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RESEARCH PAPER

Analysis of a model to control the co-dynamics of Chlamydia and Gonorrhea using Caputo fractional derivative

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Abstract

This paper investigates a fractional derivative model of Chlamydia-Gonorrhea co-infection using Caputo derivative definition. The positivity boundedness of the model is established using Laplace transform. Additionally, we investigated the existence and uniqueness of the model using methods established by some fixed point theorems. We concluded that the model is Ulam-Hyers-Rassias stable. Furthermore, we obtained plots of the model at different fractional derivative orders, which show the significant role played by the fractional order on various classes of the model as it varies. We observe distinct results for each class in different orders, highlighting the importance of considering the fractional order in modeling Chlamydia-Gonorrhea co-infection. Moreover, the fractional model presented in this paper can be used to study the dynamics of Chlamydia-Gonorrhea co-infection in a more accurate and realistic way compared to traditional integer-order models.

Keywords: Chlamydia; Gonorrhea; fractional derivative; co-infection; control

AMS 2020 Classification: 26A33; 34A08; 34D20; 34D23; 92B05

1 Introduction

Sexually Transmitted Diseases (STD), such as chlamydia, are a major public health concern in the United States. Despite being asymptomatic in most cases, chlamydia is one of the most frequently reported bacterial infections [1]. The number of reported cases of chlamydia remains high, with up

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to four million new infections estimated to have occurred in 2018, and youth between the ages of 15 and 24 accounting for two-thirds of these cases. In fact, it is believed that one in twenty sexually active young women between the ages of 14 and 24 have chlamydia [1]. Untreated chlamydia infections can lead to serious health consequences, such as Pelvic Inflammatory Disease (PID), which can cause complications such as blocked Fallopian tubes, ectopic pregnancy, and longer pelvic and abdominal pain. Although the number of reported cases of chlamydia may have decreased in 2020 due to COVID-19 restrictions, such as clinic closures and reduced testing, it is important to continue researching this disease to gain further insights on how to reduce its burden on public health.

According to reports, gonorrhea is the second most common sexually transmitted disease in the United States, with young people being the majority of those infected [2]. This STD causes infections in the genitals, throat, and rectum, similar to chlamydia, and is particularly prevalent among people aged 15-24 years. Gonorrhea commonly affects the cervix, uterus, and Fallopian tubes in women, and the urethra in both genders. It can also spread from mother to child during pregnancy. The Centers for Disease Control and Prevention (CDC) recommend a single intramuscular dose of 500mg of ceftriaxone for treatment of gonorrhea, but there have been cases of resistance to some antibiotics. This highlights the importance of continued research and vigilance in addressing the growing concern of antibiotic-resistant gonorrhea.

Co-infection of both gonorrhea and chlamydia trachomatis is not uncommon, and understanding the implications of this co-infection is necessary to manage such situations effectively. According to some reports [3], individuals with this co-infection are more likely to develop pelvic inflammatory disease (PID), which is a severe complication that can result from untreated STDs. Therefore, it is important to study the dynamics of this co-infection and gain insights into how to manage it. Furthermore, with increased drug resistance from some of these STDs, vaccination of individuals may be a more effective solution. This will be investigated in this work to enable an informed decision on how to tackle these diseases in the future.

The use of mathematical models in infectious disease research has proven to be valuable, but it is crucial to use models that capture the memory effect. Fractional calculus has shown promise in developing such models, and more research is needed to improve the accuracy of disease predictions and control measures, hence this is our motivation in carrying out this research. Some integer models have also been developed to understand the dynamics of specific infections, such as Chlamydia trachomatis, Gonorrhea, or their co-infections and other diseases can be seen in [4–17]. The work by Odionyenma et al. [4] presented an SVEIRT epidemiological model, which centered on investigating the role of a vaccination class in the general dynamics of the model. Optimal control analysis of the model was also carried out, showing that the most cost-effective strategy in dealing with the transmission dynamics of the model. A study in [17] looked at a co-infection model of Chlamydia and Gonorrhea, with target interest on the effect of treatment for each disease on the co-infection on the population. It showed that implementing female Chlamydia treatment and male Chlamydia treatment resulted in a significant decrease in the total number of females and males co-infected with Chlamydia and Gonorrhea.

The use of non-integer order derivatives in modeling contagious illnesses has gained increasing attention from scholars and analysts. Traditional epidemiological models can only be designated via a fixed order, which is not applicable to fractional order derivatives. One study by Omame et al. [18] investigated a new mathematical model for co-infection of COVID-19 and Hepatitis B virus using the Atangana-Baleanu fractional derivative. The authors solved the model analytically using the Laplace-Adomian decomposition method and discussed the stability of the iterative scheme to approximate the solution. The numerical analysis showed that prevention and control measures for either COVID-19 or Hepatitis B could significantly reduce the burden of co-infection.

The dynamics of tuberculosis model using the Caputo-Fabrizio fractional derivative was studied by Ullah in [19]. The study utilized data from reported cases of TB in the national TB program Khyber Pakhtunkhwa, Pakistan from 2002 to 2017. The model was used to derive the reproduction number R_0, and other relevant variables. The Adams-Bashforth method was used to compute the solution of the model iteratively. The study concluded that the fractional model provided helpful information on TB and a better way to view the spread of the disease. There have been so many studies on the use of fractional derivatives in modeling infectious diseases as can be seen in [20–36].

Models that have utilized fractional derivatives in non-disease modeling can be seen in [37–40]. A fractional-order derivative chaotic system described by Caputo derivative was studied in the work by [37]. The effect of the fractional-order derivative was carried out, the stability analysis was utilized to determine the chaotic region where the order of the Caputo derivative presented in the system, and the nature of the chaos was established using the Lyapunov's exponents in the fractional context. Also, the work by [38] analyzes a model describing the production of mobile phone worms. This study explores the behaviors of the forced Korteweg–De Vries (KdV) equation, which describes flowing over a hole. By utilizing the q-homotopy analysis transform technique (q-HATT), the study finds solutions using a combination of the q-homotopy analysis scheme and the Laplace transform. The study employs fractional operators to generalize models associated with various characteristics. It establishes the existence, uniqueness, and convergence of the models using a fixed-point theorem. The results demonstrate the reliability and systematic nature of the solution procedure for investigating both integer and fractional-order nonlinear models.

The paper is organized as follows: Section 1 presents the introduction and model formulation. Section 2 covers the basic theory of Caputo fractional derivative, including the existence and uniqueness of solutions, the basic reproduction number, stability analysis of fractional order systems, and global stability of the disease-free equilibrium.

Sections 3 and 4 provide details on the numerical simulations and their interpretations, with plots displayed. Section 5 concludes the work with some recommendations for future research. The model considered in this paper is a modified version of an existing integer order model considered in [17]. We have modified it and also applied fractional calculus to analyze the model. This has not been considered before.

Model description

At the time t, the population is represented by $N_H(t)$ and is divided into seven compartments. These compartments are susceptible individuals who are unvaccinated (S_H), those who are vaccinated (V_{CL}), those infected with Chlamydia (I_{CL}), those treated for Chlamydia (T_{CL}), those infected with Gonorrhea (I_G), those treated for Gonorrhea (T_G), and those infected with both Chlamydia and Gonorrhea (I_{GCL}). The unvaccinated susceptible group, S_H, is increased by the recruitment rate Λ_H. Individuals within this group can contract Chlamydia and/or Gonorrhea from infected individuals at rates β_{CL} and β_G, respectively, with acquisition rates of λ_{CL} and λ_G. Furthermore, the parameter η_L and η_G account for the increased infectiousness of individuals who are infected with both Chlamydia and Gonorrhea, where other parameters are defined in Table 1. Additionally, individuals who receive treatment move to the T_{CL} or T_G compartments, depending on their infection status. Vaccinated individuals move from the V_{CL} compartment to the S_H compartment at a rate proportional to their vaccine efficacy. The total population at time t can be expressed as $N_H = S_H + V_{CL} + I_{CL} + T_{CL} + I_G + T_G + I_{GCL}$.

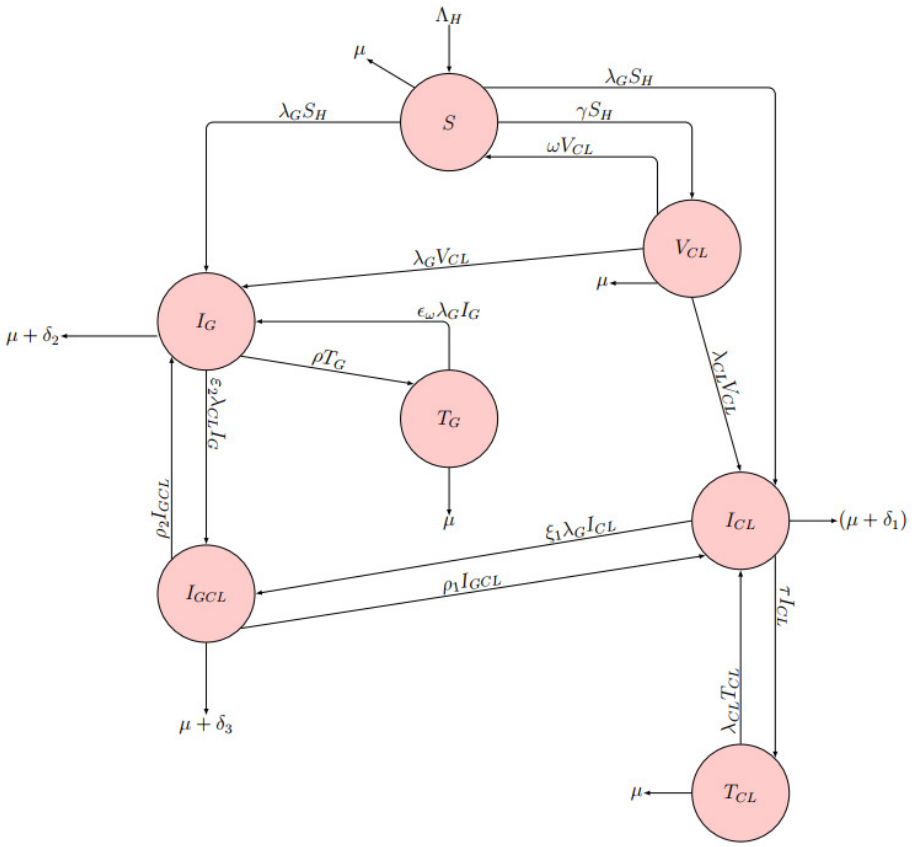

Figure 1. Schematic diagram of model (1).

Table 1. Parameters and description

Parameter	Description
ϵ_W,ϵ_L	Reinfection rates for Chlamydia and Gonorrhea
μ	Natural death rate
$\rho_1(\rho_2)$	Rate at which dually infected gets treated of Gonorrhea(Chlamydia)
ρ	Treatment rate for gonorrhea
ω	Waning of vaccine rate
γ	Vaccination rate
τ	Treatment rate for chlamydia
δ_1	Disease-induced death rate for chlamydia infected individuals
δ_2	Disease-induced death rate for gonorrhea infected individuals
δ_3	Disease-induced death rate for co-infected individuals
Λ_H	Recruitment rate
β_{CL},β_G	Contact rate for Chlamydia and Gonorrhea infected individuals
ξ_1,ξ_2	Rate at which singly infected individuals becomes dually infected with Chlamydia and Gonorrhea
η_L,η_g	Modification parameter accounting for increased infectiousness of individuals dually infected with chlamydia and gonorrhea

The Chlamydia-Gonorrhea co-infection model is given by the following system of fractional order differential equations based on the aforementioned formulations and assumptions:

$$
\begin{aligned}
D_t^{\zeta} S_H(t) &= \Lambda_H - (\lambda_G + \lambda_{CL}) S_H + \omega V_{CL} - (\mu + \gamma) S_H, \\
D_t^{\zeta} V_{CL}(t) &= \gamma S_H - (\mu + \omega) V_{CL} - V_{CL}(\lambda_G + \lambda_{CL}), \\
D_t^{\zeta} I_{CL}(t) &= \lambda_{CL} S_H + \epsilon_L \lambda_{CL} T_{CL} - (\mu + \tau + \delta_1) I_{CL} - \xi_1 \lambda_G I_{CL} + \rho_1 I_{GCL} + V_{CL} \lambda_{CL}, \\
D_t^{\zeta} T_{CL}(t) &= \tau I_{CL} - (\mu + \epsilon_L \lambda_{CL}) T_{CL}, \\
D_t^{\zeta} I_G(t) &= \lambda_G S_H - (\mu + \delta_2 + \rho) I_G + \rho_2 I_{GCL} + \epsilon_W \lambda_G T_G - \xi_2 \lambda_{CL} I_G + V_{CL} \lambda_G, \\
D_t^{\zeta} T_G(t) &= \rho I_G - (\mu + \epsilon_W \lambda_G) T_G, \\
D_t^{\zeta} I_{GCL}(t) &= \xi_1 \lambda_G I_{CL} + \xi_2 \lambda_{CL} I_G - (\mu + \delta_3 + \rho_1 + \rho_2) I_{GCL},
\end{aligned}
\tag{1}
$$

$$
\begin{aligned}
\lambda_{CL} &= \frac{\beta_{CL}(I_{CL} + \eta_L I_{GCL})}{N_H}, \\
\lambda_G &= \frac{\beta_G(I_G + \eta_g I_{GCL})}{N_H},
\end{aligned}
\tag{2}
$$

with the corresponding initial conditions

$$
S_H(0) \geq 0,\ V_{CL}(0) \geq 0,\ I_{CL}(0) \geq 0,\ T_{CL}(0) \geq 0,\ T_G(0) \geq 0\ , I_G(0) \geq 0,\ I_{GCL}(0) \geq 0\ . \tag{3}
$$

2 Preliminaries and basic properties of the model

Caputo fractional derivative has been widely used in modeling disease dynamics due to its ability to capture the memory effect and long-range dependence in the system. This section provides some basic definitions related to fractional calculus, with a focus on Caputo fractional derivative. Specifically, we introduce the Riemann-Liouville fractional integral of order $\zeta > 0$, which can be obtained by replacing n with ζ in the integral formula. We also discuss the Ulam-Hyers-Rassias stability of fractional order systems and derive the basic reproduction number for the co-infection model. Furthermore, we establish the existence and uniqueness of the model's solution, which is essential for numerical simulation and analysis.

Definition 1 *[41] The Caputo fractional derivative of order $\zeta > 0$ of a function $f(t)$ of order $\zeta \in \mathbb{R}^+$ is defined by*

$$
D_t^{\zeta} f(t) = J_t^{n-\zeta} D^n f(t) = \frac{1}{\Gamma(n-\zeta)} \int_0^t (t-\tau)^{n-\zeta-1} f^{(n)}(\tau) d\tau,
$$

where n is an integer whose definition is $n-1 < \zeta \leq n$. Actually, where $0 < \zeta \leq 1$, from the derivative above, where $\zeta > 0$ gives

$$
D_t^{\zeta} f(t) \quad = \quad \frac{1}{\Gamma(1-\zeta)} \int_0^t (t-\tau)^{-\zeta} f'(\tau) d\tau. \tag{4}
$$

Definition 2 *[34] The fractional integral of order $\zeta > 0$ of a function $f \in C^1(0,T)$ is given by*

$$J_t^{\zeta} f(t) = \frac{1}{\Gamma(\zeta)} \int_0^t (t-\tau)^{\zeta-1} f(\tau) d\tau, \qquad t > 0,$$

if the integral exists in $\mathbb{R}^+$. For convenience, suppose $f(t) = K$, where K is a constant then;

$$J_t^{\zeta}(K) = \frac{1}{\Gamma(\zeta)} \int_0^t (t-\tau)^{\zeta-1}(K) d\tau = K \frac{t^{\zeta}}{\Gamma(\zeta+1)}.$$

Definition 3 *[41] The Laplace transform of Caputo fractional derivative is given by*

$$\mathcal{L}\left\{D_t^{\zeta} f(t)\right\} = s^{\zeta} \tilde{f}(s) - s^{\zeta-1} f(0), \quad 0 < \zeta \leq 1, \tag{5}$$

where $\mathcal{L}$ is the operator of the Laplace transform..

Lemma 1 *[42] Let $\zeta \in \mathbb{R}^+$, $\phi_1(t)$ and $\phi_2(t)$ represent positive functions and $\phi_3(t)$ represent an increasing and positive function for $0 \leq t \leq T$, $T > 0$, $\phi_3(t) \leq M$, where M is a constant. Suppose*

$$\phi_1 \leq \phi_2 + \phi_3(t) \int_0^T (t-\tau)^{\zeta-1} \phi_1(t) d\tau,$$

then

$$\phi_1 \leq \phi_2 E_{\zeta}\left(\phi_3(t) \frac{\pi}{\Gamma(1-\zeta)\sin(\pi\zeta)} T^{\zeta}\right).$$

Theorem 1 *Suppose $S_H(t)$, $V_{CL}(t)$, $I_{CL}(t)$, $T_{CL}(t)$, $T_G(t)$, $I_G(t)$, $I_{GCL}(t)$ are any solution of the system (1)- (3), then the set*

$$\begin{aligned} \Delta &= \left\{(S_H(t), V_{CL}(t), I_{CL}(t), T_{CL}(t), T_G(t), I_G(t), I_{GCL}(t)) \in \mathbb{R}_+^7 : S_H \right. \\ &+ \left. V_{CL} + I_{CL} + T_{CL} + T_G + I_G + I_{GCL} \leq \frac{\Lambda_H}{\mu}\right\}, \end{aligned} \tag{6}$$

is positively invariant.

Proof:. When all of the equations in (1) are added, we have

$$\begin{aligned} D_t^{\zeta} N_H(t) &= D_t^{\zeta} S_H(t) + D_t^{\zeta} V_{CL}(t) + D_t^{\zeta} I_{CL}(t) + D_t^{\zeta} T_{CL}(t) + D_t^{\zeta} T_G(t) + D_t^{\zeta} I_G + D_t^{\zeta} I_{GCL} \\ &= \Lambda_H - \mu\left(S_H + V_{CL} + I_{CL} + T_{CL} + T_G + I_G + I_{GCL}\right) \\ &\quad - \left(\gamma S_H + \omega V_{CL} + \delta_1 I_{CL} + \delta_2 I_G + \delta_3 I_{GCL}\right) \\ &\leq \Lambda_H - \mu N_H. \end{aligned}$$

If we apply the Laplace Transform to the above equation, we have

$$s^{\zeta}\tilde{N}_H(s) - s^{\zeta-1}N_H(0) \leq \frac{\Lambda_H}{s} - \mu\tilde{N}_H(s),$$

from which

$$\tilde{N}_H(s) \leq \frac{\Lambda_H}{s(s^{\zeta}+\mu)} + N_H(0)\frac{s^{\zeta-1}}{s^{\zeta}+\mu}.$$

The expression above gives the following result after partial fraction decomposition

$$\tilde{N}_H(s) \leq \frac{\Lambda_H}{\mu}\left(\frac{1}{s}\right) - \left(\frac{\Lambda_H}{\mu} - N_H(0)\right)\sum_{k=0}^{\infty}\frac{(-\mu)^k}{s^{\zeta k+1}}.$$

The inverse Laplace transform gives

$$N_H(t) \leq \frac{\Lambda_H}{\mu} - \left(\frac{\Lambda_H}{\mu} - N_H(0)\right)E_{\zeta}\left(-\mu t^{\zeta}\right),$$

as $t \to \infty$, we have

$$N_H \leq \frac{\Lambda_H}{\mu}, \tag{7}$$

giving the condition for Eqs. (1)-(3) to be bounded and mathematically posed within the region.

Positivity of solution of the model

Using the approach by [34], and assuming that I_{CL} class is not positively invariant. Let $t_1 = \min\{t : S_H(t), V_{CL}(t), I_{CL}(t), T_{CL}(t), I_G(t), T_G(t), I_{GCL}\}$. Suppose $I_{CL}(t_1) = 0$, it gives that $S_H(t) > 0$, $V_{CL}(t) > 0$, $T_{CL}(t) > 0$, $I_G(t) > 0$, $I_{GCL}(t) > 0$ for all $[0, t_1]$. If the following expression exists,

$$\theta_1 = \min_{0\leq t\leq t_1}\left\{\frac{(\lambda_{CL}S_H + \epsilon_L\lambda_{CL}T_{CL} + \rho_1 I_{GCL} + V_{CL}\lambda_{CL})}{I_{CL}} - (\mu + \tau + \delta_1 + \xi_1\lambda_G)\right\}.$$

It will result in that

$$D_t^{\zeta}I_{CL}(t) - \theta_1 I_{CL}(t) > 0. \tag{8}$$

With Ω_1, a continuous function, we can say that the following equation is ascertained

$$D_t^{\zeta}I_{CL}(t) - \theta_1 I_{CL}(t) = -\Omega_1(t).$$

with Laplace transform applied to the inequality, it gives

$$s^{\zeta}\tilde{I}_{CL}(s) - s^{\zeta-1}I_{CL}(0) - \theta_1\tilde{I}_{CL}(s) = -\tilde{\Omega}_1(s),$$

from which

$$\begin{aligned}\tilde{I}_{CL}(s) &= I_{CL}(0)\frac{s^{\zeta-1}}{s^{\zeta}-\theta_1}-\frac{\Omega_1(s)}{s^{\zeta}-\theta_1}=\frac{I_{CL}(0)}{s}\left(1-\frac{\theta_1}{s^{\zeta}}\right)^{-1}-\frac{\Omega_1(s)}{s^{\zeta}}\left(1-\frac{\theta_1}{s^{\zeta}}\right)^{-1}\\ &= I_s(0)\sum_{k=0}^{\infty}\frac{\theta_1^k}{s^{\zeta k+1}}-\Omega_1(s)\sum_{k=0}^{\infty}\frac{\theta_1^k}{s^{\zeta k+\zeta}}.\end{aligned}$$

The Mittag-Leffler function and the inverse Laplace transform yield the solution to Eq. (8) satisfying the following expression;

$$I_{CL}(t) > I_{CL}(0)\sum_{k=0}^{\infty}\frac{(\theta_1 t^{\zeta})^k}{\Gamma(\zeta k+1)}=I_{CL}(0)E_{\zeta}\left(\theta_1 t^{\zeta}\right).$$

Then the positivity of I_{CL} is given by

$$I_{CL}(t) > I_{CL}(0)E_{\zeta}\left(\theta_1 t^{\zeta}\right) > 0,$$

which contradicts $I_{CL}(t_1) = 0$ Similarly, suppose $T_{CL}(t_1) = 0$ which implies that $S_H(t) > 0$, $I_{CL}(t) > 0$, $V_{CL}(t) > 0$, $I_G(t) > 0$, $T_G(t) > 0$, $I_{GCL}(t) > 0$, $\forall\ 0 \le t \le t_1$. Making the following assumption

$$\theta_2 = \min_{0\le t\le t_1}\left\{\frac{\tau I_{CL}}{T_{CL}}-(\mu+\epsilon_L\lambda_{CL})\right\},$$

then

$$D_t^{\zeta}T_{CL}(t) > \theta_2 T_{CL}(t). \tag{9}$$

With Ω_2, a continuous function, the following equation is ascertained

$$D_t^{\zeta}T_{CL}(t)-\theta_2 T_{CL}(t) = -\Omega_2(t).$$

Applying Laplace transform to the above inequality we get

$$s^{\zeta}\tilde{T}_{CL}(s)-s^{\zeta-1}T_{CL}(0)-\theta_2\tilde{T}_{CL}(s)=-\tilde{\Omega}_2(s),$$

from which

$$\tilde{T}_{CL}(s) = T_{CL}(0)\frac{s^{\zeta-1}}{s^{\zeta}-\theta_2}-\frac{\Omega_2(s)}{s^{\zeta}-\theta_2}=T_{CL}(0)\sum_{k=0}^{\infty}\frac{\theta_2^k}{s^{\zeta k+1}}-\Omega_2(s)\sum_{k=0}^{\infty}\frac{\theta_2^k}{s^{\zeta k+\zeta}}.$$

The solution of Eq. (9) is provided by utilizing the Mittag-Leffler function and the inverse Laplace transform, satisfying the following expression

$$T_{CL}(t) \quad > \quad T_{CL}(0)\sum_{k=0}^{\infty}\frac{(\theta_2 t^{\zeta})^k}{\Gamma(\zeta k+1)} = T_{CL}(0)E_{\zeta}\left(\theta_2 t^{\zeta}\right).$$

which gives the positivity of solution of T_{CL} as

$$T_{CL}(t) > T_{CL}(0)E_{\zeta}\left(\theta_2 t^{\zeta}\right) > 0,$$

which contradicts $T_{CL}(t_1) = 0$. If we follow the same method above, and assume $T_G(t_1) = 0$ which implies that $S_H(t) > 0$, $V_{CL}(t) > 0$, $I_{CL}(t) > 0$, T_{CL}, $I_G(t) > 0$, $I_{GCL}(t) > 0$, for all $0 \leq t \leq t_1$. Assuming that the following expression exist

$$\theta_3 \quad = \quad \min_{0\leq t\leq t_1}\left\{\frac{\rho I_G}{T_G} - (\mu + \epsilon_w \lambda_G)\right\},$$

so that

$$D_t^{\zeta} T_G(t) \quad > \quad \theta_3 T_G(t). \tag{10}$$

With Ω_3, a continuous function, we can say that the following equation is ascertained

$$D_t^{\zeta} T_G(t) - \theta_3 T_G(t) \quad = \quad -\Omega_3(t).$$

With Laplace transform applied to the inequality, it gives;

$$s^{\zeta}\tilde{T}_G(s) - s^{\zeta-1}T_G(0) - \theta_3\tilde{T}_G(s) = -\tilde{\Omega}_3(s),$$

which gives the following;

$$\tilde{T}_G(s) \quad = \quad T_G(0)\sum_{k=0}^{\infty}\frac{\theta_3^k}{s^{\zeta k+1}} - \Omega_3(s)\sum_{k=0}^{\infty}\frac{\theta_3^k}{s^{\zeta k+\zeta}}.$$

The solution of Eq. (10) is provided by utilizing the Mittag-Leffler function and the inverse Laplace transform. satisfying the following expression

$$T_G(t) \quad > \quad T_G(0)\sum_{k=0}^{\infty}\frac{(\theta_3 t^{\zeta})^k}{\Gamma(\zeta k+1)} = I_c(0)E_{\zeta}\left(\theta_3 t^{\zeta}\right). \tag{11}$$

Hence the positivity of the solution T_G is shown as $T_G(t) > T_G(0)E_{\zeta}\left(\theta_3 t^{\zeta}\right) > 0$, which contradicts $T_G(t_1) = 0$. Again, we suppose $I_{GCL}(t_1) = 0$ which implies that $S_H(t) > 0$, $V_{CL}(t) > 0$, $I_{CL}(t) > 0$, $T_{CL} > 0$, $I_G(t) > 0$, $T_G > 0$ for all $0 \leq t \leq t_1$. Assuming the expression below exists;

$$\theta_4 \quad = \quad \min_{0\leq t\leq t_1}\left\{\frac{(\xi_1\lambda_G I_{CL} + \xi_2\lambda_{CL}T_G)}{I_{GCL}} - (\mu + \delta_3 + \rho_1 + \rho_2)\right\},$$

such that

$$D_t^{\zeta} I_{GCL}(t) \quad > \quad \theta_4 I_{GCL}(t). \tag{12}$$

Also, Ω_4 can be gotten, such that the expression below is ascertained

$$D_t^{\zeta} I_{GCL}(t) - \theta_4 I_{GCL}(t) \quad = \quad -\Omega_4(t).$$

If Laplace transform is applied to the above inequality, we have;

$$s^{\zeta} \tilde{I}_{GCL}(s) - s^{\zeta-1} I_{GCL}(0) - \theta_4 \tilde{I}_{GCL}(s) = -\tilde{\Omega}_p(s),$$

from which

$$\tilde{I}_{GCL}(s) \quad = \quad I_{GCL}(0) \sum_{k=0}^{\infty} \frac{\theta_4^k}{s^{\zeta k+1}} - \Omega_4(s) \sum_{k=0}^{\infty} \frac{\theta_4^k}{s^{\zeta k+\zeta}}.$$

The following expressions are satisfied by the solution of (12) when the negative term is ignored when using the Mittag-Leffler function and the inverse Laplace transform.

$$I_{GCL}(t) \quad > \quad I_{GCL}(0) \sum_{k=0}^{\infty} \frac{(\theta_p t^{\zeta})^k}{\Gamma(\zeta k+1)} = I_{GCL}(0) E_{\zeta}\left(\theta_4 t^{\zeta}\right).$$

and the positivity of solution of I_{GCL}, is shown as;

$$I_{GCL}(t) > I_{GCL}(0) E_{\zeta}\left(\theta_4 t^{\zeta}\right) > 0,$$

which contradicts $I_{GCL}(t_1) = 0$. Using the same method shows that the positivity of the solutions S_H, V_{CL}, and I_G respectively are given by

$$S_H(t) > S_H(0) E_{\zeta}\left(\theta_5 t^{\zeta}\right) > 0, \; V_{CL}(t) > V_{CL}(0) E_{\zeta}\left(\theta_6 t^{\zeta}\right) > 0,$$

$$I_G(t) > I_G(0) E_{\zeta}\left(\theta_7 t^{\zeta}\right) > 0.$$

Existence and uniqueness of the solution of the model

In this section, we will demonstrate the existence and uniqueness of the solution to the fractional model (1). To achieve this, we adopt a similar approach to the one used in [32], where the Banach fixed point theorem was used. Additionally, we will apply Schaefer's fixed point theorem to establish the existence of the solution and demonstrate its boundedness. The fractional integral will be applied to the Caputo fractional derivative model (1) of order $\zeta > 0$, along with its respective initial conditions (3). This process will yield Volterra-integral equations of the second

kind, which will serve as the solution to the fractional model. Given that F, G, H, K, Q, U, and V are the right side of the various classes of (1) respectively.

$$\begin{aligned}
S_H(t) - S_H(0) &= \frac{1}{\Gamma(\zeta)} \int_0^t (t-\tau)^{\zeta-1} F\left(t, S_H(t)\right) d\tau, \\
V_{CL}(t) - V_{CL}(0) &= \frac{1}{\Gamma(\zeta)} \int_0^t (t-\tau)^{\zeta-1} G\left(t, V_{CL}(t)\right) d\tau, \\
I_{CL}(t) - I_{CL}(0) &= \frac{1}{\Gamma(\zeta)} \int_0^t (t-\tau)^{\zeta-1} H\left(t, I_{CL}()\right) d\tau, \\
T_{CL}(t) - T_{CL}(0) &= \frac{1}{\Gamma(\zeta)} \int_0^t (t-\tau)^{\zeta-1} K\left(t, T_{CL}(t)\right) d\tau, \\
I_G(t) - I_G(0) &= \frac{1}{\Gamma(\zeta)} \int_0^t (t-\tau)^{\zeta-1} Q\left(t, I_G(t)\right) d\tau, \\
T_G(t) - T_G(0) &= \frac{1}{\Gamma(\zeta)} \int_0^t (t-\tau)^{\zeta-1} U\left(t, T_G(t)\right) d\tau, \\
I_{GCL}(t) - I_{GCL}(0) &= \frac{1}{\Gamma(\zeta)} \int_0^t (t-\tau)^{\zeta-1} V\left(t, I_{GCL}(t)\right) d\tau.
\end{aligned} \tag{13}$$

The functions $(F, G, H, K, Q, U, V) : [0, T] \times \mathbb{D} \to \mathbb{D}$ are assumed to be continuous such that $(\mathbb{D}, \|.\|)$ is the Banach space and $\mathbb{H}^1([0, T])$ is the Banach space of all the continuous function defined in $[0, T] \to \mathbb{D}$ shaped with Chebyshev norm.

The continuous functions F, G, H, K, Q, U and V satisfy the Lipschitz condition if

$$\sup_{0<t\leq T} \left\| \frac{S_H}{N_H} \right\| \leq \Theta_1, \quad \sup_{0<t\leq T} \left\| \frac{V_{CL}}{N_H} \right\| \leq \Theta_2 \quad , \sup_{0<t\leq T} \left\| \frac{I_{CL}}{N_H} \right\| \leq \Theta_3, \quad \sup_{0<t\leq T} \left\| \frac{T_{CL}}{N_H} \right\| \leq \Theta_4,$$

$$\sup_{0<t\leq T} \left\| \frac{I_G}{N_H} \right\| \leq \Theta_5, \quad \sup_{0<t\leq T} \left\| \frac{T_G}{N_H} \right\| \leq \Theta_6, \quad \sup_{0<t\leq T} \left\| \frac{I_{GCL}}{N_H} \right\| \leq \Theta_7.$$

Thus, firstly we have

$$\begin{aligned}
\|F(S_{H1}) - F(S_{H2})\| &= \left\| \Lambda_H + \omega V_{CL} - \left(\frac{\beta_{CL}\left(I_{CL} + \eta_C I_{GCL}\right)}{N_H} + \frac{\beta_G\left(I_G + \eta_g I_{GCL}\right)}{N_H} + \mu + \gamma \right) S_{H1} \right. \\
&- \left. \left(\Lambda_H + \omega V_{CL} - \left(\frac{\beta_{CL}\left(I_{CL} + \eta_C I_{GCL}\right)}{N_H} + \frac{\beta_G\left(I_G + \eta_g I_{GCL}\right)}{N_H} + \mu + \gamma \right) S_{H2} \right) \right\| \\
&= \left\| -\frac{\beta_{CL} I_{CL}}{N_H}\left(S_{H1} - S_{H2}\right) - \frac{\beta_{CL}\eta_C I_{GCL}}{N_H}\left(S_{H1} - S_{H2}\right) - \frac{\beta_G I_G}{N_H}\left(S_{H1} - S_{H2}\right) \right. \\
&- \left. \frac{\beta_G \eta_g I_{GCL}}{N_H}\left(S_{H1} - S_{H2}\right) - \gamma\left(S_{H1} - S_{H2}\right) - \mu\left(S_{H1} - S_{H2}\right) \right\| \\
&\leq \beta_{CL} \sup_{0\leq t\leq T} \left\| \frac{I_{CL}}{N_H} \right\| \|S_{H1} - S_{H2}\| + \beta_{CL}\eta_g \sup_{0\leq t\leq T} \left\| \frac{I_{GCL}}{N_H} \right\| \|S_{H1} - S_{H2}\| \\
&+ \gamma \|S_{H1} - S_{H2}\| + \mu \|S_{H1} - S_{H2}\| \\
&\leq L_F \|S_{H1} - S_{H2}\|,
\end{aligned} \tag{14}$$

where

$$L_F = \left(\beta_{CL}\Theta_3 + \beta_{CL}\eta_g\Theta_7 + \gamma + \mu\right) > 0.$$

Secondly,

$$\begin{aligned}\|G(V_{CL1}) - G(V_2)\| &= \|\gamma S_H - (\mu+\omega+\lambda_G+\lambda_{CL})V_{CL1} \\ &- (\gamma S_H - (\mu+\omega+\lambda_G+\lambda_{CL})V_{CL2})\| \\ &= \|\left(\mu+\omega+\lambda_G+\lambda_{CL}\right)\left(V_{CL1}-V_{CL2}\right)\| \\ &\leq L_G\left\|V_{CL1}-V_{CL2}\right\|,\end{aligned}$$

where

$$L_G = (\mu+\omega+\lambda_G+\lambda_{CL}) > 0.$$

With the same method, we arrive at the following

$$\begin{aligned}\|H(I_{CL1}) - H(I_{CL2})\| &= \left\|\left(\frac{\beta_{CL}(I_{CL1}+\eta_C I_{GCL})}{N_H}\right)S_H + \rho_1 I_{GCL} + \epsilon_L T_{CL}\left(\frac{\beta_{CL}(I_{CL1}+\eta_C I_{GCL})}{N_H}\right)\right. \\ &- \left(\frac{\xi_1\beta_G(T_G+\eta_g I_{GCL})}{N_H}\right)I_{CL1} + \left(\frac{\beta_{CL}(I_{CL1}+\eta_C I_{GCL})}{N_H}\right)V_{CL} \\ &- \left(\frac{\beta_{CL}(I_{CL2}+\eta_C I_{GCL})}{N_H}\right)S_H - \rho_1 I_{GL} - \epsilon_L T_{CL}\left(\frac{\beta_{CL}(I_{CL2}+\eta_C I_{GCL})}{N_H}\right) \\ &+ \left(\frac{\xi_1\beta_G(T_G+\eta_g I_{GCL})}{N_H}\right)I_{CL2} - \left(\frac{\beta_{CL}(I_{CL1}+\eta_C I_{GCL})}{N_H}\right)V_{CL} \\ &+ (\mu+\tau+\delta_1)I_{CL2} - (\mu+\tau+\delta_1)I_{CL1}\| \\ &\leq \beta_{CL}\sup_{0\leq t\leq T}\left\|\frac{S_H}{N_H}\right\|\|I_{CL1}-I_{CL2}\| \qquad (15) \\ &+ \beta_{CL}\epsilon_L\eta_g\sup_{0\leq t\leq T}\left\|\frac{T_{CL}}{N_H}\right\|\|I_{CL1}-I_{CL2}\| + \beta_{CL}\sup_{0\leq t\leq T}\left\|\frac{V_{CL}}{N_H}\right\|\|I_{CL1}-I_{CL2}\| \\ &+ \xi_1\beta_G\sup_{0\leq t\leq T}\left\|\frac{T_G}{N_H}\right\|\|I_{CL1}-I_{CL2}\| + \mu\|I_{CL1}-I_{CL2}\| \\ &+ \tau\|I_{CL1}-I_{CL2}\| + \delta_1\|I_{CL1}-I_{CL2}\| \\ &\leq L_H\|I_{CL1}-I_{CL2}\|,\end{aligned}$$

where $L_H = (\beta_{CL}\Theta_1 + \beta_{CL}\Theta_3 + \beta_G\epsilon_L\Theta_4 + \xi_1\beta_G\Theta_6 + \mu + \tau + \delta_1) > 0.$

$$\begin{aligned}\|K(T_{CL1}) - K(T_{CL2})\| &= \left\|\tau I_{CL} - \mu T_{CL1} - \frac{\epsilon_L\beta_{CL}(I_{CL}+\eta_C I_{GCL})}{N_H}T_{CL1}\right. \\ &- \left.\tau I_{CL} + \mu T_{CL2} + \frac{\epsilon_L\beta_{CL}(I_{CL}+\eta_C I_{GCL})}{N_H}T_{CL2}\right\| \qquad (16) \\ &\leq \epsilon_L\beta_{CL}\sup_{0\leq t\leq T}\left\|\frac{I_{CL}}{N_H}\right\|\|T_{CL1}-T_{CL2}\| + \epsilon_L\beta_{CL}\eta_C\sup_{0\leq t\leq T}\left\|\frac{I_{GCL}}{N_H}\right\|\|T_{CL1}-T_{CL2}\| \\ &+ \mu\|T_{CL1}-T_{CL2}\| \\ &\leq L_K\|T_{CL1}-T_{CL2}\|,\end{aligned}$$

where $L_K = (\epsilon_L \beta_{CL} \Theta_3 + \epsilon_L \beta_{CL} \Theta_7 + \mu) > 0$.

$$\begin{aligned}
\|Q(I_{G1}) - Q(I_{G2})\| &= \left\| \frac{\beta_G \left(I_{G1} + \eta_g I_{GCL}\right) S_H}{N_H} + \frac{\epsilon_W \beta_G \left(I_{G1} + \eta_g I_{GCL}\right) T_G}{N_H} - (\rho + \mu + \delta_2) I_{G1} \right. \\
&- \frac{\xi_2 \beta_{CL} \left(I_{CL} + \eta_C I_{GCL}\right)}{N_H} + \rho_2 I_{GCL} + \frac{\beta_G \left(I_{G1} + \eta_g I_{GCL}\right) V_{CL}}{N_H} \\
&+ \frac{\beta_G \left(I_{G2} - \eta_g I_{GCL}\right) S_H}{N_H} - \frac{\epsilon_W \beta_G \left(I_{G2} - \eta_g I_{GCL}\right) T_G}{N_H} + (\rho + \mu + \delta_2) I_{G2} \\
&+ \left. \frac{\xi_2 \beta_{CL} \left(I_{CL} - \eta_C I_{GCL}\right)}{N_H} - \rho_2 I_{GCL} - \frac{\beta_G \left(I_{G1} + \eta_g I_{GCL}\right) V_{CL}}{N_H} \right\| \\
&\leq \beta_G \sup_{0 \leq t \leq T} \left\| \frac{S_H}{N_H} \right\| \|I_{G1} - I_{G2}\| + \epsilon_W \beta_G \sup_{0 \leq t \leq T} \left\| \frac{T_G}{N_H} \right\| \|I_{G1} - I_{G2}\| \\
&+ \mu \|I_{G1} - I_{G2}\| + \rho \|I_{G1} - I_{G2}\| + \delta_2 \|I_{G1} - I_{G2}\| + \beta_G \sup_{0 \leq t \leq T} \left\| \frac{V_{CL}}{N_H} \right\| \|I_{G1} - I_{G2}\| \\
&\leq L_Q \|I_{G1} - I_{G2}\|,
\end{aligned}$$

where $L_Q = (\beta_G \Theta_1 + \beta_G \Theta_2 + \epsilon_W \beta_G \Theta_6 + \mu + \rho + \delta_2) > 0$.

$$\begin{aligned}
\|U(T_{G1}) - U(T_{G2})\| &= \left\| \rho I_{G1} - \left(\mu + \frac{\epsilon \beta_G \left(I_G + \eta_g I_{GCL}\right)}{N_H} \right) T_{G1} + \frac{V_{CL} \beta_G \left(I_G + \eta_g I_{GCL}\right)}{N_H} \right. \\
&- \left. \rho I_{G2} + \left(\mu + \frac{\epsilon \beta_G \left(I_G + \eta_g I_{GCL}\right)}{N_H} \right) T_{G2} - \frac{V_{CL} \beta_G \left(I_G + \eta_g I_{GCL}\right)}{N_H} \right\| \\
&\leq \epsilon \beta_G \sup_{0 \leq t \leq T} \left\| \frac{I_G}{N_H} \right\| \|T_{G1} - T_{G2}\| + \epsilon \beta_G \eta_g \sup_{0 \leq t \leq T} \left\| \frac{I_{GCL}}{N_H} \right\| \|T_{G1} - T_{G2}\| \\
&+ \mu \|T_{G1} - T_{G2}\|, \\
&\leq L_U \|T_{G1} - T_{G2}\|,
\end{aligned}$$

where $L_U = \left(\epsilon \beta_G \Theta_4 + \epsilon \beta_G \eta_g \Theta_6 + \mu\right) > 0$.

$$\begin{aligned}
\|V(I_{GCL1}) - V(I_{GCL2})\| &= \left\| \frac{\xi_1 \beta_{CL} \left(I_{CL} + \eta_C I_{GCL1}\right) I_{CL}}{N_H} + \frac{\xi_2 \beta_G \left(I_G + \eta_g I_{GCL1}\right) I_G}{N_H} \right. \\
&- (\mu + \delta_1 + \rho_1 + \rho_2) I_{GCL1} + (\mu + \delta_1 + \rho_1 + \rho_2) I_{GCL2} \\
&- \left. \frac{\xi_1 \beta_{CL} \left(I_{CL} + \eta_C I_{GCL2}\right) I_{CL}}{N_H} - \frac{\xi_2 \beta_G \left(I_G + \eta_g I_{GCL2}\right) I_G}{N_H} \right\| \\
&\leq \xi_1 \beta_{CL} \eta_C \sup_{0 \leq t \leq T} \left\| \frac{I_{CL}}{N_H} \right\| \|I_{GCL1} - I_{GCL2}\| \qquad (17) \\
&+ \xi_2 \beta_G \eta_g \sup_{0 \leq t \leq T} \left\| \frac{I_G}{N_H} \right\| \|I_{GCL1} - I_{GCL2}\| \\
&+ \mu \|I_{GCL1} - I_{GCL2}\| + \delta_1 \|I_{GCL1} - I_{GCL2} \\
&+ \rho_1 \|I_{GCL1} - I_{GCL2} + \rho_2 \|I_{GCL1} - I_{GCL2}\| \\
&\leq L_V \|I_{GCL1} - I_{GCL2}\|,
\end{aligned}$$

where $L_V = \left(\xi_1\beta_{CL}\eta_C\theta_3 + \xi_2\beta_G\eta_g\Theta_5 + \mu_4 + \delta_1 + \rho_1 + \rho_2\right) > 0$.

Theorem 2 *Suppose* $\left(L_F, L_G, L_H, L_K, L_Q, L_U, L_V\right)\frac{\Gamma(1-\zeta)\sin(\pi\zeta)T^\zeta}{\zeta\pi} < 1$, *then the model* (1)-(3) *has a unique solution on* $[0,T]$ *assuming that* $(F,G,H,K,Q,U,V) : [0,T]\times\mathbb{D} \to \mathbb{D}$ *are continuous and satisfies the Lipschitz criteria.*

Proof Considering the mapping $\vartheta : \mathbb{H}^1([0,T],\mathbb{D}) \to \mathbb{H}^1([0,T],\mathbb{D})$, where ϑ is defined in $(F,G,H,K,Q,U,V) : [0,T]\times\mathbb{D} \to \mathbb{D}$. Using (15)-(17) and for all $((S_{H1},S_{H2}),\ (V_{CL1},V_{CL2}),\ (I_{CL1},I_{CL2}),\ (T_{CL1},T_{CL2}),\ (I_{G1},I_{G2}),\ (T_{G1},T_{G2}),\ (I_{GCL1},I_{GCL2}),\ ) \in \mathbb{H}^1([0,T],\mathbb{D})$ and $0 \le t \le T$ we get

$$\begin{aligned}
\|\vartheta(S_{H1}(t)) - \vartheta(S_{H2}(t))\| &= \Bigg\| S_H(0) + \frac{1}{\Gamma(\zeta)}\int_0^T (t-\tau)^{\zeta-1} F\left(t, S_{H1}(t)\right) d\tau - \\
&- \left(S_H(0) + \frac{1}{\Gamma(\zeta)}\int_0^T (t-\tau)^{\zeta-1} F\left(t, S_{H2}(t)\right) d\tau \right) \Bigg\| \\
&\le \frac{1}{\Gamma(\zeta)}\int_0^T (t-\tau)^{\zeta-1} \|F(t,S_{H1}(t)) - F(t,S_{H2}(t))\|\, d\tau \\
&\le \frac{L_F}{\Gamma(\zeta)}\int_0^T (t-\tau)^{\zeta-1} \|S_{H1}(t) - S_{H2}(t)\|\, d\tau \\
&\le L_F\left(\frac{T^\zeta}{\Gamma(\zeta+1)}\right)\|S_{H1} - S_{H2}\|_{\mathbb{H}^1}.
\end{aligned}$$

Similar process yields

$$\begin{aligned}
\|\vartheta(V_{CL1}(t)) - \vartheta(V_{CL2}(t))\| &\le L_G\left(\frac{T^\zeta}{\Gamma(\zeta+1)}\right)\|V_{CL1} - V_{CL2}\|_{\mathbb{H}^1}, \\
\|\vartheta(I_{CL1}(t)) - \vartheta(I_{CL2}(t))\| &\le L_H\left(\frac{T^\zeta}{\Gamma(\zeta+1)}\right)\|I_{CL1} - I_{CL2}\|_{\mathbb{H}^1}, \\
\|\vartheta(T_{CL1}(t)) - \vartheta(T_{CL2}(t))\| &\le L_K\left(\frac{T^\zeta}{\Gamma(\zeta+1)}\right)\|T_{CL1} - T_{CL2}\|_{\mathbb{H}^1}, \\
\|\vartheta(I_{G1}(t)) - \vartheta(I_{G2}(t))\| &\le L_Q\left(\frac{T^\zeta}{\Gamma(\zeta+1)}\right)\|I_{G1} - I_{G2}\|_{\mathbb{H}^1}, \\
\|\vartheta(T_{G1}(t)) - \vartheta(T_{G2}(t))\| &\le L_U\left(\frac{T^\zeta}{\Gamma(\zeta+1)}\right)\|T_{G1} - T_{G2}\|_{\mathbb{H}^1}, \\
\|\vartheta(I_{GCL1}(t)) - \vartheta(I_{GCL2}(t))\| &\le L_V\left(\frac{T^\zeta}{\Gamma(\zeta+1)}\right)\|I_{GCL1} - I_{GCL2}\|_{\mathbb{H}^1}.
\end{aligned} \tag{18}$$

.

It is evident from the condition that $\left(L_F, L_G, L_H, L_K, L_Q, L_U, L_V\right)\frac{\Gamma(1-\zeta)\sin(\pi\zeta)T^\zeta}{\zeta\pi} < 1$. The application of the Banach contraction mapping principle reveals that the parameter ϑ has a distinct fixed point in $0 \le t \le T$. since it is a contraction mapping. ■

Utilizing Schaefer's fixed point theorem, we investigate the fractional model (1)-(3) existence of solutions.

Theorem 3 *Given that* $(F, G, H, K, Q, U, V) : [0, T] \times \mathbb{D} \to \mathbb{D}$ *are continuous and that there exist constants* $(L_{F1}, L_{G1}, L_{H1}, L_{K1}, L_{Q1}, L_{U1}, L_{V1}) > 0$ *such that*

$$\|F(t, S_H)\| \leq L_{F1}(c + \|S_H\|), \ \|G(t, V_{CL})\| \leq L_{G1}(c + \|V_{CL}\|),$$

$$\|H(t, I_{CL})\| \leq L_{H1}(c + \|I_{CL}\|), \ \|K(t, T_{CL})\| \leq L_{K1}(c + \|T_{CL}\|),$$

$$\|Q(t, I_G)\| \leq L_{Q1}(c + \|I_G\|), \ \|U(t, T_G)\| \leq L_{U1}(c + \|T_G\|), \ \|V(t, I_{GCL})\| \leq L_{V1}(c + \|I_{GCL}\|),$$

where $0 < c \leq 1$ *is an arbitrary number, then* (1)-(3) *has at least one solution.*

Proof From (18) we have that the operator ϑ is continuous. Let $\{S_H^{m+1}\}_\infty$, $\{V_{CL}^{m+1}\}_\infty$, $\{I_{CL}^{m+1}\}_\infty$, $\{T_{CL}^{m+1}\}_\infty$, $\{I_G^{m+1}\}_\infty$, $\{T_G^{m+1}\}_\infty$, $\{I_{GCL}^{m+1}\}_\infty$, be sequences such that $S_H^{m+1} \to S_H^m$, $V_{CL}^{m+1} \to V_C L^m$, $I_{CL}^{m+1} \to I_{CL}^m$, $T_{CL}^{m+1} \to T_{CL}^m$, $I_G^{m+1} \to I_G^m$, $T_G^{m+1} \to T_G^m$, $I_{GCL}^{m+1} \to I_{GCL}^m$, in $\mathbb{H}^1([0, T], \mathbb{D})$. For each $0 \leq t \leq T$ we have that

$$\begin{aligned}
\left\|\vartheta S_H^{m+1}(t) - \vartheta S_H^m(t)\right\| &= \frac{1}{\Gamma(\zeta)} \left\| \int_0^t (t-\tau)^{\zeta-1} F\left(t, S_H^{m+1}(t)\right) d\tau - \int_0^t (t-\tau)^{\zeta-1} F\left(t, S_H^m(t)\right) d\tau \right\| \\
&\leq \frac{1}{\Gamma(\zeta)} \int_0^t (t-\tau)^{\zeta-1} \left\| F(t, S_H^{m+1}(t)) - F(t, S_H^m(t)) \right\| d\tau \qquad (19) \\
&\leq \frac{L_{F1} T^\zeta}{\Gamma(\zeta+1)} \left\| S_H^{m+1} - S_H^m \right\|,
\end{aligned}$$

where $\left\|S_H^{m+1} - S_H^m\right\|_{\mathbb{H}} \to 0$ as $m \to \infty$. Using the same methodology yields

$$\begin{aligned}
\left\|\vartheta V_{CL}^{m+1}(t) - \vartheta V_{CL}^m(t)\right\| &\leq L_{G1}\left(\frac{T^\zeta}{\Gamma(\zeta+1)}\right) \left\|V_{CL}^{m+1} - V_{CL}^m\right\|_{\mathbb{H}^1}, \\
\left\|\vartheta I_{CL}^{m+1}(t) - \vartheta I_{CL}^m(t)\right\| &\leq L_{H1}\left(\frac{T^\zeta}{\Gamma(\zeta+1)}\right) \left\|I_{CL}^{m+1} - I_{CL}^m\right\|_{\mathbb{H}^1}, \\
\left\|\vartheta T_{CL}^{m+1}(t) - \vartheta T_{CL}^m(t)\right\| &\leq L_{K1}\left(\frac{T^\zeta}{\Gamma(\zeta+1)}\right) \left\|T_{CL}^{m+1} - T_{CL}^m\right\|_{\mathbb{H}^1}, \\
\left\|\vartheta I_G^{m+1}(t) - \vartheta I_G^m(t)\right\| &\leq L_{Q1}\left(\frac{T^\zeta}{\Gamma(\zeta+1)}\right) \left\|I_G^{m+1} - I_G^m\right\|_{\mathbb{H}^1}, \\
\left\|\vartheta T_G^{m+1}(t) - \vartheta T_G^m(t)\right\| &\leq L_{U1}\left(\frac{T^\zeta}{\Gamma(\zeta+1)}\right) \left\|T_G^{m+1} - T_G^m\right\|_{\mathbb{H}^1}, \\
\left\|\vartheta I_{GCL}^{m+1}(t) - \vartheta I_{GCL}^m(t)\right\| &\leq L_{F1}\left(\frac{T^\zeta}{\Gamma(\zeta+1)}\right) \left\|I_{GCL}^{m+1} - I_{GCL}^m\right\|_{\mathbb{H}^1},
\end{aligned}$$

where $\left\|V_{CL}^{m+1} - V_{CL}^m\right\|_{\mathbb{H}^1} \to 0$, $\left\|I_{CL}^{m+1} - I_{CL}^m\right\|_{\mathbb{H}^1} \to 0$, $\left\|T_{CL}^{m+1} - T_{CL}^m\right\|_{\mathbb{H}^1} \to 0$, $\left\|I_G^{m+1} - I_G^m\right\|_{\mathbb{H}^1} \to 0$, $\left\|T_G^{m+1} - T_G^m\right\|_{\mathbb{H}^1} \to 0$, $\left\|I_{GCL}^{m+1} - I_{GCL}^m\right\|_{\mathbb{H}^1} \to 0$, as $m \to \infty$. Thus the operator ϑ is continuous. ■

Next, we show that the operator ϑ is a one-to-one bounded function on the set of $\mathbb{H}^1([0, T], \mathbb{D})$. For each $S_H \in B_{S_H}$, $V_{CL} \in B_{V_{CL}}$, $I_{CL} \in B_{I_{CL}}$, $T_{CL} \in B_{T_{CL}}$, $I_G \in B_{I_G}$, $T_G \in B_{T_G}$, $I_{GCL} \in B_{I_{GCL}}$

and for $a > 0$, there corresponds a value $b > 0$ where $\|\vartheta S_h\| \leq b$, $\|\vartheta S_H\| \leq b$, $\|\vartheta V_{CL}\| \leq b$, $\|\vartheta I_{CL}\| \leq b$, $\|\vartheta T_{CL}\| \leq b$, $\|\vartheta I_G\| \leq b$, $\|\vartheta T_G\| \leq b$, $\|\vartheta I_{GCL}\| \leq b$, and the subset of Banach space of all continuous functions on the interval $0 \leq t \leq T$ are defined by

$$B_{S_H} = \left\{ S_H \in \mathbb{H}^1([0,T],\mathbb{D}) : \|S_H\| \leq a \right\}, \quad B_{V_{CL}} = \left\{ V_{CL} \in \mathbb{H}^1([0,T],\mathbb{D}) : \|V_{CL}\| \leq a \right\},$$

$$B_{I_{CL}} = \left\{ I_{CL} \in \mathbb{H}^1([0,T],\mathbb{D}) : \|I_{CL}\| \leq a \right\}, \quad B_{T_{CL}} = \left\{ T_{CL} \in \mathbb{H}^1([0,T],\mathbb{D}) : \|T_{CL}\| \leq a \right\},$$

$$B_{I_G} = \left\{ I_G \in \mathbb{H}^1([0,T],\mathbb{D}) : \|I_G\| \leq a \right\}, \quad B_{T_G} = \left\{ T_G \in \mathbb{H}^1([0,T],\mathbb{D}) : \|T_G\| \leq a \right\},$$

$$B_{I_{GCL}} = \left\{ I_{GCL} \in \mathbb{H}^1([0,T],\mathbb{D}) : \|I_{GCL}\| \leq a \right\}.$$

So for any $0 \leq t \leq T$,

$$\begin{aligned}
\|\vartheta S_H\| &\leq \|S_H(0)\| + \frac{1}{\Gamma(\zeta)} \int_0^t (t-\tau)^{\zeta-1} \|F(t, S_H(t))\| \, d\tau \\
&\leq \|S_H(0)\| + \frac{\|F(t, S_H(t))\|}{\Gamma(\zeta)} \int_0^t (t-\tau)^{\zeta-1} d\tau \\
&\leq \|S_H(0)\| + L_{F1} \left(c + \|S_H\|\right) \left(\frac{T^\zeta}{\Gamma(\zeta+1)} \right) \\
&\leq \|S_H(0)\| + L_{F1} \left(c + a\right) \left(\frac{T^\zeta}{\Gamma(\zeta+1)} \right).
\end{aligned}$$

Following a similar approach we have

$$\begin{aligned}
\|\vartheta V_{CL}\| &\leq \|V_{CL}(0)\| + L_{G1} \left(c + a\right) \left(\frac{T^\zeta}{\Gamma(\zeta+1)} \right), \\
\|\vartheta I_{CL}\| &\leq \|I_{CL}(0)\| + L_{H1} \left(c + a\right) \left(\frac{T^\zeta}{\Gamma(\zeta+1)} \right), \\
\|\vartheta T_{CL}\| &\leq \|T_{CL}(0)\| + L_{K1} \left(c + a\right) \left(\frac{T^\zeta}{\Gamma(\zeta+1)} \right), \\
\|\vartheta I_G\| &\leq \|I_G(0)\| + L_{Q1} \left(c + a\right) \left(\frac{T^\zeta}{\Gamma(\zeta+1)} \right), \\
\|\vartheta T_G\| &\leq \|T_G(0)\| + L_{U1} \left(c + a\right) \left(\frac{T^\zeta}{\Gamma(\zeta+1)} \right), \\
\|\vartheta I_{GCL}\| &\leq \|I_{GCL}(0)\| + L_{V1} \left(c + a\right) \left(\frac{T^\zeta}{\Gamma(\zeta+1)} \right).
\end{aligned}$$

On the other hand, let Ω maps bounded set into equal continuous sets in $\mathbb{H}^1([0,T],\mathbb{D})$. If $0 \leq t_1 \leq t_2 \leq T$, $S_H \in B_{S_H}$, $V_{CL} \in B_{V_{CL}}$, $I_{CL} \in B_{I_{CL}}$, $T_{CL} \in B_{T_{CL}}$, $I_G \in B_{I_G}$, $T_G \in B_{T_G}$, $I_{GCL} \in B_{I_{GCL}}$,

where $t_1, t_2 \in [0, T]$, then

$$\begin{aligned}
\|\vartheta S_H(t_1) - \vartheta S_H(t_2)\| &= \frac{1}{\Gamma(\zeta)} \left\| \int_0^{t_1} (t_1-\tau)^{\zeta-1} F(t, S_H(t)) - \int_0^{t_2} (t_2-\tau)^{\zeta-1} F(t, S_H(t)) \right\| d\tau \\
&\leq \frac{1}{\Gamma(\zeta)} \left\| \int_0^{t_1} \left((t_1-\tau)^{\zeta-1} - (t_2-\tau)^{\zeta-1} \right) F(t, S_H(t)) d\tau \right\| \\
&+ \frac{1}{\Gamma(\zeta)} \left\| \int_{t_1}^{t_2} (t_2-\tau)^{\zeta-1} F(t, S_h(t)) d\tau \right\| \\
&\leq \frac{L_{F1}\,(c+a)}{\Gamma(\zeta)} \left\| \int_0^{t_1} \left((t_1-\tau)^{\zeta-1} - (t_2-\tau)^{\zeta-1} \right) d\tau + \int_{t_1}^{t_2} (t_2-\tau)^{\zeta-1} d\tau \right\| \\
&\leq \left(\frac{L_{F1}\,(c+a)\,T^{\zeta}}{\Gamma(\zeta+1)} \right) \left(t_1^{\zeta} - t_2^{\zeta} + 2(t_2-t_1)^{\zeta} \right).
\end{aligned}$$

Similar approach produces

$$\begin{aligned}
\|\vartheta V_{CL}(t_1) - \vartheta V_{CL}(t_2)\| &\leq \left(\frac{L_{G1}\,(c+a)\,T^{\zeta}}{\Gamma(\zeta+1)} \right) \left(t_1^{\zeta} - t_2^{\zeta} + 2(t_2-t_1)^{\zeta} \right), \\
\|\vartheta I_{CL}(t_1) - \vartheta I_{CL}(t_2)\| &\leq \left(\frac{L_{H1}\,(c+a)\,T^{\zeta}}{\Gamma(\zeta+1)} \right) \left(t_1^{\zeta} - t_2^{\zeta} + 2(t_2-t_1)^{\zeta} \right), \\
\|\vartheta T_{CL}(t_1) - \vartheta T_{CL}(t_2)\| &\leq \left(\frac{L_{K1}\,(c+a)\,T^{\zeta}}{\Gamma(\zeta+1)} \right) \left(t_1^{\zeta} - t_2^{\zeta} + 2(t_2-t_1)^{\zeta} \right), \\
\|\vartheta I_{CL}(t_1) - \vartheta I_{CL}(t_2)\| &\leq \left(\frac{L_{Q1}\,(c+a)\,T^{\zeta}}{\Gamma(\zeta+1)} \right) \left(t_1^{\zeta} - t_2^{\zeta} + 2(t_2-t_1)^{\zeta} \right), \\
\|\vartheta T_{G}(t_1) - \vartheta T_{G}(t_2)\| &\leq \left(\frac{L_{U1}\,(c+a)\,T^{\zeta}}{\Gamma(\zeta+1)} \right) \left(t_1^{\zeta} - t_2^{\zeta} + 2(t_2-t_1)^{\zeta} \right), \\
\|\vartheta I_{GCL}(t_1) - \vartheta I_{GCL}(t_2)\| &\leq \left(\frac{L_{V1}\,(c+a)\,T^{\zeta}}{\Gamma(\zeta+1)} \right) \left(t_1^{\zeta} - t_2^{\zeta} + 2(t_2-t_1)^{\zeta} \right).
\end{aligned}$$

As $t_1 \to t_2$ on the right side of the inequality, the expression tends to zero. ϑ is a continuous function, following the Arzela-Ascoli theorem. Now, to show that $R(\vartheta) = \{(S_H, V_{CL}, I_{CL}, T_{CL}, I_G, T_G, I_{GCL}) \in \mathbb{H}^1([0,T], \mathbb{D}) : (S_H, V_{CL}, I_{CL}, T_{CL}, I_G, T_G, I_{GCL}) = \lambda (S_H, V_{CL}, I_{CL}, T_{CL}, I_G, T_G, I_{GCL})\}$ is bounded for some $\lambda \in (0,1)$ by (1). Suppose $(S_H, V_{CL}, I_{CL}, T_{CL}, I_G, T_G, I_{GCL}) \in R(\vartheta)$, such that $(S_H, V_{CL}, I_{CL}, T_{CL}, I_G, T_G, I_{GCL}) = \lambda\vartheta (S_H, V_{CL}, I_{CL}, T_{CL}, I_G, T_G, I_{GCL})$, for each $t \in [0, T]$ gives

$$\begin{aligned}
\|S_H(t)\| &\leq S_H(0) + \frac{1}{\Gamma(\zeta)} \int_0^T (t-\tau)^{\zeta-1} \|F(t, S_H(t))\| d\tau \\
&\leq S_H(0) + \frac{L_{F1}}{\Gamma(\zeta)} \int_0^T (t-\tau)^{\zeta-1} (c + \|S_H(t)\|) d\tau \\
&\leq S_H(0) + \frac{cL_{F1}}{\Gamma(\zeta)} \int_0^T (t-\tau)^{\zeta-1} d\tau + \frac{L_{F1}}{\Gamma(\zeta)} \int_0^T (t-\tau)^{\zeta-1} \|S_H(t)\| d\tau \qquad (20) \\
&\leq S_H(0) + \left(L_{F1} \frac{T^{\zeta}}{\Gamma(\zeta+1)} \right) + \left(\frac{L_{F1} T^{\zeta}}{\Gamma(\zeta+1)} \right) \int_0^T (t-\tau)^{\zeta-1} \|S_H(t)\| d\tau \\
&\leq \left(S_H(0) + \frac{L_{F1} T^{\zeta}}{\Gamma(\zeta+1)} E_{\zeta} \left(L_{F1} T^{\zeta} \right) \right) < \infty.
\end{aligned}$$

Following a similar approach we have

$$\begin{aligned}
\|V_{CL}(t)\| &\leq \left(V_{CL}(0) + \frac{L_{G1}T^{\zeta}}{\Gamma(\zeta+1)}E_{\zeta}\left(L_{G1}T^{\zeta}\right)\right) < \infty,\\
\|I_{CL}(t)\| &\leq \left(I_{CL}(0) + \frac{L_{H1}T^{\zeta}}{\Gamma(\zeta+1)}E_{\zeta}\left(L_{H1}T^{\zeta}\right)\right) < \infty,\\
\|T_{CL}(t)\| &\leq \left(T_{CL}(0) + \frac{L_{FK1}T^{\zeta}}{\Gamma(\zeta+1)}E_{\zeta}\left(L_{K1}T^{\zeta}\right)\right) < \infty,\\
\|I_{G}(t)\| &\leq \left(I_{G}(0) + \frac{L_{FQ1}T^{\zeta}}{\Gamma(\zeta+1)}E_{\zeta}\left(L_{Q1}T^{\zeta}\right)\right) < \infty,\\
\|T_{G}(t)\| &\leq \left(T_{G}(0) + \frac{L_{U1}T^{\zeta}}{\Gamma(\zeta+1)}E_{\zeta}\left(L_{U1}T^{\zeta}\right)\right) < \infty,\\
\|I_{GCL}(t)\| &\leq \left(I_{GCL}(0) + \frac{L_{V1}T^{\zeta}}{\Gamma(\zeta+1)}E_{\zeta}\left(L_{V1}T^{\zeta}\right)\right) < \infty.
\end{aligned}$$

Since we have proved that $R(\vartheta)$ is bounded, ϑ has a fixed point given by Schaefer's fixed point theorem and hence the solution of the model.

Basic reproduction number of the model

The Disease Free Equilibrium (DFE) of the co-infection model is given by

$$\begin{aligned}
\xi_0 =& (S_h^0, V_{CL}^0, I_{CL}^0, T_{CL}^0, I_G^0, T_G^0, I_{GCL}^0)\\
=& \Big(\frac{\Lambda_H + \omega V_{CL}}{\mu+\gamma}, \frac{\gamma S_H}{\mu+\omega}, 0, 0, 0, 0, 0, 0\Big),
\end{aligned} \tag{21}$$

$$V = \begin{bmatrix} (\mu+\tau+\delta_1) & 0 & -\rho_1 \\ 0 & (\mu+\rho+\delta_2) & 0 \\ 0 & 0 & (\mu+\delta_3+\rho_1+\rho_2) \end{bmatrix}, \qquad F = \begin{pmatrix} \frac{S_H\beta_{CL}}{N_H} & 0 & \frac{S_H\eta_L\beta_{CL}}{N_H} \\ 0 & \frac{S_H\beta_G}{N_H} & \frac{S_H\beta_G}{N_H} \\ 0 & 0 & 0 \end{pmatrix},$$

$$\mathcal{R}_0 = \left|FV^{-1} - \lambda I\right|.$$

The basic reproduction number of the model obtained and stated below, using the approach illustrated in [43], is given by $\mathcal{R}_0 = \max\{\mathcal{R}_{0CL}, \mathcal{R}_{0Go}\}$ where $\mathcal{R}_{0CL}$ and $\mathcal{R}_{0Go}$ are, respectively, the Chlamydia and Gonorrhea associated reproduction numbers, given by

$$\mathcal{R}_{0CL} = \frac{S_H^*\beta_{CL}}{N_H^*(\tau+\mu+\delta_1)}, \quad \text{and} \quad \mathcal{R}_{0Go} = \frac{S_H^*\beta_G}{N_H^*(\rho+\mu+\delta_2)}.$$

Generalized Ulam-Hyers-Rassias stability

We investigate the stability of the fractional model system using the Ulam-Hyers-Rassias(UHR) Stability method as given in [42] to demonstrate UHR stability of the model.

Definition 4 *The fractional model* (1)-(3) *is generalized Ulam-Hyers-Rassias (UHR) stable with respect to* $\Omega(t) \in \mathbb{H}^1([0,T],\mathbb{D})$ *if there exists a real value* $\kappa_\epsilon > 0$ *with* $\epsilon > 0$ *and for all solution*

$(S_H, V_{CL}, I_{CL}, T_{CL}, I_G, T_G, I_{GCL}) \in \mathbb{H}^1([0,T],\mathbb{D})$ *of the following inequalities*

$$\left|D_t^\zeta S_h(t) - F(t, S_H(t))\right| \leq \Omega(t), \quad \left|D_t^\zeta V_{CL}(t) - G(t, V_{CL}(t))\right| \leq \Omega(t),$$

$$\left|D_t^\zeta I_{CL}(t) - H(t, I_{CL}(t))\right| \leq \Omega(t), \quad \left|D_t^\zeta T_{CL}(t) - K(t, T_{CL}(t))\right| \leq \Omega(t),$$

$$\left|D_t^\zeta T_G(t) - Q(t, T_G(t))\right| \leq \Omega(t), \quad \left|D_t^\zeta I_{GCL}(t) - V(t, I_{GCL}(t))\right| \leq \Omega(t),$$

there exists a solution $(\bar{S}_H, \bar{V}_C L, \bar{I}_{CL}, \bar{T}_{CL}, \bar{I}_G, \bar{T}_G, \bar{I}_{GCL},) \in \mathbb{H}^1([0,T],\mathbb{D})$ *of the model* (1)-(3) *with*

$$\left|S_H(t) - \bar{S}_H(t)\right| \leq \kappa_\epsilon \Omega(t), \quad \left|V_{CL}(t) - \bar{V}_{CL}(t)\right| \leq \kappa_\epsilon \Omega(t), \quad \left|I_{CL}(t) - \bar{I}_{CL}(t)\right| \leq \kappa_\epsilon \Omega(t),$$

$$\left|T_{CL}(t) - \bar{T}_{CL}(t)\right| \leq \kappa_\epsilon \Omega(t),$$

$$\left|I_G(t) - \bar{I}_G(t)\right| \leq \kappa_\epsilon \Omega(t), \quad \left|T_G(t) - \bar{T}_G(t)\right| \leq \kappa_\epsilon \Omega(t), \quad \left|I_{GCL}(t) - \bar{I}_{GCL}(t)\right| \leq \kappa_\epsilon \Omega(t).$$

Theorem 4 *In relation to* $\Omega \in \mathbb{H}^1([0,T],\mathbb{D})$, *the fractional model* (1)-(3) *are generalized Ulam-Hyers-Rassias stable if* $\left(L_F, L_G, L_H, L_K, L_Q, L_V\right) T^\zeta < 1$.

Proof From definition (4), denoting Ω as a non-decreasing function of t, then there exists $\epsilon > 0$ such that

$$\int_0^t (t-\tau)^{\zeta-1} \Omega(t) d\tau \leq \epsilon \Omega(t),$$

for every $t \in [0,T]$. Where It has been demonstrated that the functions F, G, H, K, Q, V are continuous and $(L_F, L_G, L_H, L_K, L_Q, L_V) > 0$ satisfies the Lipschitz condition as described in the preceding section. The fractional model (1)-(3) unique solution comes from Theorem (2)

$$\bar{S}_H(t) = S_H(0) + \frac{1}{\Gamma(\zeta)} \int_0^t (t-\tau)^{\zeta-1} F(t, \bar{S}_H(t)) d\tau.$$

Integrating the inequalities in Definition (4) we get

$$\begin{aligned} \left|S_H(t) - S_H(0) - \frac{1}{\Gamma(\zeta)} \int_0^t (t-\tau)^{\zeta-1} F(t, S_H(t)) d\tau\right| &\leq \frac{1}{\Gamma(\zeta)} \int_0^t (t-\tau)^{\zeta-1} \Omega(t) d\tau, \\ &\leq \frac{\epsilon \Omega(t) T^\zeta}{\Gamma(\zeta+1)}. \end{aligned} \tag{22}$$

Using Lemma 1 and Eq. (22), we get

$$\begin{aligned}
\left|S_H(t)-\bar{S}_H(t)\right| &\leq \left|S_H(t)-\left(S_H(0)+\frac{1}{\Gamma(\zeta)}\int_0^t(t-\tau)^{\zeta-1}F(t,\bar{S}_H(t))d\tau\right)\right| \\
&\leq \left|S_H(t)-S_H(0)-\left(\frac{1}{\Gamma(\zeta)}\int_0^t(t-\tau)^{\zeta-1}F(t,\bar{S}_H(t))d\tau\right.\right. \\
&+ \left.\left.\frac{1}{\Gamma(\zeta)}\int_0^t(t-\tau)^{\zeta-1}F(t,S_H(t))d\tau-\frac{1}{\Gamma(\zeta)}\int_0^t(t-\tau)^{\zeta-1}F(t,S_H(t))d\tau\right)\right| \\
&\leq \left|S_H(t)-S_H(0)-\frac{1}{\Gamma(\zeta)}\int_0^t(t-\tau)^{\zeta-1}F(t,S_H(t))d\tau\right| \\
&+ \frac{1}{\Gamma(\zeta)}\int_0^t(t-\tau)^{\zeta-1}\left|F(t,S_H(t))-F(t,\bar{S}_H(t))\right|d\tau \\
&\leq \frac{\epsilon\Omega(t)T^\zeta}{\Gamma(\zeta+1)}+\frac{L_FT^\zeta}{\Gamma(\zeta+1)}\int_0^t(t-\tau)^{\zeta-1}\left|S_H(t)-\bar{S}_H(t)\right|d\tau \\
&\leq \frac{\epsilon\Omega(t)T^\zeta}{\Gamma(\zeta+1)}E_\zeta\left(L_FT^\zeta\right).
\end{aligned}$$

By setting $\kappa_\zeta=\frac{\epsilon T^\zeta}{\Gamma(\zeta+1)}E_\zeta\left(L_FT^\zeta\right)$, we have

$$\left|S_H(t)-\bar{S}_H(t)\right| \leq \kappa_\zeta\Omega(t), \quad t\in[0,T].$$

Using a similar method, we obtain

$$\left|V_{CL}(t)-\bar{V}_{CL}(t)\right|\leq\kappa_\zeta\Omega(t),\ \left|I_{CL}(t)-\bar{I}_{CL}(t)\right|\leq\kappa_\zeta\Omega(t),\ \left|T_{CL}(t)-\bar{T}_{CL}(t)\right|\leq\kappa_\zeta\Omega(t),$$

$$\left|I_G(t)-\bar{I}_G(t)\right|\leq\kappa_\zeta\Omega(t),\left|I_{GCL}(t)-\bar{I}_{GCL}(t)\right|\leq\kappa_\zeta\Omega(t),\ t\in[0,T].$$

Consequently, this indicates that the model is UHR-stable in general with regard to $\Omega(t)$. ■

The disease-free equilibrium's global asymptotic stability (GAS)

Theorem 5 *Consider the model equation (1) with the DFE (21) given by $\mathcal{E}_0$, the DFE $\mathcal{E}_0$ of the model is globally asymptotically stable in $\mathcal{D}$ whenever $\mathcal{R}_0\leq 1$.*

We use the method presented in [44] to investigate global stability.

$$D_t^\zeta X=F(X,W)=\begin{bmatrix}\Lambda_H-(\lambda_G+\lambda_{CL})S_H+\omega V_{CL}-(\mu+\gamma)S_H\\ \gamma S_H-(\mu+\omega)V_{CL}-V_{CL}(\lambda_G+\lambda_{CL})\\ \tau I_{CL}-(\mu+\epsilon_L\lambda_{CL})T_{CL}+V_{CL}\lambda_{CL}\\ pI_G-(\mu+\epsilon_W\lambda_G)T_G+V_{CL}\lambda_G\end{bmatrix}, \tag{23}$$

$$F(X,0)=\begin{bmatrix}\Lambda_H+\omega V_{CL}-(\mu+\gamma)S_H\\ \gamma S_H-(\mu+\omega)V_{CL}\\ -\mu T_{CL}\\ -\mu T_G\end{bmatrix}, \tag{24}$$

$$\frac{dW}{dt} = \begin{bmatrix} \lambda_{CL}S_H + \epsilon_L\lambda_{CL}T_{CL} - (\mu+\tau+\delta_1)I_{CL} - \xi_1\lambda_G I_{CL} + \rho_1 I_{GCL} \\ \lambda_G S_H + \epsilon_W\lambda_G T_G - (\rho+\mu+\delta_2)I_G - \xi_2\lambda_{CL}I_G + \rho_2 I_{GCL} \\ \xi_1\lambda_G I_{CL} + \xi_2\lambda_{CL}I_G - (\mu+\delta_3+\rho_1+\rho_2)I_{GCL} \end{bmatrix}, \tag{25}$$

$$A = \begin{bmatrix} \frac{\beta_{CL}(S_H+\epsilon_L T_C)}{N} - (\mu+\tau+\delta_1) & 0 & \frac{\beta_C L\eta_c(S_H+\epsilon_L T_{CL})}{N} + \rho_1 \\ 0 & \frac{\beta_G(S_H+\epsilon_W T_G)}{N} - (p+\mu+\delta_2) & \frac{\beta_G\eta_g(S_H+\epsilon_W T_G)}{N} + \rho_2 \\ \xi_1\lambda_G & \xi_2\lambda_{CL} & -(\mu+\delta_3+\rho_1+\rho_2) \end{bmatrix}, \tag{26}$$

$$AW = \begin{bmatrix} \frac{I_{CL}\beta_{CL}(S_H+\epsilon_L T_C)}{N} - I_{CL}(\mu+\tau+\delta_1) + \frac{I_{GCL}\beta_C L\eta_c(S_H+\epsilon_L T_{CL})}{N} + I_{GCL}\rho_1 \\ \frac{I_G\beta_G(S_H+\epsilon_W T_G)}{N} - I_G(p+\mu+\delta_2) + \frac{I_{GCL}\beta_G\eta_g(S_H+\epsilon_W T_G)}{N} + I_{GCL}\rho_2 \\ I_{CL}\xi_1\lambda_G + I_G\xi_2\lambda_{CL} - I_{GCL}(\mu+\delta_3+\rho_1+\rho_2) \end{bmatrix}, \tag{27}$$

$$\tilde{G}(X,W) = AW - G(X,W) = \begin{bmatrix} \xi_1\lambda_G I_{CL} \\ \xi_2\lambda_{CL}I_G \\ 0 \end{bmatrix}.$$

Since $\hat{G}(X,W) \geq 0$, this gives that the DFE is globally asymptotically stable.

3 Numerical scheme and algorithms

If $t_k = kh, k = 0,1,2,\ldots,m$ be the uniform grid points represented by some integer m and the grid step size represented by $(h = T/m)$. Then, using piece-wise interpolation and knots and nodes located at $t_j, j = 0,1,2,\ldots,k+1,$, the fractional version of the one-step Adam-Moulton method is reduced to equation (13) (Corrector formula as described in [45]);

$$\begin{aligned}
S_H(t_{k+1}) - S_H(0) &= \frac{h^\zeta}{\Gamma(\zeta+2)}\left(\sum_{j=0}^{k} u_{j,k+1}F\left(t_j, S_H(t_j)\right) + F\left(t_{k+1}, S_H^p(t_{k+1})\right)\right), \\
V_{CL}(t_{k+1}) - V_{CL}(0) &= \frac{h^\zeta}{\Gamma(\zeta+2)}\left(\sum_{j=0}^{k} u_{j,k+1}G\left(t_j, V_{CL}(t_j)\right) + G\left(t_{k+1}, I_{CL}^p(t_{k+1})\right)\right), \\
I_{CL}(t_{k+1}) - I_{CL}(0) &= \frac{h^\zeta}{\Gamma(\zeta+2)}\left(\sum_{j=0}^{k} u_{j,k+1}H\left(t_j, I_{CL}(t_j)\right) + H\left(t_{k+1}, I_{CL}^p(t_{k+1})\right)\right), \\
T_{CL}(t_{k+1}) - T_{CL}(0) &= \frac{h^\zeta}{\Gamma(\zeta+2)}\left(\sum_{j=0}^{k} u_{j,k+1}K\left(t_j, T_{CL}(t_j)\right) + K\left(t_{k+1}, T_{CL}^p(t_{k+1})\right)\right), \\
I_G(t_{k+1}) - I_G(0) &= \frac{h^\zeta}{\Gamma(\zeta+2)}\left(\sum_{j=0}^{k} u_{j,k+1}Q\left(t_j, I_G(t_j)\right) + Q\left(t_{k+1}, I_G^p(t_{k+1})\right)\right), \\
T_G(t_{k+1}) - T_G(0) &= \frac{h^\zeta}{\Gamma(\zeta+2)}\left(\sum_{j=0}^{k} u_{j,k+1}U\left(t_j, T_G(t_j)\right) + U\left(t_{k+1}, T_G^p(t_{k+1})\right)\right),
\end{aligned} \tag{28}$$

$$I_{GCL}(t_{k+1}) - I_{GCL}(0) = \frac{h^{\zeta}}{\Gamma(\zeta+2)}\left(\sum_{j=0}^{k} u_{j,k+1} V\left(t_j, I_{GCL}(t_j)\right) + V\left(t_{k+1}, I^{p}_{GCL}(t_{k+1})\right)\right),$$

given the weight

$$u_{j,k+1} = \begin{cases} k^{\zeta+1} - (k-\zeta)(k+1)^{\zeta}, & j = 0. \\ (k-j+2)^{\zeta+1} + (k-j)^{\zeta+1} - 2(k-j+1)^{\zeta+1}, & 1 \le j \le k. \\ 1, & j = k+1. \end{cases}$$

Based on the well-known one-step Adams-Bashforth method, the predictor formula is provided by

$$\begin{aligned}
S^{p}_{H}(t_{k+1}) - S_H(0) &= \frac{1}{\Gamma(\zeta)}\sum_{j=0}^{k} v_{j,k+1} F\left(t_j, S_H(t_j)\right), \\
V^{p}_{CL}(t_{k+1}) - V_{CL}(0) &= \frac{1}{\Gamma(\zeta)}\sum_{j=0}^{k} v_{j,k+1} G\left(t_j, V_{CL}(t_j)\right), \\
I^{p}_{CL}(t_{k+1}) - I_{CL}(0) &= \frac{1}{\Gamma(\zeta)}\sum_{j=0}^{k} v_{j,k+1} H\left(t_j, I_{CL}(t_j)\right), \\
T^{p}_{CL}(t_{k+1}) - T_{CL}(0) &= \frac{1}{\Gamma(\zeta)}\sum_{j=0}^{k} v_{j,k+1} K\left(t_j, T_{CL}(t_j)\right), \\
I^{p}_{G}(t_{k+1}) - I_G(0) &= \frac{1}{\Gamma(\zeta)}\sum_{j=0}^{k} v_{j,k+1} Q\left(t_j, I_G(t_j)\right), \\
T^{p}_{G}(t_{k+1}) - T_G(0) &= \frac{1}{\Gamma(\zeta)}\sum_{j=0}^{k} v_{j,k+1} U\left(t_j, T_G(t_j)\right), \\
I^{p}_{GCL}(t_{k+1}) - I_{GCL}(0) &= \frac{1}{\Gamma(\zeta)}\sum_{j=0}^{k} v_{j,k+1} V\left(t_j, I_{GCL}(t_j)\right),
\end{aligned} \tag{29}$$

given the weight

$$v_{j,k+1} = \zeta^{-1} h^{\zeta}\left((k-j+1)^{\zeta} - (k-j)^{\zeta}\right).$$

4 Numerical simulations

In this numerical simulation, we employ a Caputo-based predictor-corrector method derived from the Adams-Bashforth linear multistep method to solve the model described in model (1). The parameter values used in the simulation are listed in Table 2, unless otherwise specified.
We set the initial conditions for the variables as follows: $S_H(0) = 35,000,000$, $V_{CL}(0) = 3000$, $I_{CL}(0) = 3500$, $T_{CL}(0) = 3000$, $I_G(0) = 3000$, $T_G(0) = 3000$, and $I_{GCL}(0) = 3500$. These initial conditions are chosen arbitrarily based on the non-availability of data on the co-infection of Chlamydia and Gonorrhea study.

Table 2. Parameter values and corresponding references

Parameter	Values	Reference
ϵ_W	1.1	[9, 17]
ϵ_L	1.1	[9, 17]
μ	0.0122	[17]
β_C, β_G	1.1	[9, 17]
ω	0.5	[9]
γ	0.895	Assumed
τ	0.9	Assumed
δ_1	0.5	[15]
δ_2	0.05	Assumed
δ_3	0.05	Assumed
η_c	1.2	[17]
η_g	1.2	[17]
ξ_1, ξ_2	1.2	Assumed
ρ_1, ρ_2	0.9	Assumed
ρ	0.9	Assumed
Λ_H	596620.9	Assumed

We present the results of a numerical simulation of model (1) over a period of 10 years, using different fractional order values $\zeta = 0.98, 0.88, 0.78, 0.68, 0.58$. The simulation results for the different classes $I_{CL}(t)$, $I_G(t)$, and $I_{GCL}(t)$ are shown in Figs. (2), (3), and (8), respectively. The fractional order of the model is of high significance in modeling, as demonstrated by the simulation results. Fig. (2) shows that the population of people infected with Chlamydia initially decreases as the order increases, but after one and a half years, the trend reverses and remains uniform for the rest of the period. The same trend is observed for the population of infectious Gonorrhea class as the order of the model is varied. Fig. (4) demonstrates the importance of vaccination in reducing the burden of Chlamydia. At a fractional order of $\zeta = 0.98$, an effective reduction of individuals in the class is achieved for the first four years, leading to the total eradication of people infected with Chlamydia when the vaccination is sustained afterwards. Fig. (5) shows that effective treatment of people infected with Chlamydia can also reduce the burden of the disease. At a fractional order of $\zeta = 0.78$, the desired result of reducing the burden of the disease is achieved in a shorter period than that required for vaccination. When individuals with dual infection are treated for Gonorrhea, there is a corresponding reduction in the population of individuals in the Chlamydia class, as seen in Fig. (6). This is likely due to the use of similar antibiotics in treating some common STDs. Similarly, Fig. (7) shows that effective treatment of one STD can lead to the collapse of other similar diseases, whether detected at that point or not. Finally, Fig. (9) shows that increasing the treatment rate leads to a corresponding decrease in the population of infected Gonorrhea patients, as expected.

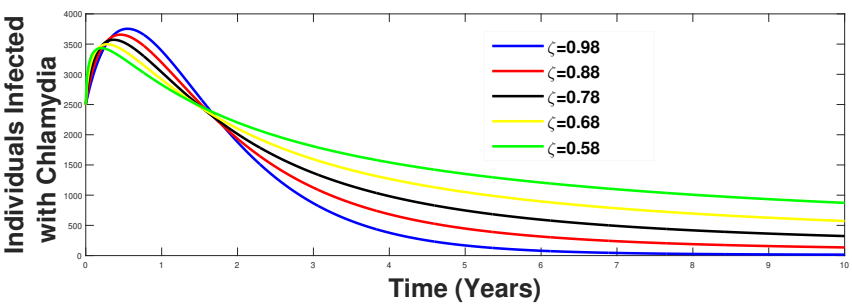

Figure 2. Varying the fractional order and its effects on the dynamics of the infectious Chlamydia class

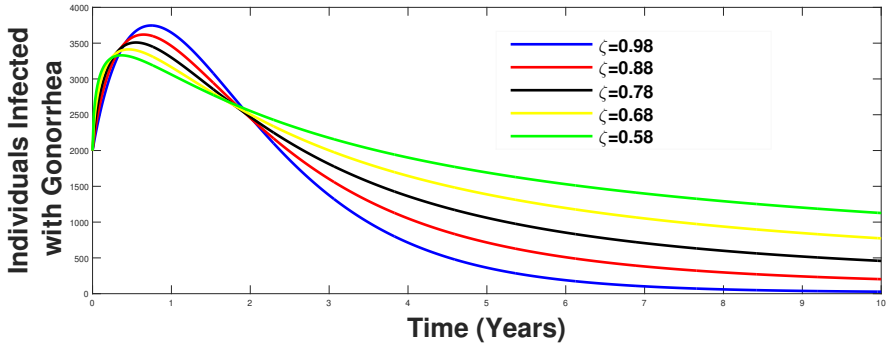

Figure 3. Varying the fractional order and its effects on the dynamics of the infectious Gonorrhea class

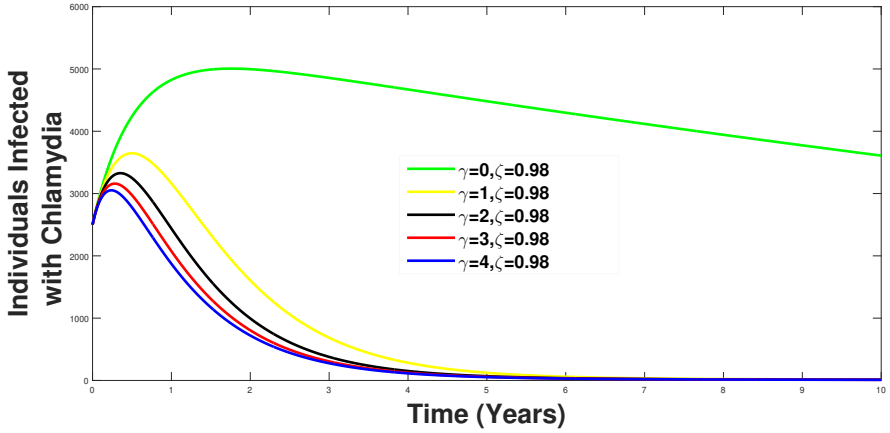

Figure 4. Effect of rate of vaccination on individuals infected with Chlamydia at $\zeta = 0.98$

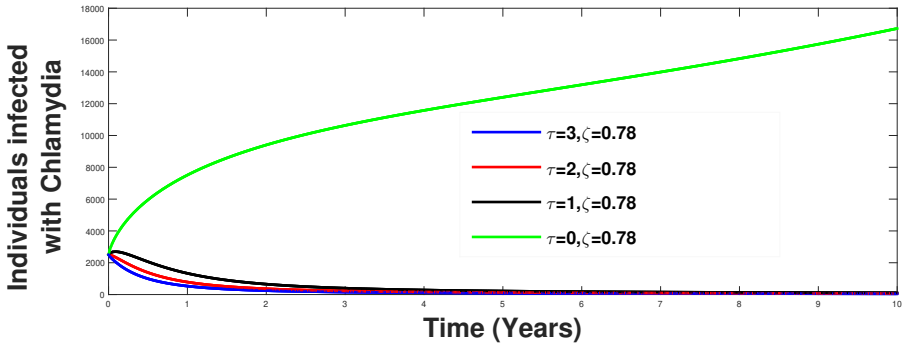

Figure 5. The effect of treatment rate on the infectious Chlamydia class $\zeta = 0.78$

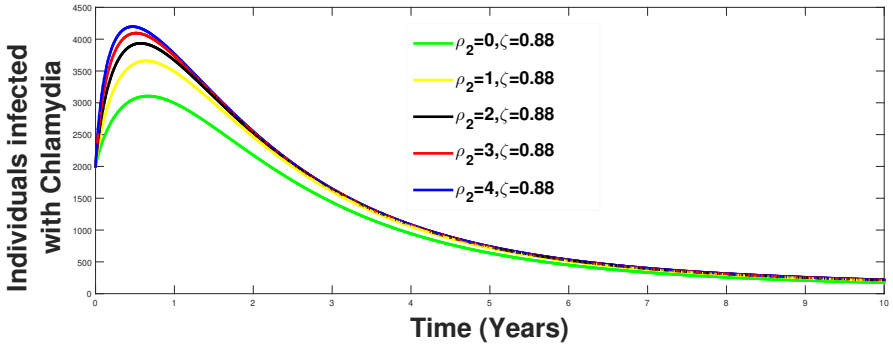

Figure 6. The effect of varying the rate at which dually infected individuals gets treated of Gonorrhea at $\zeta = 0.88$

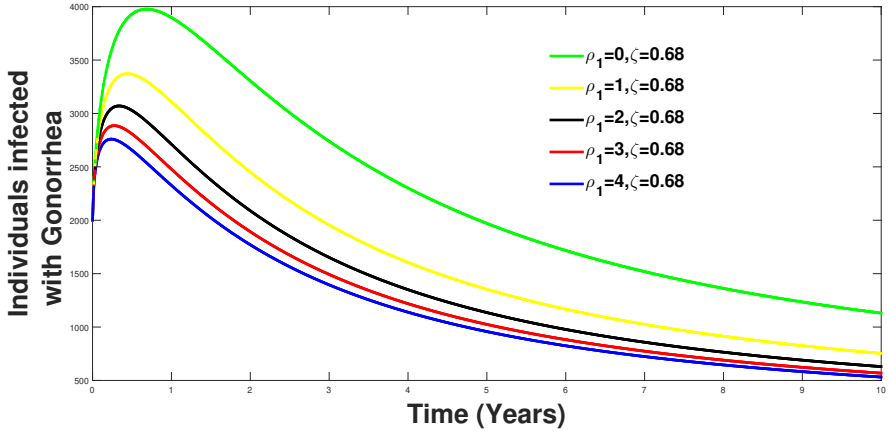

Figure 7. Varying the rate which dually infected individuals gets treated of Chlamydia

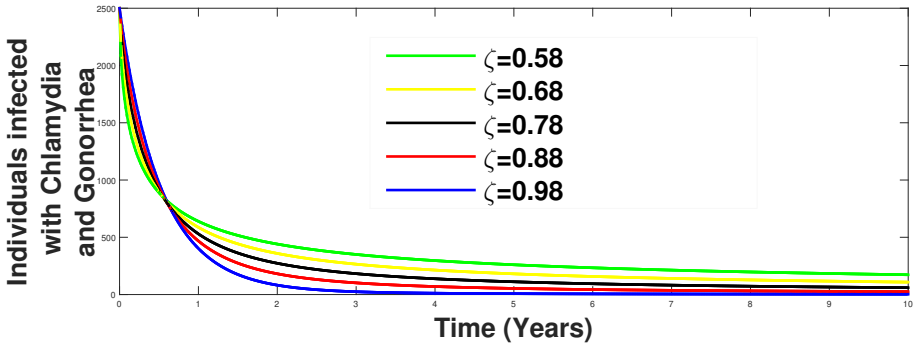

Figure 8. The effect of varying the fractional order of the dually infected individuals and its effects on the disease dynamics

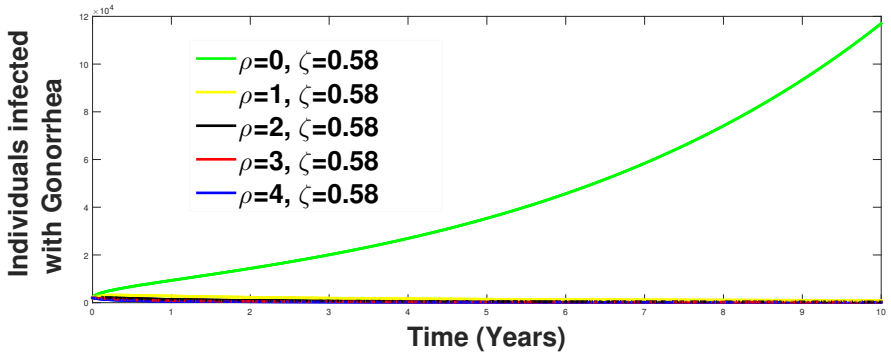

Figure 9. Varying the treatment rate of Gonorrhea and its effects on the Gonorrhea class

5 Conclusions

The results of the numerical simulation for model (1) are presented in Figs. (2)–(9), which show the dynamics of the different classes of infections over a 10-year period for varying fractional order values ζ. One important observation from Figs. (2) and (3) is that the populations of individuals infected with chlamydia and gonorrhea, respectively, decrease as the fractional order increases for the first year and a half, but then increase as the fractional order further increases. In contrast, the population of individuals who are co-infected with both chlamydia and gonorrhea decreases initially as the fractional order decreases, but increases thereafter.

Another key finding from the simulation is that increasing the vaccination rate leads to a decrease in the population of individuals infected with chlamydia, as shown in Fig. (4). On the other hand, varying the treatment rate for gonorrhea (Fig. (6)) and chlamydia (Fig. (7)) yields different outcomes depending on the fractional order. Furthermore, the results in Fig. (9) indicate that increasing the treatment rate for individuals infected with gonorrhea can reduce the burden of gonorrhea infection in the population of individuals with gonorrhea.

It is worth noting that the Caputo-based predictor-corrector method was used for the numerical simulation, and the Laplace transform was used to show that the model is bounded and positively invariant. The existence and uniqueness of the model were established using methods based on Banach and Schaefer's fixed point theorem. Additionally, the model was found to be Ulam-Hyers-Rassias stable.

Based on the results obtained from this study, we can conclude the following:

* Effective treatment of individuals infected with both Chlamydia and Gonorrhea is crucial in achieving the desired outcome of reducing the burden of the diseases in a general sense.
* Vaccination has been shown to play a significant role in the fight against the investigated diseases, and more efforts should be made in developing and administering vaccines if the long-term goal of eradicating the diseases is desired

These conclusions are based on the simulation results of the proposed fractional model of Chlamydia-Gonorrhea co-infection.

In summary, the results suggest that the effective treatment of individuals infected with both Chlamydia and Gonorrhea, coupled with vaccination programs, can significantly reduce the burden of the diseases. The findings of this study provide insights that can guide policymakers and health-care providers in developing and implementing effective strategies for controlling and managing Chlamydia-Gonorrhea co-infection. The results obtained from this study can aid in the development of more effective treatment strategies for this type of co-infection. Additionally, the methods used in this study can be applied to investigate the dynamics of other infectious

diseases modeled using fractional calculus. Overall, this paper contributes to the growing body of research on the application of fractional calculus in modeling infectious diseases, highlighting the importance of considering the fractional order in the modeling process.

Declarations

Ethical approval

The authors state that this research complies with ethical standards. This research does not involve either human participants or animals.

Consent for publication

Not applicable.

Conflicts of interest

The authors declare that they have no conflict of interest.

Data availability statement

Data availability is not applicable to this article as no new data were created or analysed in this study.

Author's contributions

U.B.O.: Conceptualization, Supervision, Project Administration, Formal Analysis, Resources, Visualization, Acquisition. N.I.: Formal Analysis, Investigation, Data Curation, Software, Validation, Writing-Original Draft, Writing-Review & Editing. B.B.: Methodology, Writing-Original Draft, Validation, Project Administration. All authors discussed the results and contributed to the final manuscript.

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