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ON A LIE ALGEBRA RELATED TO SOME TYPES OF DERIVATIONS AND THEIR DUALS

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Abstract. Let A be an associative algebra over a commutative ring R, $\mathrm{BiL}(A)$ the set of R-bilinear maps from $A \times A$ to A, and arbitrarily elements x, y in A. Consider the following R-modules:

$$\Omega(A) = \{(f,\ \alpha) \mid f \in \mathrm{Hom}_R(A,\ A),\ \alpha \in \mathrm{BiL}(A)\},$$
$$\mathrm{TDer}(A) = \{(f,\ f',\ f'') \in \mathrm{Hom}_R(A,\ A)^3 \mid f(xy) = f'(x)y + xf''(y)\}.$$

$\mathrm{TDer}(A)$ is called the set of triple derivations of A. We define a Lie algebra structure on $\Omega(A)$ and $\mathrm{TDer}(A)$ such that $\varphi_A : \mathrm{TDer}(A) \to \Omega(A)$ is a Lie algebra homomorphism.

Dually, for a coassociative R-coalgebra C, we define the R-modules $\Omega(C)$ and $\mathrm{TCoder}(C)$ which correspond to $\Omega(A)$ and $\mathrm{TDer}(A)$, and show that the similar results to the case of algebras hold. Moreover, since $C^* = \mathrm{Hom}_R(C,\ R)$ is an associative R-algebra, we give that there exist anti-Lie algebra homomorphisms $\theta_0 : \mathrm{TCoder}(C) \to \mathrm{TDer}(C^*)$ and $\theta_1 : \Omega(C) \to \Omega(C^*)$ such that the following diagram is commutative :

$$\begin{array}{ccc} \mathrm{TCoder}(C) & \xrightarrow{\psi_C} & \Omega(C) \\ \downarrow{\scriptstyle \theta_0} & & \downarrow{\scriptstyle \theta_1} \\ \mathrm{TDer}(C^*) & \xrightarrow{\varphi_{C^*}} & \Omega(C^*). \end{array}$$

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1. Introduction

Throughout the following, R is a commutative ring with an identity 1, A an associative R-algebra and C a coassociative R-coalgebra. We do not assume that A has an identity 1_A and C has a counit $\varepsilon : C \to R$. For any R-modules X and Y, we denote the set of R-linear maps from X to Y by $\mathrm{Hom}(X,\ Y)$ and the symbol $\otimes$ means the tensor product $\otimes_R$ over R. For an A-bimodule M, an R-linear map

Dedicated to the memory of Prof. Manabu Harada.

$d : A \to M$ is called a *derivation* if $d(xy) = d(x)y + xd(y)$ for any $x, y \in A$, and there are several variations on this concept. For examples, $f \in \mathrm{Hom}(A,\ M)$ is called a B-*derivation* (i.e., Bresar's derivation, cf. [1] and [8]) if there exists a derivation $d \in \mathrm{Hom}(A,\ M)$ such that $f(xy) = f(x)y + xd(y)$, and f is called an N-*derivation* if there exists an element $m \in M$ such that $f(xy) = f(x)y + xf(y) + x(my)$ (cf.[8] and [12]). These sets of derivations, B-derivations and N-derivations from A to M are denoted by $\mathrm{Der}(A,\ M)$, $\mathrm{BDer}(A,\ M)$ and $\mathrm{NDer}(A,\ M)$, respectively. The properties of these derivations were discussed in many papers.

An R-linear map $f : A \to M$ is called a *generalized derivation* if there exist elements $f',\ f'' \in \mathrm{Hom}(A,\ M)$ such that $f(x)y + xf'(y) = f''(xy)$. This concept was introduced by G. F. Leger and E. M. Luks in a non-associative algebra in [10], and many properties of the generalized derivations for Lie algebras were given. Since f' and f'' are not uniquely determined by f, we define that a triple $(f,\ f',\ f'') \in \mathrm{Hom}(A,\ M)^3$ is called a *triple derivation* if $f(xy) = f'(x)y + xf''(y)$, and denote the set of triple derivations from A to M by $\mathrm{TDer}(A,\ M)$ (cf. [8]). It is easy to see that the derivations, B-derivations and N-derivations are represented by $(d,\ d,\ d)$, $(f,\ f,\ d)$ and $(f,\ f,\ f + m_\ell)$ as triple derivations, respectively, and $f + m_\ell$ is a derivation, where $m_\ell(x) = mx$. We have the following relations for these R-modules:

$$\mathrm{Der}(A,\ M) \subseteq \mathrm{NDer}(A,\ M) \subseteq \mathrm{BDer}(A,\ M) \subseteq \mathrm{TDer}(A,\ M) \subset \mathrm{Hom}(A,\ M)^3,$$

where $\mathrm{Hom}(A,\ M)^3 = \mathrm{Hom}(A,\ M) \times \mathrm{Hom}(A,\ M) \times \mathrm{Hom}(A,\ M)$ is the direct product of R-module $\mathrm{Hom}(A,\ M)$.

Let C be an R-coalgebra with a comultiplication $\Delta : C \to C \otimes C$. An R-module N is called a C-*bicomodule* if there exist C-comodule structure maps $\rho^+ : N \to N \otimes C$ and $\rho^- : N \to C \otimes N$ such that the following relations hold:

$$(I \otimes \Delta)\rho^+ = (\rho^+ \otimes I)\rho^+,\ \ (\Delta \otimes I)\rho^- = (I \otimes \rho^-)\rho^-,\ \ (\rho^- \otimes I)\rho^+ = (I \otimes \rho^+)\rho^- \quad (1.0)$$

where the letter I always stands for the identity map (here, the identity map $C \to C$). An R-linear map $d : N \to C$ is called a *coderivation* if

$$\Delta d = (d \otimes I)\rho^+ + (I \otimes d)\rho^- : N \to C \otimes C \quad (1.1)$$

(cf. [4] and [11]). The notion of a coderivation is also extended as follows. An R-linear map $f : N \to C$ is called a B-*coderivation* if there exists a coderivation $d : N \to C$ such that

$$\Delta f = (f \otimes I)\rho^+ + (I \otimes d)\rho^- \quad (1.2)$$

and f is called an N-*coderivation* if there exists an R-linear map $\xi : N \to R$ such that

$$\Delta f = (f \otimes I)\rho^+ + (I \otimes f)\rho^- + (I \otimes \xi \otimes I)(I \otimes \rho^+)\rho^-. \tag{1.3}$$

A triple $(f,\ f',\ f'') \in \mathrm{Hom}(N,\ C)^3$ is called a *triple coderivation* if

$$\Delta f = (f' \otimes I)\rho^+ + (I \otimes f'')\rho^-. \tag{1.4}$$

These sets of coderivations, B-coderivations, N-coderivations and triple coderivations from N to C are denoted by $\mathrm{Coder}(N,\ C)$, $\mathrm{BCoder}(N,\ C)$, $\mathrm{NCoder}(N,\ C)$ and $\mathrm{TCoder}(N,\ C)$, respectively. Similarly to the case of derivations, these coderivations, B-coderivations and N-coderivations are represented by $(d,\ d,\ d)$, $(f,\ f,\ d)$ and $(f,\ f,\ f+(\xi\otimes I)\rho^+)$ as triple coderivations, respectively, we have the following relations for these R-modules

$$\mathrm{Coder}(N,\ C) \subseteq \mathrm{NCoder}(N,\ C) \subseteq \mathrm{BCoder}(N,\ C) \subseteq \mathrm{TCoder}(N,\ C).$$

The middle sign $\subseteq$ is proven by Lemma 3.1. These R-modules are also R-submodules of the direct product $\mathrm{Hom}(N,\ C)^3 = \mathrm{Hom}(N,\ C) \times \mathrm{Hom}(N,\ C) \times \mathrm{Hom}(N,\ C)$. The properties of these coderivations were discussed in [9] and [15]. Note that when C has a counit $\varepsilon : C \to R$, then we will show that

$$\mathrm{NCoder}(N,\ C) = \mathrm{BCoder}(N,\ C) = \mathrm{TCoder}(N,\ C)$$

in Section 3, Lemma 3.1.

In Section 2, we treat an associative R-algebra A and an A-bimodule M. Writing $\mathrm{BiL}(A,\ M)$ for the set of R-bilinear maps from $A \times A \to M$, we consider the following R-module as the direct product of R-modules $\mathrm{Hom}(A,\ M)$ and $\mathrm{BiL}(A,\ M)$:

$$\Omega(A,\ M) = \mathrm{Hom}(A,\ M) \times \mathrm{BiL}(A,\ M),$$

and define

$$\varphi_M : \mathrm{TDer}(A,\ M) \to \Omega(A, M).$$

Then we see that $\Omega(A) = \Omega(A,\ A)$ and $\mathrm{TDer}(A) = \mathrm{TDer}(A,\ A)$ have Lie algebra structures such that $\varphi_A : \mathrm{TDer}(A) \to \Omega(A)$ is a Lie algebra homomorphism. Moreover, we show that the set of generalized Lie (resp. Jordan) derivations $\mathrm{GLDer}(A)$ (resp. $\mathrm{GJDer}(A)$) from A to A is also a Lie subalgebra of $\Omega(A)$. In the final of Section 2, we discuss the subset of $\mathrm{BiL}(A,\ M)$ consisting of the biderivations in the sense of [2] and [17].

In Section 3, for a coassociative R-coalgebra C and a C-bicomodule N, we define the direct product of R-modules $\mathrm{Hom}(N,\ C)$ and $\mathrm{Hom}(N,\ C \otimes C)$

$$\Omega(N,\ C) = \mathrm{Hom}(N,\ C) \times \mathrm{Hom}(N,\ C \otimes C).$$

Then we show that results similar to those from Section 2 hold. Moreover, since $C^* = \mathrm{Hom}(C,\ R)$ is an associative R-algebra and $N^* = \mathrm{Hom}(N,\ R)$ is a C^*-bimodule, there exist R-module homomorphisms

$$\theta_0 : \mathrm{TCoder}(N,\ C) \to \mathrm{TDer}(C^*,\ N^*) \quad \text{and} \quad \theta_1 : \Omega(N,\ C) \to \Omega(C^*,\ N^*)$$

such that the following diagram is commutative:

$$\begin{array}{ccc} \mathrm{TCoder}(N,\ C) & \xrightarrow{\psi_N} & \Omega(N,\ C) \\ \downarrow{\scriptstyle \theta_0} & & \downarrow{\scriptstyle \theta_1} \\ \mathrm{TDer}(C^*,\ N^*) & \xrightarrow{\varphi_{N^*}} & \Omega(C^*,\ N^*) \end{array}$$

where ψ_N is defined in Lemma 3.2. Especially, if $N = C$, then ψ_N and φ_{N^*} are Lie algebra homomorphisms whereas θ_0 and θ_1 are anti-Lie algebra homomorphisms.

2. The case of algebras

In this section, A is an associative R-algebra and the letters $x,\ y,\ z$ denote arbitrary elements in A. M is an A-bimodule, that is, M is an R-module, and a left and a right A-module such that

$$x(my) = (xm)y, \quad r(xm) = (rx)m = x(rm) = (xm)r \quad \text{and} \quad rm = mr$$

for any $m \in M$ and $r \in R$. An A-bimodule M is said to be *unital* if $\{m \in M \mid AmA = 0\} = 0$. If A has an identity element 1_A and M is unital, then, for any $m \in M$, we have $A(1_A m - m)A = 0$, and hence $1_A m = m$, similarly $m 1_A = m$.

An R-bilinear map $\alpha \in \mathrm{BiL}(A,\ M)$ is called a *factor set* or *Hochschild* 2-*cocycle* if

$$x\alpha(y,\ z) - \alpha(xy,\ z) + \alpha(x,\ yz) - \alpha(x,\ y)z = 0,$$

and the factor set α is called a *split factor set* if there exists $f \in \mathrm{Hom}(A,\ M)$ such that

$$f(xy) = f(x)y + xf(y) + \alpha(x,\ y)$$

(cf. [3, (72.13) and (72.14)] and [14]). Although f is not uniquely determined by α, we consider the set of above pairs, and denote it by

$$\Lambda(A,\ M) = \{(f,\ \alpha) \in \Omega(A,\ M) \mid f(xy) = f(x)y + xf(y) + \alpha(x,\ y)\}.$$

As is easily seen, $\Lambda(A,\ M)$ is an R-submodule of $\Omega(A,\ M)$, and for any $(f,\ f',\ f'') \in \mathrm{TDer}(A,\ M)$, we have a bilinear map

$$A \times A \ni (x,\ y) \mapsto f(xy) - f(x)y - xf(y) = (f' - f)(x)y + x(f'' - f)(y) \in M.$$

Then we have the following.

Lemma 2.1. *For any $(f,\ f',\ f'') \in TDer(A,\ M)$, define an R-bilinear map $\alpha(f)$ by*

$$\alpha(f) : A \times A \ni (x,\ y) \mapsto f(xy) - f(x)y - xf(y) \in M. \tag{2.1}$$

Then the map

$$\varphi_M : TDer(A,\ M) \ni (f,\ f',\ f'') \mapsto (f,\ \alpha(f)) \in \Lambda(A,\ M) \subset \Omega(A,\ M) \tag{2.2}$$

is an R-module homomorphism with

$$Ker\ \varphi_M = \{(0,\ f',\ f'') \in TDer(A,\ M) \mid f'(x)y + xf''(y) = 0 \quad for\ any \quad x,\ y \in A\}.$$

Especially, if M is a unital A-bimodule, then φ_M is a monomorphism on $BDer(A,\ M)$.

Proof. We note that for any $(f,\ f',\ f'') \in \mathrm{TDer}(A,\ M)$, $\alpha(f)$ is a split factor set by f and thus $\varphi_M(\mathrm{TDer}(A,\ M))$ is contained in $\Lambda(A,\ M)$. By (2.1) and the definition of φ_M, it is clear that φ_M is an R-module homomorphism with kernel defined above. Assume that M is a unital A-bimodule. If $(f,\ f,\ d)$ is a B-derivation, then by $\varphi_M(f,\ f,\ d) = (f,\ \alpha(f)) = 0$, we have $f = 0$ and $\alpha(f)(x,\ y) = x(f - d)(y) = -xd(y) = 0$ for any x, $y \in A$. Since M is a unital A-bimodule, we see $d = 0$, which shows that φ_M is a monomorphism on $\mathrm{BDer}(A,\ M)$. □

For any triple derivations $(f,\ f',\ f'')$, $(g,\ g',\ g'') \in \mathrm{TDer}(A) = \mathrm{TDer}(A,\ A)$, there holds $[f,\ g](xy) = (fg - gf)(xy) = [f',\ g'](x)y + x[f'',\ g''](y)$ for any x, $y \in A$. This shows that the operation

$$[(f,\ f',\ f''),\ (g,\ g',\ g'')] = ([f,\ g],\ [f',\ g'],\ [f'',\ g''])$$

gives a Lie algebra structure on $\mathrm{TDer}(A)$. It is easy to see that $\mathrm{BDer}(A) = \mathrm{BDer}(A,\ A)$ is a Lie subalgebra of $\mathrm{TDer}(A)$ by the above operation.

Now, we define a Lie algebra structure on $\Omega(A) = \Omega(A,\ A)$, and show that the map φ_A defined by (2.2) is a non-trivial Lie algebra homomorphism.

Let $(f,\ \alpha)$ be in $\Omega(A)$ and define an R-bilinear map α_f by

$$\alpha_f : A \times A \ni (x,\ y) \to \alpha(f(x),\ y) + \alpha(x,\ f(y)) - f\alpha(x,\ y) \in A,$$

that is,

$$\alpha_f = \alpha(f \times I + I \times f) - f\alpha. \tag{2.3}$$

First, we have the following.

Lemma 2.2. *For any $f,\ g \in Hom(A,\ A)$ and $\alpha,\ \beta \in BiL(A) = BiL(A,\ A)$ and $r \in R$, the following relations hold:*

(1) $\alpha_{(rf)} = (r\alpha)_f = r(\alpha_f)$, (2) $(\alpha+\beta)_f = \alpha_f + \beta_f$, (3) $\alpha_{(f+g)} = \alpha_f + \alpha_g$,
(4) $(\alpha_f)_g - (\alpha_g)_f = \alpha_{[f,\ g]}$.

Proof. Since (1), (2) and (3) are easily seen by definition (2.3), we only show (4). By (2.3), we have for any $x,\ y \in A$

$$\begin{aligned}&(\alpha_f)_g - (\alpha_g)_f\\ &= \alpha_f(g\times I + I\times g - g\alpha_f - \alpha_g(f\times I + I\times f) + f\alpha_g\\ &= \{\alpha(f\times I + I\times f) - f\alpha\}(g\times I + I\times g) - g\{\alpha(f\times I + I\times f) - f\alpha\}\\ &-\{\alpha(g\times I + I\times g) - g\alpha\}(f\times I + I\times f) + f\{\alpha(g\times I + I\times g) - g\alpha\}\\ &= \alpha(fg\times I - gf\times I) + \alpha(I\times fg - I\times gf) - (fg-gf)\alpha = \alpha_{[f,\ g]}.\end{aligned}$$ □

Theorem 2.3. *For any $(f,\ \alpha),\ (g,\ \beta) \in \Omega(A)$, we define*

$$[(f,\ \alpha),\ (g,\ \beta)] = ([f,\ g],\ \alpha_g - \beta_f). \tag{2.4}$$

Then $\Omega(A)$ is a Lie algebra with a Lie subalgebra $\Lambda(A) = \Lambda(A,\ A)$, and the map

$$\varphi_A : TDer(A) \ni (f,\ f',\ f'') \mapsto (f,\ \alpha(f)) \in \Lambda(A) \subset \Omega(A)$$

defined by (2.2) is a Lie algebra homomorphism.

Proof. First, we show that $\Omega(A)$ is a Lie algebra. Let $u = (f,\ \alpha)$, $v = (g,\ \beta)$ and $w = (h,\ \gamma)$ be in $\Omega(A)$. Then by Lemma 2.2 and the definition (2.4), the following is easily seen.

$$[u,\ v] + [v,\ u] = 0 \quad\text{and}\quad [u+v,\ w] = [u,\ w] + [v,\ w].$$

Therefore, it is enough to show that the Jacobi identity

$$[u,\ [v,\ w]] + [v,\ [w,\ u]] + [w,\ [u,\ v]] = 0 \tag{$*$}$$

holds. By the associativity of R-linear maps $f,\ g,\ h \in \mathrm{Hom}(A,\ A)$, the first component of the above relation $(*)$ is

$$[f,\ [g,\ h]] + [g,\ [h,\ f]] + [h,\ [f,\ g]] = 0.$$

And by (2.4), since

$$\begin{aligned}[(f,\ \alpha),\ [(g,\ \beta),\ (h,\ \gamma)]] &= [(f,\ \alpha),\ ([g,\ h],\ \beta_h - \gamma_g)]\\ &= ([f,\ [g,\ h]], \alpha_{[g,\ h]} - (\beta_h - \gamma_g)_f),\end{aligned}$$

then by Lemma 2.2, the second component of $(*)$ is

$$\begin{aligned}&\{\alpha_{[g,\ h]} - (\beta_h - \gamma_g)_f\} + \{\beta_{[h,\ f]} - (\gamma_f - \alpha_h)_g\} + \{\gamma_{[f,\ g]} - (\alpha_g - \beta_f)_h\}\\ &= \alpha_{[g,\ h]} - \{(\alpha_g)_h - (\alpha_h)_g\} + \beta_{[h,\ f]} - \{(\beta_h)_f - (\beta_f)_h\}\\ &+ \gamma_{[f,\ g]} - \{(\gamma_f)_g - (\gamma_g)_f\} = 0.\end{aligned}$$

Therefore $\Omega(A)$ is a Lie algebra.

Next, we show that $\Lambda(A)$ is a Lie subalgebra of $\Omega(A)$. Let $(f,\ \alpha)$ and $(g,\ \beta)$ be in $\Lambda(A)$. Then by (2.3), we have

$$\begin{aligned}[f,\ g](xy) &= f(g(x)y + xg(y) + \beta(x,\ y)) - g(f(x)y + xf(y) + \alpha(x,\ y))\\ &= (fg(x))y + xfg(y) + \alpha(g(x),\ y) + \alpha(x,\ g(y)) - g\alpha(x,\ y)\\ &- \{(gf(x))y + xgf(y) + \beta(f(x),\ y) + \beta(x,\ f(y)) - f\beta(x,\ y)\}\\ &= [f,\ g](x)y + x[f,\ g](y) + (\alpha_g - \beta_f)(x,\ y).\end{aligned}$$

Thus $\Lambda(A)$ is a Lie subalgebra of $\Omega(A)$.

Finally, we show that φ_A is a Lie algebra homomorphism. Let $(f,\ f',\ f'')$ and $(g,\ g',\ g'')$ be in $\mathrm{TDer}(A)$. Then by

$$\varphi_A([(f,\ f',\ f''),\ (g,\ g',\ g'')]) = \varphi_A([f,\ g],\ [f',\ g'],\ [f'',\ g'']) = ([f,\ g],\ \alpha([f,\ g]))$$

and

$$[\varphi_A(f,\ f',\ f''),\ \varphi_A(g,\ g',\ g'')] = [(f,\ \alpha(f)),\ (g,\ \alpha(g))] = ([f,\ g],\ \alpha(f)_g - \alpha(g)_f),$$

it is enough to show that $\alpha([f,\ g]) = \alpha(f)_g - \alpha(g)_f$. Since $(f,\ f',\ f'')$ and $(g,\ g',\ g'')$ are triple derivations, then by (2.1) and (2.4), we see

$$\alpha(f)_g(x,\ y) = \alpha(f)\{(g(x),\ y) + (x,\ g(y))\} - g(\alpha(f)(x,\ y)),$$

and so

$$\begin{aligned}\alpha(f)_g(x,\ y) &= (f' - f)(g(x))y + g(x)(f'' - f)(y) + (f' - f)(x)g(y)\\ &+ x(f'' - f)(g(y)) - (g'(f' - f)(x))y - (f' - f)(x)g''(y)\\ &- g'(x)(f'' - f)(y) - xg''((f'' - f)(y))\\ &= -\{fg(x)y + g'f'(x)y + xfg(y) + xg''f''(y)\}\\ &+ f'g(x)y + g(x)f''(y) - g(x)f(y) + f'(x)g(y) - f(x)g(y)\\ &+ xf''g(y) + g'f(x)y - f'(x)g''(y) + f(x)g''(y) - g'(x)f''(y)\\ &+ g'(x)f(y) + xg''f(y).\end{aligned}$$

Symmetrically, we see

$$\begin{aligned}\alpha(g)_f(x,\ y) = &-\{gf(x)y + f'g'(x)y + xgf(y) + xf''g''(y)\}\\ &+ g'f(x)y + f(x)g''(y) - f(x)g(y) + g'(x)f(y) - g(x)f(y)\\ &+ xg''f(y) + f'g(x)y - g'(x)f''(y) + g(x)f''(y) - f'(x)g''(y)\\ &+ f'(x)g(y) + xf''g(y).\end{aligned}$$

Hence we have

$$\begin{aligned}(\alpha(f)_g - \alpha(g)_f)(x,\ y) &= (f'g'(x) - g'f'(x))y - (fg(x) - gf(x))y\\ &\quad + x(f''g''(y) - g''f''(y)) - x(fg(y) - gf(y))\\ &= ([f',\ g'] - [f,\ g])(x)y + x([f'',\ g''] - [f,\ g])(y)\\ &= \alpha([f,\ g])(x,\ y).\end{aligned}$$

Therefore φ_A is a Lie algebra homomorphism. □

A bilinear map $\alpha \in \mathrm{BiL}(A,\ M)$ is called *symmetric* (resp. *skew symmetric*) if $\alpha(x,\ y) = \alpha(y,\ x)$ (resp. $\alpha(x,\ y) = -\alpha(y,\ x)$) for any $x,\ y \in A$. We denote the set of symmetric (resp. skew symmetric) bilinear maps from $A \times A$ to M by $\mathrm{BiL}_{Sy}(A,\ M)$ (resp. $\mathrm{BiL}_{sSy}(A,\ M)$). Define

$$\Omega_{Sy}(A,\ M) = \{(f,\ \alpha) \in \Omega(A,\ M) \mid \alpha \in \mathrm{BiL}_{Sy}(A,\ M)\},$$
$$\Omega_{sSy}(A,\ M) = \{(f,\ \alpha) \in \Omega(A,\ M) \mid \alpha \in \mathrm{BiL}_{sSy}(A,\ M)\}.$$

These sets are R-submodules of $\Omega(A,\ M)$ and we have the following R-submodules of $\Lambda(A,\ M)$:

$$\Lambda_{Sy}(A,\ M) = \{(f,\ \alpha) \in \Lambda(A,\ M) \mid \alpha \in \mathrm{BiL}_{Sy}(A,\ M)\},$$
$$\Lambda_{sSy}(A,\ M) = \{(f,\ \alpha) \in \Lambda(A,\ M) \mid \alpha \in \mathrm{BiL}_{sSy}(A,\ M)\}.$$

Moreover, for any $(f,\ \alpha), (g,\ \beta) \in \Omega_{Sy}(A,\ M)$ (resp. $(f,\ \alpha), (g,\ \beta) \in \Omega_{sSy}(A,\ M)$), α_g and β_f are symmetric (resp. skew symmetric) by (2.3) and thus we have the following.

Corollary 2.4. *$\Omega_{Sy}(A) = \Omega_{Sy}(A,\ A)$ and $\Omega_{sSy}(A) = \Omega_{sSy}(A,\ A)$ are Lie subalgebras of $\Omega(A)$. Especially, $\Lambda_{Sy}(A) = \Lambda_{Sy}(A,\ A)$ and $\Lambda_{sSy}(A) = \Lambda_{sSy}(A,\ A)$ are Lie subalgebras of $\Lambda(A)$.*

In [12], we showed that the set of N-derivations

$$\mathrm{NDer}(A) = \{(f,\ a) \in \mathrm{Hom}(A,\ A) \times A \mid f(xy) = f(x)y + xf(y) + xay\}$$

from A to A is a Lie algebra by the following operation

$$[(f,\ a),\ (g,\ b)] = ([f,\ g],\ f(b) - g(a)) \quad (a,\ b \in A). \tag{2.5}$$

Since an N-derivation $(f,\ a)$ is represented by $(f,\ f,\ f + a_\ell)$ as a triple derivation and the Lie algebra structure of $\mathrm{TDer}(A)$ is given by

$$[(f,\ f,\ f + a_\ell),\ (g,\ g,\ g + b_\ell)] = ([f,\ g],\ [f,\ g],\ [f + a_\ell,\ g + b_\ell]),$$

then we see by (2.1)

$$\begin{aligned}\alpha([f,\ g])(x,\ y) &= x([(f + a_\ell),\ (g + b_\ell)] - [f,\ g])(y)\\ &= x(fb_\ell + a_\ell g + a_\ell b_\ell - ga_\ell - b_\ell f - b_\ell a_\ell)(y)\\ &= x(f(b) - g(a))y\end{aligned}$$

as a triple derivation. Thus the Lie algebra structure of $\mathrm{NDer}(A)$ defined by (2.5) is the same as (2.1).

It is known the following types of derivations which are not triple derivations. Let G be a multiplicative subsemigroup of the set of R-algebra endomorphisms of A and $f \in \mathrm{Hom}(A,\ M)$. f is called a *G-derivation* if $f(xy) = f(x)\sigma(y) + \tau(x)f(y)$ for some $\sigma,\ \tau \in G$. It is a generalization of $(\sigma,\ \tau)$-derivation. The properties of $(\sigma,\ \tau)$-derivations were discussed in many papers (cf. [5]). We denote the set of G-derivations by $\mathrm{Der}_G(A,\ M)$. A G-derivation is not a triple derivation. Another one is as follows. $g \in \mathrm{Hom}(A,\ M)$ is called a *right derivation* if $g(xy) = g(x)y + g(y)x$. The right derivations relate to the *quasi-separable* or *differentially separable* extension A over R (cf. [7] and [16]). We denote the set of right derivations by $\mathrm{RDer}(A,\ M)$. As is easily seen, the above derivations are not triple derivations, but if we define $\alpha(f)(x,\ y) = f(xy) - f(x)y - xf(y)$ and $\beta(g)(x,\ y) = g(y)x - xg(y)$, the equality $\alpha(f)(x,\ y) = f(x)(\sigma(y) - y) + (\tau(x) - x)f(y)$ is then an easy consequence of this definition and by

$$\begin{aligned}&\mathrm{Der}_G(A,\ M) \ni f \mapsto (f,\ \alpha(f)) \in \Lambda(A,\ M)\\ &\mathrm{RDer}(A,\ M) \ni g \mapsto (g,\ \beta(g)) \in \Lambda(A,\ M),\end{aligned}$$

$\mathrm{Der}_G(A,\ M)$ and $\mathrm{RDer}(A,\ M)$ are R-submodules of $\Lambda(A,\ M)$. In general, $\mathrm{Der}_G(A) = \mathrm{Der}_G(A,\ A)$ and $\mathrm{RDer}(A) = \mathrm{RDer}(A,\ A)$ are not Lie subalgebras of $\Lambda(A)$. If we assume that G is commutative and $\sigma d = d\sigma$ for any $\sigma \in G$ and $d \in \mathrm{Der}_G(A)$, then $\mathrm{Der}_G(A)$ is a Lie subalgebra of $\Lambda(A)$. Similarly, if $[f(x),\ g(y)] + [f(y),\ g(x)] = 0$ for any $f,\ g \in \mathrm{RDer}(A)$ and $x,\ y \in A$, then $\mathrm{RDer}(A)$ is also a Lie subalgebra of $\Lambda(A)$. But we have not good examples for G-derivations and right derivations which satisfy the above relations.

A pair $(f,\ \alpha) \in \Omega(A,\ M)$ is called a *generalized Jordan derivation* if

$$f(x^2) = f(x)x + xf(x) + \alpha(x,\ x) \quad \text{for any} \quad x \in A, \tag{2.6}$$

and similarly, $(f,\ \alpha)$ is called a *generalized Lie derivation* if

$$f([x,\ y]) = [f(x),\ y] + [x,\ f(y)] + \alpha(x,\ y) - \alpha(y,\ x) \quad \text{for any} \quad x,\ y \in A. \tag{2.7}$$

These notions were introduced in [13] (cf. [14]) and some properties of them were given. The sets of generalized Jordan derivations and generalized Lie derivations from A to M are denoted by $\mathrm{GJDer}(A,\ M)$ and $\mathrm{GLDer}(A,\ M)$, respectively. Then we have the following.

Theorem 2.5. *$GJDer(A) = GJDer(A,\ A)$ and $GLDer(A) = GLDer(A,\ A)$ are Lie subalgebras of $\Omega(A)$ by the operation (2.4).*

Proof. As is easily seen, $\mathrm{GJDer}(A,\ M)$ and $\mathrm{GLDer}(A,\ M)$ are R-submodules of $\Omega(A,\ M)$.

First, we show that $\mathrm{GJDer}(A)$ is a Lie subalgebra of $\Omega(A)$. Let $(f,\ \alpha)$ and $(g,\ \beta)$ be in $\mathrm{GJDer}(A)$. Since $g(x)x + xg(x) = (g(x) + x)^2 - g(x)^2 - x^2$, then by (2.6),

$$\begin{aligned} fg(x^2) &= f(g(x)x + xg(x) + \beta(x,\ x)) \\ &= (fg(x))x + xfg(x) + f(x)g(x) + g(x)f(x) \\ &\quad + \alpha(g(x),\ x) + \alpha(x,\ g(x)) + f\beta(x,\ x), \end{aligned}$$

and symmetrically

$$\begin{aligned} gf(x^2) &= (gf(x))x + xgf(x) + g(x)f(x) + f(x)g(x) \\ &\quad + \beta(f(x),\ x) + \beta(x,\ f(x)) + g\alpha(x,\ x). \end{aligned}$$

Thus by (2.3) and (2.4), we see

$$[f,\ g](x^2) = [f,\ g](x)x + x[f,\ g](x) + (\alpha_g - \beta_f)(x,\ x),$$

which show that $([f,\ g],\ \alpha_g - \beta_f)$ is a generalized Jordan derivation in our sense.

Next, let $(f,\ \alpha)$ and $(g,\ \beta)$ be in $\mathrm{GLDer}(A)$. Then by (2.3) and (2.7), we see

$$\begin{aligned} fg([x,\ y]) &= f([g(x),\ y] + [x,\ g(y)] + \beta(x,\ y) - \beta(y,\ x)) \\ &= [fg(x),\ y] + [g(x),\ f(y)] + \alpha(g(x),\ y) - \alpha(y,\ g(x)) \\ &\quad + [f(x),\ g(y)] + [x,\ fg(y)] + \alpha(x,\ g(y)) - \alpha(g(y),\ x) \\ &\quad + f(\beta(x,\ y)) - f(\beta(y,\ x)), \end{aligned}$$

and symmetrically

$$\begin{aligned} gf([x,\ y]) &= [gf(x),\ y] + [f(x),\ g(y)] + \beta(f(x),\ y) - \beta(y,\ f(x)) \\ &\quad + [g(x),\ f(y)] + [x,\ gf(y)] + \beta(x,\ f(y)) - \beta(f(y),\ x) \\ &\quad + g(\alpha(x,\ y)) - g(\alpha(y,\ x)). \end{aligned}$$

Thus we have

$$[f,\ g]([x,\ y]) = [[f,\ g](x),\ y] + [x,\ [f,\ g](y)] + (\alpha_g - \beta_f)(x,\ y) - (\alpha_g - \beta_f)(y,\ x),$$

which shows that $([f,\ g],\ \alpha_g - \beta_f)$ is a generalized Lie derivation. These show that $\mathrm{GJDer}(A)$ and $\mathrm{GLDer}(A)$ are Lie subalgebras of $\Omega(A)$. □

In the final part of this section, we study biderivations from $A \times A$ to M. A bilinear map $B : A \times A \to M$ is called a *biderivation* if for any $x \in A$, the maps

$$B(x,\ -) : A \ni y \mapsto B(x,\ y) \in M \quad \text{and} \quad B(-,\ x) : A \ni y \mapsto B(y,\ x) \in M$$

are derivations. A biderivation B is called *symmetric* (resp. *skew symmetric*) if B is a symmetric (resp. skew symmetric) bilinear map. We denote the sets of biderivations, symmetric biderivations and skew symmetric biderivations from $A \times A$ to M by $\mathrm{BiDer}(A,\ M)$, $\mathrm{SyBiDer}(A,\ M)$ and $\mathrm{sSyBiDer}(A,\ M)$, respectively. It is known that if A is commutative and $f,\ g : A \to A$ are derivations, then the map $f \cdot g$ defined by $(f \cdot g)(x,\ y) = f(x)g(y)$ is a biderivation, and thus $f \cdot f$ is a symmetric biderivation. Moreover, the map $[-,\ -] : A \times A \ni (x,\ y) \mapsto xy - yx \in A$ is a skew symmetric biderivation. The properties of biderivations and symmetric biderivations were discussed in [2], [6] and [17].

Now, we consider the following R-module

$$\Lambda_B(A,\ M) = \{(f,\ \alpha) \in \Lambda(A,\ M) \mid \alpha \in \mathrm{BiDer}(A,\ M)\}.$$

Let $(f,\ \alpha)$ and $(g,\ \beta)$ be in $\Lambda_B(A) = \Lambda_B(A,\ A)$. Then by (2.3), $\alpha_g(xz,\ y) = \alpha(g(xz),\ y) + \alpha(xz,\ g(y)) - g\alpha(xz,\ y)$, we have

$$\begin{aligned} \alpha_g(xz,\ y) &= x\{\alpha(g(z),\ y) + \alpha(z,\ g(y)) - g\alpha(z,\ y)\} \\ &\quad + \{\alpha(g(x),\ y) + \alpha(x,\ g(y)) - g\alpha(x,\ y)\}z \\ &\quad + \alpha(\beta(x,\ z),\ y) - \beta(x,\ \alpha(z,\ y)) - \beta(\alpha(x,\ y),\ z), \end{aligned}$$

that is,

$$\alpha_g(xz,\ y) - x\alpha_g(z,\ y) - \alpha_g(x,\ y)z = \alpha(\beta(x,\ z),\ y) - \beta(x,\ \alpha(z,\ y)) - \beta(\alpha(x,\ y),\ z).$$

This means that α_g is not a biderivation in general. Similarly, we have

$$\beta_f(xz,\ y) - x\beta_f(z,\ y) - \beta_f(x,\ y)z = \beta(\alpha(x,\ z),\ y) - \alpha(x,\ \beta(z,\ y)) - \alpha(\beta(x,\ y),\ z)$$

and so

$$\begin{aligned}&(\alpha_g - \beta_f)(xz,\ y) - x(\alpha_g - \beta_f)(z,\ y) - (\alpha_g - \beta_f)(x,\ y)z \\ &= \alpha(\beta(x,\ z),\ y) - \beta(x,\ \alpha(z,\ y)) - \beta(\alpha(x,\ y),\ z) \\ &- \{\beta(\alpha(x,\ z),\ y) - \alpha(x,\ \beta(z,\ y)) - \alpha(\beta(x,\ y),\ z)\},\end{aligned}$$

$$\beta_f(xz,\ y) - x\beta_f(z,\ y) - \beta_f(x,\ y)z = \beta(\alpha(x,\ z),\ y) - \alpha(x,\ \beta(z,\ y)) - \alpha(\beta(x,\ y),\ z)$$

and so

$$\begin{aligned}&(\alpha_g - \beta_f)(xz,\ y) - x(\alpha_g - \beta_f)(z,\ y) - (\alpha_g - \beta_f)(x,\ y)z \\ &= \alpha(\beta(x,\ z),\ y) - \beta(x,\ \alpha(z,\ y)) - \beta(\alpha(x,\ y),\ z) \\ &- \{\beta(\alpha(x,\ z),\ y) - \alpha(x,\ \beta(z,\ y)) - \alpha(\beta(x,\ y),\ z)\},\end{aligned}$$

which shows that $\alpha_g - \beta_f$ is not a biderivation, too. Therefore, $\Lambda_B(A)$ is not a Lie subalgebra of $\Lambda(A)$. By these calculations, we have the following.

Theorem 2.6. (1) *For any* $(f,\ \alpha),\ (g,\ \beta) \in \Lambda_{SyB}(A) = \Lambda_{SyB}(A,\ A)$*, assume that* α *and* β *satisfy the following condition:*

$$\begin{aligned}&\alpha(x,\ \beta(y,\ z)) + \alpha(y,\ \beta(z,\ x)) + \alpha(z,\ \beta(x,\ y)) \\ &= \beta(x,\ \alpha(y,\ z)) + \beta(y,\ \alpha(z,\ x)) + \beta(z,\ \alpha(x,\ y)).\end{aligned}$$

Then $[(f,\ \alpha),\ (g,\ \beta)] \in \Lambda_{SyB}(A)$.

(2) *For any* $(f,\ \alpha),\ (g,\ \beta) \in \Lambda_{sSyB}(A) = \Lambda_{sSyB}(A,\ A)$*, assume that* α *and* β *satisfy the following condition:*

$$\begin{aligned}&\alpha(x,\ \beta(y,\ z)) - \alpha(y,\ \beta(z,\ x)) + \alpha(z,\ \beta(x,\ y)) \\ &= \beta(x,\ \alpha(y,\ z)) - \beta(y,\ \alpha(z,\ x)) + \beta(z,\ \alpha(x,\ y)).\end{aligned}$$

Then $[(f,\ \alpha),\ (g,\ \beta)] \in \Lambda_{sSyB}(A)$.

Proof. (1) Let $(f,\ \alpha)$ and $(g,\ \beta)$ be in $\Lambda_{SyB}(A)$. Since $[(f,\ \alpha),\ (g,\ \beta)] = ([f,\ g],\ \alpha_g - \beta_f)$, it is enough to show that $\alpha_g - \beta_f$ is a symmetric biderivation.

By the above calculations and using that α and β are symmetric, we have

$$\begin{aligned}&(\alpha_g-\beta_f)(xz,\ y)-x\{(\alpha_g-\beta_f)(z,\ y)\}-\{(\alpha_g-\beta_f)(x,\ y)\}z\\&=\{\alpha(\beta(x,\ z),\ y)-\beta(x,\ \alpha(z,\ y))-\beta(\alpha(x,\ y),\ z)\}\\&-\{\beta(\alpha(x,\ z),\ y)-\alpha(x,\ \beta(z,\ y))-\alpha(\beta(x,\ y),\ z)\}\\&=\alpha(x,\ \beta(y,\ z))+\alpha(y,\ \beta(z,\ x))+\alpha(z,\ \beta(x,\ y))\\&-\beta(x,\ \alpha(y,\ z))-\beta(y,\ \alpha(z,\ x))-\beta(z,\ \alpha(x,\ y))=0\end{aligned}$$

by our assumption. Moreover, α and β are symmetric, then α_g and β_f are also symmetric by (2.3), and thus

$$\begin{aligned}(\alpha_g-\beta_f)(x,\ zy)&=(\alpha_g-\beta_f)(zy,\ x)=z(\alpha_g-\beta_f)(y,\ x)+(\alpha_g-\beta_f)(z,\ x)y\\&=z(\alpha_g-\beta_f)(x,\ y)+(\alpha_g-\beta_f)(x,\ z)y,\end{aligned}$$

which shows that $(\alpha_g-\beta_f)$ is a biderivation.

(2) For any $(f,\ \alpha)$, $(g,\ \beta)\in\Lambda_{sSyB}(A)$, since α_g and β_f are skew symmetric, it is similarly proved. □

We can easily check the following. For a commutative algebra A and derivations $f,\,g:A\to A$, if $fg=gf$, then $\alpha=f\cdot f$ and $\beta=g\cdot g$ are biderivations which satisfy the assumption (2.8). And for any $c_1,\,c_2\in C(A)$, the center of a noncommutative algebra A, then $\alpha=c_1[-,\ -]$ and $\beta=c_2[-,\ -]$ are biderivations which satisfy the assumption (2.9). We have not more good examples of biderivations which satisfy the assumption in Theorem 2.6.

3. The case of coalgebras

Let C be a coassociative R-coalgebra with a comultiplication $\Delta:C\to C\otimes C$. We do not assume that C has a counit $\varepsilon:C\to R$. Let N be a C-bicomodule with a right C-comodule structure map $\rho^+:N\to N\otimes C$ and a left C-comodule structure map $\rho^-:N\to C\otimes N$ which satisfy the relation (1.0).

The notions of a coderivation, an N-coderivation, a B-coderivation and a triple coderivation are defined by (1.1), (1.3), (1.2) and (1.4), respectively. The set of triple coderivations

$$\mathrm{TCoder}(N,\ C)=\{(f,\ f_1,\ f_2)\in\mathrm{Hom}(N,\ C)^3\mid\Delta f=(f_1\otimes I)\rho^++(I\otimes f_2)\rho^-\}$$

is an R-module with R-submodules $\mathrm{Coder}(N,\ C)$, $\mathrm{NCoder}(N,\ C)$ and $\mathrm{BCoder}(N,\ C)$, where $I:C\to C$ is the identity map. First, we show that $\mathrm{NCoder}(N,\ C)$ is an R-submodule of $\mathrm{BCoder}(N,\ C)$.

Lemma 3.1. *If* $(f,\ f,\ f+(\xi\otimes I)\rho^+)$ *is an N-coderivation, then* $f+(\xi\otimes I)\rho^+$ *is a coderivation for any* $\xi\in Hom(N,\ R)$*. Therefore,* $NCoder(N,\ C)$ *is contained in* $BCoder(N,\ C)$*. Especially, if* C *and* N *have counits, then*

$$NCoder(N,\ C)=BCoder(N,\ C)=TCoder(N,\ C).$$

Proof. Since $(f,\ f,\ f+(\xi\otimes I)\rho^+)$ is an N-coderivation, then by (1.3), we see

$$\begin{aligned}\Delta f&=(f\otimes I)\rho^+ +(I\otimes f)\rho^- +(I\otimes\xi\otimes I)(I\otimes\rho^+)\rho^-\\&=(f\otimes I)\rho^+ +\{I\otimes(f+(\xi\otimes I)\rho^+)\}\rho^-.\end{aligned}$$

Noting that $\Delta(\xi\otimes I)=\xi\otimes\Delta:N\otimes C\to R\otimes C\to C\otimes C$, we have by (1.0),

$$\begin{aligned}\Delta(\xi\otimes I)\rho^+&=(\xi\otimes\Delta)\rho^+=(\xi\otimes I\otimes I)(I\otimes\Delta)\rho^+=(\xi\otimes I\otimes I)(\rho^+\otimes I)\rho^+\\&=((\xi\otimes I)\rho^+\otimes I)\rho^+,\end{aligned}$$

and thus

$$\begin{aligned}\Delta(f+(\xi\otimes I)\rho^+)&=\Delta f+\Delta(\xi\otimes I)\rho^+\\&=(f\otimes I)\rho^+ +\{I\otimes(f+(\xi\otimes I)\rho^+)\}\rho^- +\Delta(\xi\otimes I)\rho^+\\&=\{f\otimes I+\Delta(\xi\otimes I)\}\rho^+ +\{I\otimes(f+(\xi\otimes I)\rho^+)\}\rho^-\\&=\{(f+(\xi\otimes I)\rho^+)\otimes I\}\rho^+ +\{I\otimes(f+(\xi\otimes I)\rho^+)\}\rho^-,\end{aligned}$$

which shows that $f+(\xi\otimes I)\rho^+$ is a coderivation. Therefore NCoder$(N,\ C)\subseteq$ BCoder$(N,\ C)$.

Assume that C has a counit $\varepsilon:C\to R$. It is enough to show that a triple coderivation $(f,\ f_1,\ f_2)$ is an N-coderivation. By

$$(I\otimes\varepsilon)\Delta f=(f_1\otimes\varepsilon)\rho^+ +(I\otimes\varepsilon f_2)\rho^-,$$

we see $f=f_1+(I\otimes\varepsilon f_2)\rho^-$, which shows that $f_1=f-(I\otimes\varepsilon f_2)\rho^-$. Similarly, $f_2=f-(\varepsilon f_1\otimes I)\rho^+)$. Therefore,

$$\Delta f=(f\otimes I)\rho^+ +(I\otimes f)\rho^- +(I\otimes\varepsilon(-f_1-f_2)\otimes I)(I\otimes\rho^+)\rho^-.$$

This means that a triple coderivation $(f,\ f_1,\ f_2)$ is an N-coderivation. □

Now, define

$$\Omega(N,\ C)=\{(f,\ \alpha)\mid f\in\mathrm{Hom}(N,\ C),\ \alpha\in\mathrm{Hom}(N,\ C\otimes C)\}.$$

Then $\Omega(N,\ C)$ becomes an R-module as the direct product of Hom(N,C) with Hom$(N,C\otimes C)$ and for any $(f,\ f_1,\ f_2),(g,\ g_1,\ g_2)\in$ TCoder$(C)=$ TCoder$(C,\ C)$, we have by (1.4)

$$\Delta[f,\ g]=([f_1,\ g_1]\otimes I+I\otimes[f_2,\ g_2])\Delta.$$

This shows that the operation

$$[(f,\ f_1,\ f_2),\ (g,\ g_1,\ g_2)] = ([f,\ g],\ [f_1,\ g_1],\ [f_2,\ g_2])$$

gives a Lie algebra structure on TCoder(C), which contains a Lie subalgebra BCoder(C) = BCoder(C, C). Since the several properties of an R-module TCoder(N, C) and a Lie algebra TCoder(C) are discussed in [9], we consider the relations of TCoder(N, C) and $\Omega(N,\ C)$.

Lemma 3.2. *For any* $(f,\ f_1,\ f_2) \in TCoder(N,\ C)$*, define a map* $\alpha\langle f\rangle$ *by*

$$\alpha\langle f\rangle\ : N \ni n \mapsto\ (\varDelta f - (f\otimes I)\rho^+ - (I\otimes f)\rho^-)(n) \in C\otimes C. \tag{3.1}$$

Then the map

$$\psi_N : TCoder(N,\ C) \ni (f,\ f_1,\ f_2) \mapsto (f,\ \alpha\langle f\rangle) \in \Omega(N,\ C) \tag{3.2}$$

is an R-module homomorphism with

$$Ker\ \psi_N = \{(0,\ f_1,\ f_2) \in TCoder(N,\ C) \mid (f_1\otimes I)\rho^+ + (I\otimes f_2)\rho^- = 0\}.$$

Especially, if C *and* N *have counits, then* ψ_N *is a monomorphism on* $TCoder(N,\ C)$.

Proof. For any $(f,\ f_1,\ f_2)$, $(g,\ g_1,\ g_2) \in$ TCoder(N, C) and $r \in R$, it is clear that $\alpha\langle f\rangle + \alpha\langle g\rangle = \alpha\langle f+g\rangle$ and $r\alpha\langle f\rangle = \alpha\langle rf\rangle$ by (3.1). This shows that ψ_M is an R-module homomorphism and by $\varDelta f = (f_1\otimes I)\rho^+ + (I\otimes f_2)\rho^-$,

$$\text{Ker}\ \psi_N = \{(0,\ f_1,\ f_2) \in \text{TCoder}(N,\ C) \mid (f_1\otimes I)\rho^+ + (I\otimes f_2)\rho^- = 0\}$$

is clear. By Lemma 3.1, it is enough to show that ψ_M is a monomorphism on BCoder(N, C). If $(f,\ f,\ d)$ is a B-coderivation, then by $\psi_M(f,\ f,\ d) = (f,\ \alpha(f)) = 0$, we have $f = 0$ and $\alpha\langle f\rangle = (I\otimes d)\rho^- = 0$. Therefore, $(\varepsilon\otimes I)(I\otimes d)\rho^- = d = 0$, which shows that ψ_M is a monomorphism. □

For any $f \in \text{Hom}(C,\ C)$, $\alpha \in \text{Hom}(C,\ C\otimes C)$, we define

$$\alpha^f = (f\otimes I + I\otimes f)\alpha - \alpha f : C \to C\otimes C. \tag{3.3}$$

Then we have the following which corresponds to Lemma 2.2.

Lemma 3.3. *For any* $f,\ g \in Hom(C,\ C)$*,* $\alpha,\ \beta \in Hom(C,\ C\otimes C)$ *and* $r \in R$*, the following relations hold:*

(1) $\alpha^{(rf)} = (r\alpha)^f = r(\alpha^f)$, (2) $(\alpha+\beta)^f = \alpha^f + \beta^f$, (3) $\alpha^{(f+g)} = \alpha^f + \alpha^g$,

(4) $(\alpha^f)^g - (\alpha^g)^f = \alpha^{[g,\ f]}$.

Proof. By definition (3.3), it is enough to show (4). Since

$$\begin{aligned}(\alpha^f)^g &= (g\otimes I + I\otimes g)\alpha^f - (\alpha^f)g\\ &= (gf\otimes I + g\otimes f + f\otimes g + I\otimes gf)\alpha - (g\otimes I + I\otimes g)\alpha f\\ &- (f\otimes I + I\otimes f)\alpha g + \alpha f g,\end{aligned}$$

we see

$$(\alpha^f)^g - (\alpha^g)^f = ([g,\ f]\otimes I + I\otimes [g,\ f])\alpha - \alpha[g,\ f] = \alpha^{[g,\ f]}. \qquad \square$$

Now, we define an R-submodule $\Lambda(N,\ C)$ of $\Omega(N,\ C)$ by

$$\Lambda(N,\ C) = \{(f,\ \alpha)\in\Omega(N,\ C) \mid \varDelta f = (f\otimes I)\rho^+ + (I\otimes f)\rho^- + \alpha\}.$$

Then by Lemmas 3.2 and 3.3, we have the following which corresponds to Theorem 2.3.

Theorem 3.4. *For any $(f,\ \alpha)$, $(g,\ \beta)\in\Omega(C)$, we define*

$$[(f,\ \alpha),\ (g,\ \beta)] = ([f,\ g],\ \beta^f - \alpha^g). \tag{3.4}$$

Then $\Omega(C)$ is a Lie algebra with a Lie subalgebra $\Lambda(C)$, and the map

$$\psi_C : TCoder(C) \ni (f,\ f_1,\ f_2) \mapsto (f,\ \alpha\langle f\rangle) \in \Lambda(C) \subset \Omega(C)$$

defined by (3.2) is a Lie algebra homomorphism.

Proof. Let $u = (f,\ \alpha)$, $v = (g,\ \beta)$ and $w = (h,\ \gamma)$ be in $\Omega(C)$. By Lemma 3.3 and (3.4), $[u,\ v] + [v,\ u] = 0$ and $[u+v,\ w] = [u,\ w] + [v,\ w]$ are clear. Since

$$\begin{aligned}[(f,\ \alpha),\ [(g,\ \beta),\ (h,\ \gamma)]] &= [(f,\ \alpha),\ ([g,\ h],\ \gamma^g - \beta^h)]\\ &= ([f,\ [g,\ h]], (\gamma^g - \beta^h)^f - \alpha^{[g,\ h]}),\end{aligned}$$

then by Lemma 3.3, we have

$$\begin{aligned}&(\gamma^g - \beta^h)^f - \alpha^{[g,\ h]} + (\alpha^h - \gamma^f)^g - \beta^{[h,\ f]} + (\beta^f - \alpha^g)^h - \gamma^{[f,\ g]}\\ &= (\gamma^g)^f - (\gamma^f)^g - \gamma^{[f,\ g]} + (\beta^f)^h - (\beta^h)^f - \beta^{[h,\ f]} + (\alpha^h)^g - (\alpha^g)^h - \alpha^{[g,\ h]} = 0.\end{aligned}$$

This shows that $[u,\ [v,\ w]] + [v,\ [w,\ u]] + [w,\ [u,\ v]] = 0$. Therefore $\Omega(C)$ is a Lie algebra. Especially, if $(f,\ \alpha)$, $(g,\ \beta)\in\Lambda(C)$, then by (3.4), we see that

$[(f,\ \alpha),\ (g,\ \beta)] = ([f,\ g],\ \beta^f - \alpha^g)$ and by (3.3),

$$\begin{aligned}\Delta([f,\ g]) &= (\Delta f)g - (\Delta g)f \\ &= \{(f\otimes I + I\otimes f)\Delta + \alpha\}g - \{(g\otimes I + I\otimes g)\Delta + \beta\}f \\ &= ([f,\ g]\otimes I + I\otimes [f,\ g])\Delta + (f\otimes I + I\otimes f)\beta + \alpha g - (g\otimes I + I\otimes g)\alpha - \beta f \\ &= ([f,\ g]\otimes 1 + I\otimes [f,\ g])\Delta + \beta^f - \alpha^g.\end{aligned}$$

Thus $\Lambda(C)$ is a Lie subalgebra of $\Omega(C)$.

Next, we show that ψ_C is a Lie algebra homomorphism. Let $(f,\ f_1,\ f_2)$ and $(g,\ g_1,\ g_2)$ be in $\mathrm{TCoder}(C)$. Then

$$\psi_C([(f,\ f_1,\ f_2),\ (g,\ g_1,\ g_2)]) = \psi_C([f,\ g],\ [f_1,\ g_1],\ [f_2,\ g_2]) = ([f,\ g],\ \alpha\langle [f,\ g]\rangle)$$

and by

$$[\psi_C(f,\ f_1,\ f_2),\ \psi_C(g,\ g_1,\ g_2)] = [(f,\ \alpha\langle f\rangle),\ (g,\ \alpha\langle g\rangle)] = ([f,\ g],\ (\alpha\langle g\rangle)^f - (\alpha\langle f\rangle)^g),$$

it is enough to show that $\alpha\langle [f,\ g]\rangle = (\alpha\langle g\rangle)^f - (\alpha\langle f\rangle)^g$. Since $(f,\ f_1,\ f_2)$ and $(g,\ g_1,\ g_2)$ are triple coderivations, then by (3.1) and (3.3), we see

$$\begin{aligned}(\alpha\langle g\rangle)^f &= (f\otimes I + I\otimes f)\alpha\langle g\rangle - (\alpha\langle g\rangle)f \\ &= (f\otimes I + I\otimes f)\{(g_1 - g)\otimes I + I\otimes (g_2 - g)\}\Delta \\ &\quad - \{(g_1 - g)\otimes I + I\otimes (g_2 - g)\}(f_1\otimes I + I\otimes f_2)\Delta\end{aligned}$$

and symmetrically,

$$\begin{aligned}(\alpha\langle f\rangle)^g &= (g\otimes I + I\otimes g)\alpha\langle f\rangle - (\alpha\langle f\rangle)g \\ &= (g\otimes I + I\otimes g)\{(f_1 - f)\otimes I + I\otimes (f_2 - f)\}\Delta \\ &\quad - \{(f_1 - f)\otimes I + I\otimes (f_2 - f)\}(g_1\otimes I + I\otimes g_2)\Delta.\end{aligned}$$

Then we have

$$\begin{aligned}(\alpha\langle g\rangle)^f - (\alpha\langle f\rangle)^g &= \{([f_1,\ g_1] - [f,\ g])\otimes I + I\otimes ([f_2,\ g_2] - [f,\ g])\}\Delta \\ &= \alpha\langle [f,\ g]\rangle,\end{aligned}$$

which shows that ψ_C is a Lie algebra homomorphism. □

In [13], we showed that if $f : N \to C$ is an N-coderivation such that

$$\Delta f = (f\otimes I)\rho^+ + (I\otimes f)\rho^- + (I\otimes \xi\otimes I)(I\otimes \rho^+)\rho^-, \qquad \xi \in \mathrm{Hom}(N,\ C)$$

then $f^* : C^* = \mathrm{Hom}(C,\ R) \to N^* = \mathrm{Hom}(N,\ R)$ defined by $(f^*(c_1^*))(n) = c_1^*(f(n))$ is an N-derivation, where $c_1^* \in C^*$, $n \in N$ (cf. [15, Section 4]). We finally give some relations for coderivations and their dual derivations, which is essentially obtained [8, Section 6].

Theorem 3.5. *If* $(f,\ f_1,\ f_2) \in TCoder(N,\ C)$*, then* $(f^*,\ f_1^*,\ f_2^*) \in TDer(C^*,\ N^*)$*. Especially, if* $(f,\ f,\ d)$ *is a B-coderivation, then* $(f^*,\ f^*,\ d^*)$ *is a B-derivation.*

Proof. Let $c_1^*,\ c_2^* \in C^*$, $c \in C$ and $n \in N$. The algebra structure $\circ$ of C^* is given by $(c_1^* \circ c_2^*)(c) = \sum (c_1^* \otimes c_2^*)\Delta(c)$, and C^*-bimodule structure of N^* are defined by

$$(n_1^* c_1^*)(n) = (n_1^* \otimes c_1^*)\rho^+(n) \quad \text{and} \quad (c_1^* n_1^*)(n) = (c_1^* \otimes n_1^*)\rho^-(n), \tag{3.5}$$

where $n_1^* \in N^*$. If $(f,\ f_1,\ f_2) \in \mathrm{TCoder}(N,\ C)$, then by $\Delta f = (f_1 \otimes I)\rho^+ + (I \otimes f_2)\rho^-$, we have

$$\begin{aligned} f^*(c_1^* \circ c_2^*) &= (c_1^* \otimes c_2^*)\Delta f = (c_1^* \otimes c_2^*)\{(f_1 \otimes I)\rho^+ + (I \otimes f_2)\rho^-\} \\ &= (f_1^*(c_1^*) \otimes c_2^*)\rho^+ + (c_1^* \otimes f_2^*(c_2^*))\rho^- = f_1^*(c_1^*) \circ c_2^* + c_1^* \circ f_2^*(c_2^*). \end{aligned}$$

Therefore $(f^*,\ f_1^*,\ f_2^*)$ is a triple derivation from C^* to N^*. □

Now, as is easily seen, the map

$$\theta_0 : \mathrm{TCoder}(N,\ C) \ni (f,\ f_1,\ f_2) \mapsto (f^*,\ f_1^*,\ f_2^*) \in \mathrm{TDer}(C^*,\ N^*)$$

is an R-module homomorphism and if $N = C$, then by

$$\begin{aligned} \theta_0[(f,\ f_1,\ f_2),\ (g,\ g_1,\ g_2)] &= \theta_0([f,\ g],\ [f_1,\ g_1],\ [f_2,\ g_2]) \\ &= ([f,\ g]^*,\ [f_1,\ g_1]^*,\ [f_2,\ g_2]^*) \\ &= ([g^*,\ f^*],\ [g_1^*,\ f_1^*],\ [g_2^*,\ f_2^*]) \\ &= [\theta_0((g,\ g_1,\ g_2)),\ \theta_0((f,\ f_1,\ f_2))], \end{aligned}$$

$\theta_0 : \mathrm{TCoder}(C) \to \mathrm{TDer}(C^*)$ is a Lie algebra anti-homomorphism. And for any $(f,\ \alpha) \in \Lambda(N,\ C)$, we have a bilinear map

$$\alpha^* : C^* \times C^* \ni (c_1^*,\ c_2^*) \to \alpha^*(c_1^*,\ c_2^*) \in N^*$$

defined by $\alpha^*(c_1^*,\ c_2^*)(n) = (c_1^* \otimes c_2^*)\alpha(n)$. Since $\Delta f = (f_1 \otimes I)\rho^+ + (I \otimes f_2)\rho^- + \alpha$, we have

$$\begin{aligned} f^*(c_1^*)c_2^* + c_1^* f^*(c_2^*) + \alpha^*(c_1^*,\ c_2^*) &= (c_1^* f \otimes c_2^*)\rho^+ + c_1^* \otimes (c_2^* f)\rho^- + (c_1^* \otimes c_2^*)\alpha \\ &= (c_1^* \otimes c_2^*)\{(f \otimes I)\rho^+ + (I \otimes f)\rho^- + \alpha\} \\ &= (c_1^* \otimes c_2^*)\Delta f = f^*(c_1^* \circ c_2^*), \end{aligned}$$

and thus we can define an R-module homomorphism

$$\theta_1 : \Lambda(N,\ C) \ni (f,\ \alpha) \mapsto (f^*,\ \alpha^*) \in \Lambda(C^*,\ N^*).$$

Then we have the following.

Theorem 3.6. *The diagram of R-modules*

$$\begin{array}{ccc} TCoder(N,\ C) & \xrightarrow{\psi_N} & \Lambda(N,\ C) \\ \big\downarrow \theta_0 & & \big\downarrow \theta_1 \\ TDer(C^*,\ N^*) & \xrightarrow{\varphi_{N^*}} & \Lambda(C^*,\ N^*) \end{array}$$

is commutative. Especially, if $N = C$, then the above diagram is commutative and θ_i $(i = 0,\ 1)$ is a Lie algebra anti-homomorphism.

Proof. It is enough to show that the diagram is commutative, and θ_1 is a Lie algebra anti-homomorphism in case of $N = C$.

Let $(f,\ f_1,\ f_2) \in \mathrm{TCoder}(N,\ C)$, $c_1^*,\ c_2^* \in C^*$ and $n \in N$. By Theorem 3.5, since $(f^*,\ f_1^*,\ f_2^*) \in \mathrm{TDer}(C^*,\ N^*)$ is a triple derivation, then by Lemma 2.1(2.1), Lemma 3.2(3.1) and the definition of $\Lambda(C^*,\ N^*)$, we have

$$\begin{aligned} \alpha(f^*)(c_1^*,\ c_2^*)(n) &= \{(f_1^* - f^*)(c_1^*)c_2^* + c_1^*(f_2^* - f^*)(c_2^*)\}(n) \\ &= \{(f_1 - f)^*(c_1^*)c_2^* + c_1^*(f_2 - f)^*(c_2^*)\}(n) \\ &= \{(c_1^*(f_1 - f) \otimes c_2^*)\rho^+ + (c_1^* \otimes c_2^*(f_2 - f)\rho^-\}(n) \\ &= (c_1^* \otimes c_2^*)\{((f_1 - f) \otimes I)\rho^+ + (I \otimes (f_2 - f))\rho^-\}(n) \\ &= (\alpha\langle f\rangle)^*(c_1^*,\ c_2^*)(n). \end{aligned}$$

This shows that

$$\begin{aligned} &\theta_1\psi_N(f,\ f_1,\ f_2) = \theta_1(f,\ \alpha\langle f\rangle) = (f^*,\ (\alpha\langle f\rangle)^*) = (f^*,\ \alpha(f^*)) \\ &= \varphi_{N^*}\theta_0(f,\ f_1,\ f_1), \end{aligned}$$

that is, the above diagram is commutative.

Next, assume that $N = C$. Then for any $(f,\ \alpha)$, $(g,\ \beta) \in \Lambda(C)$, we have

$$\begin{aligned} (\beta^f)^* &= ((f \otimes I + I \otimes f)\beta - \beta f)^* = \beta^*(f^* \otimes I^* + I^* \otimes f^*) - f^*\beta^*, \\ (\beta^*)_{f^*} &= \beta^*(f^* \times I_{C^*} + I_{C^*} \times f^*) - f^*\beta^*. \end{aligned}$$

Since the algebra structure of C^* is defined by $(c_1^* \circ c_2^*) = (c_1^* \otimes c_2^*)\Delta$, we see $(\beta^f)^* = (\beta^*)_{f^*}$ and thus

$$\begin{aligned}\theta_1([(f,\ \alpha),\ (g,\ \beta)]) &= \theta_1([f,\ g],\ \beta^f - \alpha^g) = ([g^*,\ f^*],\ (\beta^f)^* - (\alpha^g)^*),\\ &= ([g^*,\ f^*],\ (\beta^*)_{f^*} - (\alpha^*)_{g^*}) = [(g^*,\ \beta^*),\ (f^*,\ \alpha^*)]\\ &= [\theta_1((g,\ \beta)),\ \theta_1((f,\ \alpha))].\end{aligned}$$

Therefore θ_1 is a Lie algebra anti-homomorphism. □

Finally, we show that the notion of a biderivation is dualized as follows.

Theorem 3.7. *Let $B_c : N \to C \otimes C$ be an R-bilinear map which satisfy the following relations:*

$$(1)\quad (\Delta \otimes I)B_c = (I \otimes B_c)\rho^- + (I \otimes t)(B_c \otimes I)\rho^+,$$

$$(2)\quad (I \otimes \Delta)B_c = (B_c \otimes I)\rho^+ + (t \otimes I)(I \otimes B_c)\rho^-,$$

where $t : C \otimes C \ni x \otimes y \mapsto y \otimes x \in C \otimes C$. Then the map $B_C^* : C^* \otimes C^* \to N^*$ defined by

$$B_C^*(c_2^*,\ c_3^*)(n) = (c_2^* \otimes c_3^*)B_c(n) \quad \textit{for any} \quad c_2^*,\ c_3^* \in C^*,\ n \in N$$

is a biderivation.

Proof. Since C^*-bimodule structures of N^* are given by (3.5), we have

$$\begin{aligned}c_1^* B_c^*(c_2^*,\ c_3^*) &= c_1^* \otimes B_c^*(c_2^*,\ c_3^*))\rho^- = (c_1^* \otimes c_2^* \otimes c_3^*)(I \otimes B_c)\rho^-,\\ B_c^*(c_1^*,\ c_3^*)c_2^* &= (B_c^*(c_1^*,\ c_2^*) \otimes c_3^*))\rho^+ = (c_1^* \otimes c_3^* \otimes c_2^*)(B_c \otimes I)\rho^+\\ &= (c_1^* \otimes c_2^* \otimes c_3^*)(I \otimes t)(B_c \otimes I)\rho^+\end{aligned}$$

for any $c_1^* \in C^*$. Then by (1)

$$\begin{aligned}&c_1^* B_c^*(c_2^*,\ c_3^*) + B_c^*(c_1^*,\ c_3^*)c_2^* = (c_1^* \otimes c_2^* \otimes c_3^*)\{I \otimes B_q)\rho^- + (I \otimes t)(B_c \otimes I)\rho^+)\\ &= (c_1^* \otimes c_2^* \otimes c_3^*)(\Delta \otimes I)B_c = ((c_1^* \circ c_2^*) \otimes c_3^*)B_c = B_c^*(c_1^* \circ c_2^*,\ c_3^*).\end{aligned}$$

Similarly by (2), we have

$$B_c^*(c_1^*,\ c_2^* \circ c_3^*) = c_2^* B_c^*(c_1^*,\ c_3) + B_c^*(c_1^*,\ c_2)^* c_3^*.$$

These show that $B_C^* : C^* \otimes C^* \to N^*$ is a biderivation. □

For an A-bimodule M, by the above theorem, the map $B_c : N \to C \otimes C$ corresponds to the notion of a biderivation $B : A \times A \to M$.

We can define some other coderivations which correspond to the G-derivations and the right derivations as follows. Let G_c be the set of R-coalgebra endomorphisms of C. For a C-bicomodule N, an R-linear map $f : N \to C$ is called a G_c-*coderivation* if

$$\Delta f = (f \otimes \sigma)\rho^{+} + (\tau \otimes f)\rho^{-}$$

for some $\sigma,\ \tau \in G_c$. Similarly, $g : N \to C$ is a *right coderivation* if

$$\Delta g = (g \otimes I)\rho^{+} + (g \otimes I)t\rho^{-},$$

where $t : C \otimes N \ni x \otimes m \mapsto m \otimes x \in N \otimes C$ is the twisted map. Since σ^* and τ^* are R-algebra endomorphisms of C^*, then we have

$$\begin{aligned} f^*(c_1^* \circ c_2^*) &= (c_1^* \otimes c_2^*)\Delta f = (c_1^* \otimes c_2^*)\{(f \otimes \sigma)\rho^{+} + (\tau \otimes f)\rho^{-}\} \\ &= c_1^* f^* \otimes c_2^* \sigma + c_1^* \tau \otimes c_2^* f^* = f^*(c_1^*)\sigma^*(c_2^*) + \tau^*(c_1^*) f^*(c_2^*), \end{aligned}$$

which show that $f^* : C^* \to N^*$ is a $(\sigma^*,\ \tau^*)$-derivation and by

$$\begin{aligned} g^*(c_1^* \circ c_2^*) &= (c_1^* \otimes c_2^*)\Delta g = (c_1^* \otimes c_2^*)\{(g \otimes I)\rho^{+} + (g \otimes I)t\rho^{-}\} \\ &= c_1^* f^* \otimes c_2^* \sigma + c_1^* \tau \otimes c_2^* f^* = g^*(c_1^*)c_2^* + g^*(c_2^*)c_1^*, \end{aligned}$$

$g^* : C^* \to M^*$ is a right derivation. Therefore, the above G_c-coderivation and the right coderivation are considered as the dual notions of the G-derivation and the right derivation. We will be able to discuss the several properties of $(\sigma,\ \tau)$-coderivations and right coderivations which correspond to the properties of $\mathrm{Der}_{\sigma^*,\ \tau^*}(A,\ M)$ and $\mathrm{RDer}(C^*,\ M^*)$.

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