Z}_{p^k})^s$, where p is an odd prime

To complete our characterization, consider $H \cong (\mathbb{Z}_{p^k})^s$, $k, s > 0$, where p is an odd prime and $\gcd(p, q) = 1$. Recall that in the case $p = 2$, there is a chance that the q-cyclotomic classes of H are divided exactly into classes of type I and type II. It is interesting to note that it is a totally different situation when p is odd. Specifically, we show that all non-zero elements in H belong to just one type of q-cyclotomic classes. Moreover, the necessary and sufficient conditions for them to be of type I or type II are determined. Recall that H_{p^i} is the set containing all elements of H of order p^i, $i = 0, 1, \ldots, k$ and $H = H_1 \cup H_p \cup \cdots \cup H_{p^k}$. Note that $S_q(0) = \{0\} = H_1$ is of type I. We start with H_p the characterization of q-cyclotomic classes of H.

Proposition 3.10. *Let $h \in H_p$. Then $S_q(h)$ is of type I if and only if $\text{ord}_p(q)$ is odd and $\text{ord}_p(q_0)$ is even. Equivalently, $S_q(h)$ is of type II if and only if $\text{ord}_p(q)$ is even or $\text{ord}_p(q_0)$ is odd.*

Proof. Following the notation introduced above, let $\nu_p = \text{ord}_p(q)$. If $h \in H_p$, then $q^{\nu_p} \cdot h = h$.

Assume $S_q(h)$ is of type I. Then $-q_0 \cdot h = q^i \cdot h = q_0^{2i} \cdot h$ for some $0 \leq i < \nu_p$. It follows that $h = -q_0^{2i-1} \cdot h = -q_0^{2i-2}(q_0 \cdot h) = -q_0^{2i-2}(-q_0^{2i} \cdot h) = q_0^{2(2i-1)} \cdot h = q^{(2i-1)} \cdot h$ which implies $\nu_p | (2i - 1)$. Hence, ν_p is odd. We note that $\text{ord}_p(q_0) \in \{\nu_p, 2\nu_p\}$. If $\text{ord}_p(q_0) = \nu_p$, then $h = q_0^{\nu_p} \cdot h = q_0^{2i-1} \cdot h = -h$, which implies that $h = 0$, a contradiction. Hence, $\text{ord}_p(q_0) = 2\nu_p$, which is even.

Conversely, assume that $\text{ord}_p(q)$ is odd and $\text{ord}_p(q_0)$ is even. It follows that $\text{ord}_p(q) = \nu_p$ and $\text{ord}_p(q_0) = 2\nu_p$. Then $h = q^{\nu_p} \cdot h = q_0^{2\nu_p} \cdot h$, i.e., $(q_0^{\nu_p} - 1)(q_0^{\nu_p} + 1) \cdot h = 0$. Since $\text{ord}_p(q_0) = 2\nu_p$, we have $p \nmid (q_0^{\nu_p} - 1)$, and hence, $(q_0^{\nu_p} + 1) \cdot h = 0$. It follows that $q_0(q_0^{\nu_p} + 1) \cdot h = (q^{\frac{\nu_p+1}{2}} + q_0) \cdot h = 0$. Since ν_p is odd, $\nu_p + 1$ is even. Which implies that $-q_0 \cdot h = q^{\frac{\nu_p+1}{2}} \cdot h \in S_q(h)$. Therefore, $S_q(h)$ is of type I as desired. □

Next, we show that all q-cyclotomic classes of $H \setminus \{0\}$ are of the same type. Because of this, the q-cyclotomic classes of H are completely characterized.

Proposition 3.11. *Let $a \in H_p$ and $b \in H_{p^i}$, for any given $1 \leq i \leq k$. Then, $S_q(a)$ is of type I if and only if $S_q(b)$ is of type I. Equivalently, $S_q(a)$ is of type II if and only if $S_q(b)$ is of type II.*

Proof. Let $a \in H_p$ and assume that $S_q(a)$ is of type I. Then, by Proposition 3.10, $\nu_p = \mathrm{ord}_p(q)$ is odd and $\mu_p = \mathrm{ord}_p(q_0) = 2\nu_p$ is even. We show that $p^i \mid \left(q^{\nu_p \cdot p^{i-1}} - 1\right)$ by induction on i. It is clear when $i = 1$. Now, assume $p^{i-1} \mid \left(q^{\nu_p \cdot p^{i-2}} - 1\right)$, for $1 < i \leq k$. Then, $q^{\nu_p \cdot p^{i-2}} \equiv 1 \,(\mathrm{mod}\; p^{i-1})$ and hence, $q^{\nu_p \cdot p^{i-2} \cdot j} \equiv 1 \,(\mathrm{mod}\; p^{i-1})$ for all $j \geq 0$. Thus, $\sum_{j=0}^{p-1} q^{\nu_p \cdot p^{i-2} \cdot j} \equiv \sum_{j=0}^{p-1} 1 \,(\mathrm{mod}\; p^{i-1})$. This implies that $p \mid \left(\sum_{j=0}^{p-1} q^{\nu_p \cdot p^{i-2} \cdot j}\right)$. Since $q^{\nu_p \cdot p^{i-1}} - 1 = \left(q^{\nu_p \cdot p^{i-2}} - 1\right)\left(\sum_{j=0}^{p-1} q^{\nu_p \cdot p^{i-2} \cdot j}\right)$, $p^{i-1} \mid \left(q^{\nu_p \cdot p^{i-2}} - 1\right)$ and $p \mid \left(\sum_{j=0}^{p-1} q^{\nu_p \cdot p^{i-2} \cdot j}\right)$, it follows that $p^i \mid \left(q^{\nu_p \cdot p^{i-1}} - 1\right)$. Therefore, $\nu_{p^i} \mid \nu_p \cdot p^{i-1}$ and means ν_{p^i} is odd. Note that $\mu_{p^i} \in \{\nu_{p^i}, 2\nu_{p^i}\}$. Since μ_p is even, ν_{p^i} is odd and $\mu_p \mid \mu_{p^i}$ hence, $\mu_{p^i} = 2\nu_{p^i}$. Hence, $p^i \mid \left(q_0^{2\nu_{p^i}} - 1\right)$ and $p^i \nmid \left(q_0^{\nu_{p^i}} - 1\right)$. It follows that $p^i \mid \left(q_0^{\nu_{p^i}} + 1\right)$. In other words, $q_0(q_0^{\nu_{p^i}} + 1) \cdot b = 0$ or $-q_0 \cdot b = q_0^{\nu_{p^i}+1} \cdot b = q^{\frac{\nu_{p^i}+1}{2}} \cdot b \in S_q(b)$ for each $b \in H_{p^i}$.

Conversely, assume that $S_q(b)$ is of type I, for all $b \in H_{p^i}$. Then, $-q_0 \cdot b = q^j \cdot b$ for some $0 \leq j < \nu_{p^i}$. It follows that $-q_0(p^{i-1} \cdot b) = q^j(p^{i-1} \cdot b)$, which implies $S_q(p^{i-1} \cdot b)$ is of type I. Since $p^{i-1} \cdot b \in H_p$, $S_q(a)$ and $S_q(p^{i-1} \cdot b)$ are of the same type. □

Combining Propositions 3.10 and 3.11, the corollary below follows immediately.

Corollary 3.12. *Let h be a non-zero element in $H \cong (\mathbb{Z}_{p^k})^s$, p is odd and $\gcd(p,q) = 1$. Then $S_q(h)$ is of type I if and only if $\mathrm{ord}_p(q)$ is odd and $\mathrm{ord}_p(q_0)$ is even. Equivalently, $S_q(h)$ is of type II if and only if $\mathrm{ord}_p(q)$ is even or $\mathrm{ord}_p(q_0)$ is odd.*

We are now ready to obtain a decomposition for $\mathbb{F}_q[H]$. This entails computing for r_I and r_{II}. If there exists $h \in H \setminus \{0\}$ such that $S_q(h)$ is of type I, then by Corollary 3.12, $r_{II} = 0$ and

$$r_I = \sum_{i=0}^{k} \frac{\mid H_{p^i} \mid}{\nu_{p^i}} = \sum_{i=0}^{k} \frac{p^{is} - p^{(i-1)s}}{\nu_{p^i}},$$

where $\nu_{p^0} = \nu_1 = 1$ and $p^{is} - p^{(i-1)s}$ is equal to 1 when $i = 0$. On the other hand, if there exists $h \in H \setminus \{0\}$ such that $S_q(h)$ is of type II, then Corollary 3.12 implies that $r_I = |H_1| = 1$ and

$$r_{II} = \sum_{i=1}^{k} \frac{\mid H_{p^i} \mid}{2\nu_{p^i}} = \sum_{i=1}^{k} \frac{p^{is} - p^{(i-1)s}}{2\nu_{p^i}}.$$

Recall that $\nu_p := \mathrm{ord}_p(q)$ and $\mu_p := \mathrm{ord}_p(q_0)$. From the above calculations, together with Corollary 3.12 and (1), we have

$$\mathbb{F}_q[H] \cong \begin{cases} \mathbb{F}_q \times \left(\prod_{i=1}^{k} \left(\prod_{j'=1}^{\frac{2^{is} - 2^{(i-1)s}}{\nu_{p^i}}} \mathbb{F}_{q^{\nu_{p^i}}} \right) \right) & \text{if } \nu_p \text{ is odd and } \mu_p \text{ is even,} \\ \mathbb{F}_q \times \left(\prod_{i=1}^{k} \left(\prod_{j=1}^{\frac{2^{is} - 2^{(i-1)s}}{2\nu_{p^i}}} \left(\mathbb{F}_{q^{\nu_{p^i}}} \times \mathbb{F}_{q^{\nu_{p^i}}} \right) \right) \right) & \text{if } \nu_p \text{ is even or } \mu_p \text{ is odd.} \end{cases}$$

It also implies that an H-quasi-abelian code C in $\mathbb{F}_q[G]$ can be decomposed as

$$C \cong \begin{cases} C_1 \times \left(\prod_{i=1}^{k} \left(\prod_{j'=1}^{\frac{2^{is}-2^{(i-1)s}}{\nu_{p^i}}} C_{i,j'} \right) \right) & \text{if } \nu_p \text{ is odd and } \mu_p \text{ is even,} \\ C_1 \times \left(\prod_{i=1}^{k} \left(\prod_{j=1}^{\frac{2^{is}-2^{(i-1)s}}{2\nu_{p^i}}} (D_{i,j} \times D'_{i,j}) \right) \right) & \text{if } \nu_p \text{ is even or } \mu_p \text{ is odd,} \end{cases} \tag{9}$$

where C_1 and $C_{i,j'}$ are linear codes of length l over $\mathbb{F}_q$ and $\mathbb{F}_{q^{\nu_{p^i}}}$, respectively, for $i = 1, 2, \ldots, k$ and $j' = 1, 2, \ldots, (2^{is} - 2^{(i-1)s})/\nu_{p^i}$. Similarly, both $D_{i,j}$ and $D'_{i,j}$ are linear codes of length l over $\mathbb{F}_{q^{\nu_{p^i}}}$, for $i = 1, 2, \ldots, k$ and $j = 1, 2, \ldots, (2^{is} - 2^{(i-1)s})/2\nu_{p^i}$. The above decomposition of the code C will lead us to the following proposition.

Proposition 3.13. *Let $H \leq G$ be finite abelian groups such that $H \cong (Z_{p^k})^s$, p is odd, $\gcd(|H|, q) = 1$ and $l = [G : H]$ is even. The number of Hermitian self-dual H-quasi-abelian codes in $\mathbb{F}_q[G]$ is*

$$\begin{cases} \left(\prod_{i=0}^{\frac{l}{2}-1} (q^{i+\frac{1}{2}} + 1) \right) \left(\prod_{i=1}^{k} \left(\prod_{r=0}^{\frac{l}{2}-1} \left((q^{\nu_{p^i}})^{r+\frac{1}{2}} + 1 \right) \right)^{\frac{p^{is}-p^{(i-1)s}}{\nu_{p^i}}} \right) & \textit{if } \nu_p \textit{ is odd and } \mu_p \textit{ is even,} \\ \left(\prod_{i=0}^{\frac{l}{2}-1} (q^{i+\frac{1}{2}} + 1) \right) \left(\prod_{i=1}^{k} \left(\sum_{r=0}^{l} \prod_{j=0}^{r-1} \frac{(q^{\nu_{p^i}})^l - (q^{\nu_{p^i}})^j}{(q^{\nu_{p^i}})^r - (q^{\nu_{p^i}})^j} \right)^{\frac{p^{is}-p^{(i-1)s}}{2\nu_{p^i}}} \right) & \textit{if } \nu_p \textit{ is even or } \mu_p \textit{ is odd.} \end{cases}$$

Proof. Apply the same arguments as in the proof of Proposition 3.8 to (9). □

An example is given when $H \cong (\mathbb{Z}_p)^s$ is an elementary p-group.

Example 3.14. *Let $H \leq G$ be finite abelian groups such that $H \cong (\mathbb{Z}_p)^s$, p is odd, $\gcd(|H|, q) = 1$ and the index $l = [G : H]$ is even. Then the number of Hermitian self-dual H-quasi-abelian codes in $\mathbb{F}_q[G]$ is*

$$\begin{cases} \prod_{i=0}^{\frac{l}{2}-1} (q^{i+\frac{1}{2}} + 1) \left((q^{\nu_p})^{i+\frac{1}{2}} + 1 \right)^{\frac{p^s-1}{\nu_p}} & \textit{if } \nu_p \textit{ is odd and } \mu_p \textit{ is even,} \\ \left(\prod_{i=0}^{\frac{l}{2}-1} (q^{i+\frac{1}{2}} + 1) \right) \left(\sum_{r=0}^{l} \prod_{j=0}^{r-1} \frac{(q^{\nu_p})^l - (q^{\nu_p})^j}{(q^{\nu_p})^r - (q^{\nu_p})^j} \right)^{\frac{p^s-1}{2\nu_p}} & \textit{if } \nu_p \textit{ is even or } \mu_p \textit{ is odd.} \end{cases}$$

See Table 3.2 for the number of Hermitian self-dual H-quasi-abelian codes when $p = 3$, $q = 4$, $l = 2$, for $k = 1, 2$ and $s = 1, 2$. In this case, $\nu_p = 1$ and $\mu_p = 2$. Then the q-cyclotomic classes of H are all of type I, and hence, this table also illustrates the 1-generator case given in Corollary 4.5 (ii), type I case.

4. Hermitian self-dual 1-generator quasi-abelian codes

In this section, we study 1-generator H-quasi-abelian codes in $\mathbb{F}_q[G]$, a cyclic $\mathbb{F}_q[H]$-module of $\mathbb{F}_q[G]$, where $H \leq G$ are finite abelian groups such that $\gcd(|H|, q) = 1$. The main idea here is to use [6, Theorem 6.1] and combine it with the characterization of Hermitian self-dual H-quasi-abelian codes obtained in

Table 2. **Number of Hermitian self-dual H-quasi-abelian codes in $\mathbb{F}_q[G]$, $H \cong (\mathbb{Z}_{3^k})^s$, $l = [G : H] = 2$ and $q = 4$.**

s	k	$\|H\|$	$\|G\|$	$A = (q_0 + 1)$	$B = \prod_{i=1}^{k} (q^{\nu_{p^i}} + 1)^{\|H_{p^i}\|/\nu_{p^i}}$	$A \cdot B$
1	1	3	6	3	9	27
	2	9	18	3	729	2187
2	1	9	18	3	6561	19683
	2	81	162	3	$3^8 \cdot 9^{24}$	$3 \cdot 3^8 \cdot 9^{24}$

Proposition 2.3. We also consider the case where $H \cong (\mathbb{Z}_{p^k})^s$, for $p = 2$ or p is odd, and obtain explicit enumeration.

From [6], we have the following characterization of 1-generator quasi-abelian codes.

Theorem 4.1 ([6, Theorem 6.1])**.** *Let q be a prime power and let $H \leq G$ be finite abelian groups with $l = [G : H]$ and $\gcd(|H|, q) = 1$. Let $e_1, e_2, \ldots, e_t$ be the primitive idempotents of $\mathbb{F}_q[H]$. In the light of (3), let*

$$C \cong \prod_{i=1}^{t} C_i$$

be an H-quasi-abelian code in $\mathbb{F}_q[G]$, where C_i is a linear code of length l over $\mathbb{L}_i \cong \mathbb{F}_q[H]e_i$. Then C is 1-generator if and only if the $\mathbb{L}_i$-dimension of C_i is at most 1, for each $i = 1, 2, \ldots, t$.

Since the $\mathbb{F}_q$-dimension of a 1-generator H-quasi-abelian code C in $\mathbb{F}_q[G]$ cannot exceed $|H|$, $C^{\perp_H}$ could never be a 1-generator if $[G : H] > 2$. In the case where $[G : H] = 2$, we have the following characterization.

Corollary 4.2. *Assume the notation in Theorem 4.1. In addition, we assume that $[G : H] = 2$. If C is a 1-generator H-quasi-abelian code in $\mathbb{F}_q[G]$, then the following statements are equivalent.*

i) *$C^{\perp_H}$ is a 1-generator H-quasi-abelian code.*

ii) *C_i has $\mathbb{L}_i$-dimension 1 for all $i = 1, 2, \ldots, t$.*

iii) *The $\mathbb{F}_q$-dimension of C is $|H|$.*

Proof. The corollary follows immediately from Theorem 4.1 and observations similar to those in [12, Corollary 3.2]. □

Combining Proposition 2.3 and Corollary 4.2, we conclude the following characterization for Hermitian self-dual 1-generator quasi-abelian codes (*cf.* [12, Theorem 3.3]).

Corollary 4.3. *A 1-generator H-quasi-abelian code C in $\mathbb{F}_q[G]$ is Hermitian self-dual if and only if $[G : H] = 2$ (i.e., $G = \mathbb{Z}_2 \times H$) and, in (3), C is decomposed as*

$$C \cong \left(\prod_{i=1}^{r_I} C_i \right) \times \left(\prod_{k=1}^{r_{II}} \left(D_j \times D_j^{\perp_E} \right) \right),$$

where

i) *C_i is Hermitian self-dual of length 2 over $\mathbb{E}_i$ for all $i = 1, 2, \ldots, r_I$, and*

ii) *D_j is a linear code of dimension 1 and length 2 over $\mathbb{K}_j$ for all $j = 1, 2, \ldots, r_{II}$.*

The enumeration of Hermitian self-dual 1-generator quasi abelian codes immediately follows.

Corollary 4.4. *Let $H \leq G$ be finite abelian groups such that $\gcd(|H|, q) = 1$, and $[G : H] = 2$. Assume that $\mathbb{F}_q[H]$ is decomposed as in (1) and contains r_I (resp., $2r_{II}$) primitive idempotents of type I (resp., II). Assume further that the primitive idempotents of type I are induced by q-cyclotomic classes of size s_i for each $i = 1, 2, \ldots, r_I$ and the primitive idempotents of type II are induced by q-cyclotomic classes of sizes t_j and t'_j, pair-wise, for each $j = 1, 2, \ldots, r_{II}$. Then the number of Hermitian self-dual 1-generator H-quasi-abelian codes in $\mathbb{F}_q[G]$ is*

$$\prod_{i=1}^{r_I}(q_0^{s_i} + 1) \prod_{j=1}^{r_{II}}(q^{t_j} + 1).$$

Proof. The corollary follows from Corollary 4.3, (6), and the fact that the number of 1-dimensional subspaces of $\mathbb{F}^2_{q^{t_j}}$ is $q^{t_j} + 1$. □

We end this paper by considering the case of Hermitian self-dual 1-generator H-quasi-abelian codes where H are some p-groups.

Corollary 4.5. *Let $H \leq G$ be finite abelian groups such that $H \cong (Z_{p^k})^s$, $\gcd(|H|, q) = 1$ and $l = [G : H] = 2$ (i.e., $G = \mathbb{Z}_2 \times H$). Then one of the following statements holds.*

i) If $p = 2$, q is odd and $0 \leq r' \leq k$ is the largest integer such that $2^{r'}|(q_0 + 1)$, then the number of Hermitian self-dual 1-generator H-quasi-abelian codes in $\mathbb{F}_q[G]$ is

$$(q_0 + 1)^{2^{r's}} \left(\prod_{r=r'+1}^{k} (q^{\nu_{2^r}} + 1)^{\frac{2^{rs} - 2^{(r-1)s}}{2\nu_{2^r}}} \right).$$

ii) If p is odd and $\gcd(p, q) = 1$, then the number of Hermitian self-dual 1-generator H-quasi-abelian codes in $\mathbb{F}_q[G]$ is

$$\begin{cases} \prod_{i=0}^{k} \left(q_0^{\nu_{p^i}} + 1\right)^{\frac{p^{is} - p^{(i-1)s}}{\nu_{p^i}}} & \text{if } \nu_p \text{ is odd and } \mu_p \text{ is even,} \\ (q_0 + 1) \left(\prod_{i=1}^{k} (q^{\nu_{p^i}} + 1)^{\frac{p^{is} - p^{(i-1)s}}{2\nu_{p^i}}} \right) & \text{if } \nu_p \text{ is even or } \mu_p \text{ is odd.} \end{cases}$$

Proof. The first statement is derived using (8) and Corollary 4.3 by getting the number of Hermitian self-dual codes C_i over $\mathbb{F}_q$ of length $l = 2$, for $i = 1, 2, \ldots, 2^{r's}$, and the number of 1-dimensional linear codes $D_{r,j'}$ of length $l = 2$ over $\mathbb{F}_{q^{\nu_{2^r}}}$ which is equal $q^{\nu_{2^r}} + 1$, for $r = r' + 1, \ldots, k$ and $j' = 1, 2, \ldots, (2^{rs} - 2^{(r-1)s})/2\nu_{2^r}$.

Suppose p is odd, $\gcd(p, q) = 1$, ν_p is odd and μ_p is even. This case follows directly from Proposition 3.13 by letting $l = 2$ and noting that $q = q_0^2$. On the other hand, suppose ν_p is even or μ_p is odd. We apply Corollary 4.3 and (9). The first factor is obtained by counting the number of Hermitian self-dual codes C_1 of length 2 over $\mathbb{F}_q$. For the second factor, we count the number of 1-dimensional linear codes $D_{i,j}$ over $\mathbb{F}_{q^{\nu_{p^i}}}$, given by $q^{\nu_{p^i}} + 1$, for each $i = 1, 2, \ldots, k$, and $j = 1, 2, \ldots, (p^{is} - p^{(i-1)s})/2\nu_{p^i}$. □

For the case where H is an elementary p-group, we have the following example.

Example 4.6. *Let $H \leq G$ be abelian groups such that $H \cong (Z_p)^s$, an elementary p-group, $\gcd(|H|, q) = 1$ and $l = [G : H] = 2$ (i.e., $G = \mathbb{Z}_2 \times H$). Then one of the following statements holds.*

i) If $p = 2$ and q is odd, then the number of Hermitian self-dual 1-generator H-quasi-abelian codes in $\mathbb{F}_q[G]$ is

$$(q_0 + 1)^{2^s}.$$

ii) If p is odd and $\gcd(p, q) = 1$, then the number of Hermitian self-dual 1-generator H-quasi-abelian codes in $\mathbb{F}_q[G]$ is

$$\begin{cases} (q_0+1)(q_0^{\nu_p}+1)^{\frac{p^s-1}{\nu_p}} & \text{if } \nu_p \text{ is odd and } \mu_p \text{ is even,} \\ (q_0+1)(q^{\nu_p}+1)^{\frac{p^s-1}{2\nu_p}} & \text{if } \nu_p \text{ is even or } \mu_p \text{ is odd.} \end{cases}$$

5. Summary

Characterization and enumeration of Hermitian self-dual quasi-abelian codes were established based on the well-known decomposition of quasi-abelian codes. Necessary and sufficient conditions for the existence of Hermitian self-dual 1-generator quasi-abelian codes were also given. For special cases where the underlying groups are some p-groups, complete classification of cyclotomic classes has been done. As a result, the actual number of resulting Hermitian self-dual quasi-abelian codes has been determined. It is interesting to note that the results in this work is restricted to $\mathbb{F}_q[H]$ being a semi-simple group algebra, i.e., the characteristic of $\mathbb{F}_q$ and $|H|$ are coprime, where H is a finite abelian group.

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