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A GG NOT FH SEMISTAR OPERATION ON MONOIDS

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Abstract. Let S be a g-monoid with quotient group $\mathrm{q}(S)$. Let $\bar{\mathrm{F}}(S)$ (resp., $\mathrm{F}(S)$, $\mathrm{f}(S)$) be the S-submodules of $\mathrm{q}(S)$ (resp., the fractional ideals of S, the finitely generated fractional ideals of S). Briefly, set $\mathrm{f} := \mathrm{f}(S)$, $\mathrm{g} := \mathrm{F}(S)$, $\mathrm{h} := \bar{\mathrm{F}}(S)$, and let $\{\mathrm{x}, \mathrm{y}\}$ be a subset of the set $\{\mathrm{f}, \mathrm{g}, \mathrm{h}\}$ of symbols. For a semistar operation $\star$ on S, if $(E+E_1)^\star = (E+E_2)^\star$ implies ${E_1}^\star = {E_2}^\star$ for every $E \in \mathrm{x}$ and every $E_1, E_2 \in \mathrm{y}$, then $\star$ is called xy-cancellative. In this paper, we prove that a gg-cancellative semistar operation need not be fh-cancellative.

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1. Introduction

A subsemigroup $S \supsetneqq \{0\}$ of a torsion-free abelian additive group is called *a grading monoid* (or, *a g-monoid*) (D. Northcott [12]). We will use, for g-monoids, the following terminologies: module, ideal, valuation, star operation analogously to them for rings (cf., [4]). Thus, let S be a g-monoid, and let X be a non-empty set. Assume that, for every $s \in S$ and $x \in X$, an element $s + x$ of X is defined so that $0 + x = x$ and, for every s_1 and s_2 in S, $(s_1 + s_2) + x = s_1 + (s_2 + x)$, then X is called *a module over* S (or, *an S-module*). For the general theory of g-monoids, we refer to [5] and [7]. The additive group $\mathrm{q}(S) := \{s - s' \mid s, s' \in S\}$ is called *the quotient group of* S. Let $\bar{\mathrm{F}}(S)$ be the set of S-submodules of $\mathrm{q}(S)$. An element E of $\bar{\mathrm{F}}(S)$ is called *a fractional ideal of* S if $s + E \subseteq S$ for some $s \in S$. Let $\mathrm{F}(S)$ be the set of fractional ideals of S. A fractional ideal I is called *an ideal of* S if $I \subseteq S$. Let $\mathrm{f}(S) := \{E \in \mathrm{F}(S) \mid E$ is a finitely generated fractional ideal$\}$. A mapping $\star\colon \bar{\mathrm{F}}(S) \longrightarrow \bar{\mathrm{F}}(S)$, $E \longmapsto E^\star$ is called *a semistar operation on* S if it satisfies the following properties for every $x \in \mathrm{q}(S)$ and $E, F \in \bar{\mathrm{F}}(S)$:

(1) $(x + E)^\star = x + E^\star$;

(2) $E \subseteq E^\star$ and $(E^\star)^\star = E^\star$;

(3) $E \subseteq F$ implies $E^\star \subseteq F^\star$.

Set f := f(S), g := F(S), h := $\bar{\mathrm{F}}(S)$, and let $\{\mathrm{x}, \mathrm{y}\}$ be a subset of the set {f, g, h}. For a semistar operation $\star$ on S, if $(E + E_1)^\star = (E + E_2)^\star$ implies ${E_1}^\star = {E_2}^\star$ for every $E \in$ x and $E_1, E_2 \in$ y, then $\star$ is called *xy-cancellative.* A mapping $\star$: $\mathrm{F}(S) \longrightarrow \mathrm{F}(S)$, $E \longmapsto E^\star$ is called *a star operation on* S if it satisfies the following properties for every $x \in \mathrm{q}(S)$ and $E, F \in \mathrm{F}(S)$:

(1) $S^\star = S$;

(2) $(x + E)^\star = x + E^\star$;

(3) $E \subseteq E^\star$ and $(E^\star)^\star = E^\star$;

(4) $E \subseteq F$ implies $E^\star \subseteq F^\star$.

We refer to M. Fontana and K. A. Loper [1] and F. Halter-Koch [6] for star and semistar operations and their Kronecker function rings.

The mapping $E \longmapsto \mathrm{q}(S)$ from $\bar{\mathrm{F}}(S)$ to $\bar{\mathrm{F}}(S)$ for every $E \in \bar{\mathrm{F}}(S)$ is a semistar operation, called *the e-semistar operation on* S. Also, a star (resp., semistar) operation $\star$ on S is said to have *finite type*, if $E^\star = \bigcup\{F^\star \mid F \in \mathrm{f}(S) \text{ with } F \subseteq E\}$ for every $E \in \mathrm{F}(S)$ (resp., $E \in \bar{\mathrm{F}}(S)$). Let Γ be a totally ordered abelian additive group, and let v be a mapping from $\mathrm{q}(S)$ onto Γ. If $v(a+b) = v(a) + v(b)$ for every $a, b \in \mathrm{q}(S)$, then v is called *a valuation on q*(S). Γ is called *the value group of* v, and the set $V := \{a \in \mathrm{q}(S) \mid v(a) \geq 0\}$ is called *the valuation semigroup belonging to* v. If $V \supseteq S$, then V is called *a valuation oversemigroup of* S.

Let $\{V_\lambda \mid \lambda \in \Lambda\}$ be a non-empty set of valuation oversemigroups of S. Then the mapping $E \longmapsto \bigcap_{\lambda \in \Lambda}(E + V_\lambda)$ from $\bar{\mathrm{F}}(S)$ to $\bar{\mathrm{F}}(S)$ is a semistar operation on S, called *the semistar operation defined by* $\{V_\lambda \mid \lambda \in \Lambda\}$. This semistar operation is fh-cancellative (cf., [4, Theorem 32.5]). If $\{V_\lambda \mid \lambda \in \Lambda\}$ is the set of all valuation oversemigroups of S, the semistar operation defined by the set is called *the b-semistar operation on* S.

In this paper, we prove that a gg-cancellative semistar operation need not be fh-cancellative.

2. Preliminary results

Various implications hold among the cancellation properties of semistar operations:

Proposition 2.1. ([3], [8], [9], [10], [11]) *Let* $\star$ *be a semistar operation on a g-monoid* S.

(1) $\star$ *is hh-cancellative if and only if* $\star$ *coincides with the e-semistar operation.*

We have the following diagram of implications:

$$\begin{array}{ccccccc} \mathrm{hh}=\mathrm{hg}=\mathrm{hf} & \Longrightarrow & \mathrm{gh} & \Longrightarrow & \mathrm{gg} & \Longrightarrow & \mathrm{gf} \\ & & \Downarrow & & \Downarrow & & \Downarrow \\ & & \mathrm{fh} & \Longrightarrow & \mathrm{fg} & \Longrightarrow & \mathrm{ff} \end{array}$$

(2) *A gh-cancellative semistar operation of finite type need not be hh-cancellative. A gf-cancellative semistar operation of finite type need not be gg-cancellative. An fh-cancellative semistar operation of finite type need not be gf-cancellative. A gg-cancellative semistar operation of finite type need not be gh-cancellative.*

(3) *A gf-cancellative semistar operation need not be fg-cancellative.*

Remark 2.2. ([2, Lemma 3]) *We have a simplified diagram of implications in the case of finite type semistar operations:*

$$\mathrm{hh}=\mathrm{hg}=\mathrm{hf} \Longrightarrow \mathrm{gh} \Longrightarrow \mathrm{gg} \Longrightarrow \mathrm{gf} \Longrightarrow \mathrm{fh}=\mathrm{fg}=\mathrm{ff}.$$

3. A gg not fh semistar operation

Throughout the section, let $u_1, u_2, u_3, \cdots$ be an infinite set of indeterminates over a torsion-free abelian additive group L. Define $S_0 := \{a + k_1u_1 + \cdots + k_nu_n \mid a \in L, 0 \leq k_i \in \mathbb{Z}, \text{ and } 0 < n \in \mathbb{Z}\}$. Then S_0 is a g-monoid, and $S_0 \supseteq M := \{a+k_1u_1+\cdots+k_nu_n \mid k_i > 0 \text{ for some } i\}$ is the unique maximal ideal of S_0. Let $\mathrm{q}(S_0)$ be the quotient group of S_0. We have $\mathrm{q}(S_0) = \{a+l_1u_1+\cdots+l_nu_n \mid a \in L, l_i \in \mathbb{Z}\}$.

First, we review the following.

Lemma 3.1. ([11]) *Let S be a g-monoid, and let S_0 be the above g-monoid.*

(1) *For every $E \in \bar{\mathrm{F}}(S)$, we have $E^b = \{x \in \mathrm{q}(S) \mid nx \in nE$ for some positive integer $n\}$, where $nE := \{x_1 + \cdots + x_n \mid$ every $x_i \in E\}$.*

(2) *The b-semistar operation on S has finite type.*

(3) *On S_0, the b-semistar operation is gg-cancellative and not gh-cancellative.*

Lemma 3.2. ([4, Proposition (32.4)]) *Let $\mathcal{S}$ be a subset of $\bar{\mathrm{F}}(S)$ with $\mathcal{S} \ni \mathrm{q}(S)$ such that, for every $x \in \mathrm{q}(S)$ and every $E \in \mathcal{S}$, $x + E \in \mathcal{S}$. For every $H \in \bar{\mathrm{F}}(S)$, set $H^\star := \bigcap\{E \in \mathcal{S} \mid E \supseteq H\}$. Then the mapping $H \longmapsto H^\star$ is a semistar operation on S.*

The semistar operation $\star$ in Lemma 3.2 is said to be defined by the set $\mathcal{S}$. Let $H_0 := (u_1 - u_2, u_2 - u_3, \cdots)$, and let $F_0 := (u_1, u_2)$, where, for a subset $X \subseteq \mathrm{q}(S_0)$, (X) denotes the S_0-submodule of $\mathrm{q}(S_0)$ generated by X. Let $\mathcal{S} :=$

$\{G^b, x + {H_0}^b, \mathrm{q}(S_0) \mid G \in \mathrm{F}(S_0), x \in \mathrm{q}(S_0)\}$, and let $\star$ be the semistar operation on S_0 defined by $\mathcal{S}$.

Let v be a valuation on $\mathrm{q}(S_0)$ non-negative on S_0 with value group Γ. Let $v(u_i) = \gamma_i$ for every i. Then we denote $v = v < u_1, u_2, u_3, \cdots > = < \gamma_1, \gamma_2, \gamma_3, \cdots >$.

Lemma 3.3. *We have the following:*

(1) $F_0 \subseteq (u_1, u_2, u_3, \cdots) \subseteq H_0$, *and* $H_0 \notin \mathrm{F}(S_0)$.

(2) $0 \notin {H_0}^b$.

Proof. (1) For every i, we have $u_i = u_{i+1} + (u_i - u_{i+1}) \in S_0 + H_0 \subseteq H_0$. Suppose that $H_0 \in \mathrm{F}(S_0)$. Then there is $x \in \mathrm{q}(S_0)$ such that $x + H_0 \subseteq S_0$. We may set $x = l_1u_1 + \cdots + l_nu_n$ with every $l_i \in \mathbb{Z}$. We have $u_n - u_{n+1} \in H_0$, hence $x + u_n - u_{n+1} \in S_0$, hence $l_1u_1 + \cdots + l_nu_n + u_n - u_{n+1} \in S_0$; a contradiction.

(2) Let $\{\alpha_i \mid i = 1, 2, \cdots\}$ be a set of positive real numbers such that $\alpha_i > \alpha_{i+1}$ for every i. Define the valuation v (and its valuation oversemigroup V) to the real numbers $\mathbb{R}$ by $v = v < u_1, u_2, \cdots > = < \alpha_1, \alpha_2, \cdots >$. Then $0 \notin H_0 + V$, hence $0 \notin {H_0}^b$. □

Lemma 3.4. *We have the following:*

(1) *We have* ${H_0}^\star = {H_0}^b$, *hence* $0 \notin {H_0}^\star$.

(2) *We have* $G^\star = G^b$ *for every* $G \in \mathrm{F}(S_0)$.

Proof. (1) Set $\{x_\sigma \mid \sigma \in \Sigma\} := \{x \in \mathrm{q}(S_0) \mid H_0 \subseteq x + {H_0}^b\}$. Since $H_0 \notin \mathrm{F}(S_0)$ by Lemma 3.3(1), there is no element $G \in \mathrm{F}(S_0)$ such that $H_0 \subseteq G^b$. Hence ${H_0}^\star = \bigcap_\sigma (x_\sigma + {H_0}^b)$. Since $H_0 \subseteq x_\sigma + {H_0}^b$, we have ${H_0}^b \subseteq (x_\sigma + {H_0}^b)^b = x_\sigma + ({H_0}^b)^b = x_\sigma + {H_0}^b$. It follows that ${H_0}^\star = {H_0}^b$.

(2) Set $\{G_\lambda \mid \lambda \in \Lambda\} := \{G' \in \mathrm{F}(S_0) \mid G \subseteq G'^b\}$, and set $\{x_\sigma \mid \sigma \in \Sigma\} := \{x \in \mathrm{q}(S_0) \mid G \subseteq x + {H_0}^b\}$. Then $G^\star = \bigcap_\lambda {G_\lambda}^b \ \bigcap_\sigma (x_\sigma + {H_0}^b)$. Since $G \subseteq {G_\lambda}^b$, we have $G^b \subseteq ({G_\lambda}^b)^b = {G_\lambda}^b$, hence $\bigcap_\lambda {G_\lambda}^b = G^b$. Since $G \subseteq x_\sigma + {H_0}^b$, we have $G^b \subseteq (x_\sigma + {H_0}^b)^b = x_\sigma + ({H_0}^b)^b = x_\sigma + {H_0}^b$, i.e., $G^b \subseteq \bigcap_\sigma (x_\sigma + {H_0}^b)$. It follows that $G^\star = G^b$. □

Lemma 3.5. $\star$ *is a gg-cancellative semistar operation on* S_0.

Proof. Let $G \in (G + G')^\star$, where $G, G' \in \mathrm{F}(S_0)$. By Lemma 3.4(2), we have $G \subseteq (G + G')^b$. By Lemma 3.1(3), we have $0 \in (G')^b$, hence $0 \in (G')^\star$. □

Lemma 3.6. *Let* $F_0 + H_0 \subseteq x + {H_0}^b$, *and let* $x = l_1u_1 + l_2u_2 + \cdots + l_nu_n$ *with every* $l_i \in \mathbb{Z}$. *Then we have*

(1) *Every* $l_i \leq 0$.

(2) $(F_0 + H_0)^\star = {H_0}^b$.

Proof. (1) Suppose the contrary. There are the following cases.

(i) The case $p_1 := l_1 > 0$. We have $-x + F_0 + H_0 \subseteq {H_0}^b$. Define the valuation v (and its valuation oversemigroup V) to $\mathbb{Z}$ by $v = v < u_1, u_2, u_3, \cdots > = < 1, 0, 0, \cdots >$. Then we have $(-p_1u_1 - l_2u_2 - \cdots - l_nu_n) + (u_2) + (u_4) \in {H_0}^b \subseteq H_0 + V$, hence $-p_1 \geq \min v(H_0) = 0$; a contradiction.

(ii) The case $p_2 := l_2 > 0$. We have $-x + F_0 + H_0 \subseteq {H_0}^b$. Define the valuation v (and its valuation oversemigroup V) to $\mathbb{Z}$ by $v = v < u_1, u_2, u_3, u_4, \cdots > = < 0, 1, 0, 0, \cdots >$. Then we have $(-l_1u_1 - p_2u_2 - \cdots - l_nu_n) + (u_1) + (u_1 - u_2) \in {H_0}^b \subseteq H_0 + V$, hence $-p_2 - 1 \geq \min v(H_0) = -1$; a contradiction.

(iii) The case $p_a := l_a > 0$ for some $a \geq 3$. We have $-x + F_0 + H_0 \subseteq {H_0}^b$. Define the valuation v (and its valuation oversemigroup V) to $\mathbb{Z}$ by $v = v < u_1, u_2, \cdots, u_a, u_{a+1}, \cdots > = < 0, 0, \cdots, 1, 0, \cdots >$. Then we have $(-l_1u_1 - l_2u_2 - \cdots - p_au_a - \cdots) + (u_1) + (u_{a-1} - u_a) \in {H_0}^b \subseteq H_0 + V$, hence $-p_a - 1 \geq \min v(H_0) = -1$; a contradiction.

(2) Set $\{x_\lambda \mid \lambda \in \Lambda\} := \{x \in \mathrm{q}(S_0) \mid F_0 + H_0 \subseteq x + {H_0}^b\}$. By Lemma 3.3(1), we have $(F_0 + H_0)^\star = \bigcap_\lambda (x_\lambda + {H_0}^b)$. By (1), we have $-x_\lambda \in S_0$ for every λ, hence ${H_0}^b \subseteq x_\lambda + {H_0}^b$. It follows that $(F_0 + H_0)^\star = {H_0}^b$. □

Lemma 3.7. *The semistar operation* $\star$ *on* S_0 *is not fh-cancellative.*

Proof. By Lemma 3.6(2), we have $F_0 \subseteq (F_0 + H_0)^\star$. On the other hand, by Lemma 3.4(1), we have $0 \notin {H_0}^\star$. □

Lemma 3.5 and Lemma 3.7 imply the following.

Proposition 3.8. *A gg-cancellative semistar operation need not be fh-cancellative.*

We finish with an easy note.

Remark 3.9. *We have that* $S_0^\star = S_0, M^\star = M$, *and that* $\star$ *is not of finite type. Further, the restriction of* $\star$ *to* $\mathrm{F}(S_0)$ *is a star operation on* S_0.

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