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Asymptotic Stability of Neutral Differential Systems with Variable Delay and Nonlinear Perturbations

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Abstract
In this paper, the problem of asymptotic stability of a kind of nonlinear perturbed neutral differential system with variable delay is discussed. The Lyapunov-Krasovskiǐ functional constructed, is used to obtain conditions for asymptotic stability of the nonlinear perturbed neutral differential system in terms of linear matrix inequality (LMI). The two new results (delay-independent and delay-dependent criteria) include and extend the existing results in the literature. Finally, an example of delay-dependent criteria is supplied and the simulation result is shown to justify the effectiveness and reliability of the used techniques.

Keywords: *Asymptotic stability, Lyapunov-Krasovskiǐ functional, Neutral delay differential system, Perturbation analysis*

AMS Subject Classification (2020): *34K05; 34K20; 34K27; 34K40; 37C75*

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1. Introduction

Qualitative behaviour of solutions of differential equations with or without delay and/or randomness of various orders have received appreciable attention in recent years, see for instance the papers in [1–9]. These significant improvements in the study of differential equations are not unconnected to umpteen areas of applications in electrical networks containing lossless transmission lines [10–12], stability properties of electrical power systems, and macroeconomic models, the motion of nuclear reactors, feedback control loops involving sensors in integrated communication and control systems, energy or signal transmission, see [13–18]. Accordingly, researchers have developed efficient and effective techniques such as the Lyapunov direct method, the technique of characteristic equation, the fixed point principle, the state trajectory method, and so on, to discuss systems of first-, second-, third-, and higher-order differential equations.

In their contributions, Hale *et al.* [19], Hale and Verdny, Lunel [20], Li [21], Slemrod and Infante [22] to mention but a few, have developed delay-independent criteria for the asymptotic stability of neutral delay differential

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systems. In addition, sufficient delay-dependent conditions were presented by Brayton and Willoughby [23] and Khusainov and Yun'kova [24] for the asymptotic stability of neural delay differential system and systems with nonlinear perturbations respectively. The obtained sufficient conditions are presented in terms of either matrix norm or matrix measure operations which by Park and Won [25], are conservative. Other relevant papers on chaotic control and hyperchaotic systems with time delay include Feng *et al.* [26], Onasanya *et al.* [27], stability tests and solution estimates for non-linear differential equations Tunç [28], among others.

In this investigation, we consider the problem of asymptotic stability of a neutral differential system with a variable delay. Lyapunov-Krasovskiĭ functionals are developed to derive sufficient conditions for the asymptotic stability of the system under investigation in terms of LMI. Two new results (i.e., sufficient criteria for delay-independent and delay-dependent) which generalize that of Park and Won are presented. The motivation for this paper comes from the work of Park and Won, where sufficient conditions for the stability of perturbed constant delay differential systems are discussed in terms of LMI. If $\tau(t) = h$, $h > 0$ is a constant delay, then the nonlinear perturb neutral variable delay differential systems considered, the Lyapunov-Krasovskiĭ functional employed and the two new results coincide with that of Park and Won.

2. Stability results

Consider a perturbed neutral variable delay differential system

$$\dot{x}(t) = Ax(t) + Bx(t-\tau(t)) + C\dot{x}(t-\tau(t)) + Q(x(t), x(t-\tau(t)), \dot{x}(t-\tau(t))) \tag{2.1}$$

with the initial condition $x(t) = \phi(t)$, where $x(t) \in \mathbb{R}^n$ is the state vector, A, B and $C \in \mathbb{R}^{n\times n}$ are constant matrices, $Q \in C(\mathbb{R}^{3n}, \mathbb{R}^n)$, the positive constants α_1, α_2 exist and satisfying the following inequalities $\tau(t) \le \alpha_1$ and its derivative $\dot{\tau}(t) \le \alpha_2$, $(0 < \alpha_2 < 1)$, $\phi(t)$ is the continuously differentiable function on $[-\alpha_1, 0]$ and for all $t \in [-\alpha_1, 0]$, $Q(t) = Q(x(t), x(t-\tau(t)), \dot{x}(t-\tau(t)))$ a nonlinear perturbation that satisfies the following estimate

$$\|Q(t)\| \le \lambda_1\|x(t)\| + \lambda_2\|x(t-\tau(t))\| + \lambda_3\|\dot{x}(t-\tau(t))\| \tag{2.2}$$

where λ_i, $(i = 1, 2, 3)$ is a positive constant.

Lemma 2.1. *(Khargonekar et al. [29])*
Let D and E be real matrices of appropriate dimensions. Then, for any scalar $\epsilon > 0$

$$DE + E^TD^T < \epsilon DD^T + \epsilon^{-1}E^TE.$$

The main tool employed in this investigation is the functional $V = V(x)$ defined as

$$V = x^T(t)Px(t) + \sum_{i=1}^{2} W_i, \quad (i = 1, 2) \tag{2.3}$$

where

$$W_1 := \int_{t-\tau(t)}^{t} \dot{x}^T(\theta)\dot{x}(\theta)d\theta \ \text{ and } \ W_2 := \int_{t-\tau(t)}^{t} x^T(\theta)Rx(\theta)d\theta.$$

Next, we shall state and prove the first stability result of this paper as follows.

Theorem 2.1. *In addition to the basic assumptions on functions τ and Q, suppose that ε_i $(i = 1, 2, 3, 4)$ are positive constants, P and R are $n \times n$ symmetric positive definite matrices satisfying the following LMI*

$$M_1 = \begin{bmatrix} A_1 & A^TB + PB & A^TC + PC \\ B^TA + B^TP & A_2 & B^TC \\ C^TA + C^TP & C^TB & A_3 \end{bmatrix} < 0$$

where

$$A_1 := D_1 + \varepsilon_1^{-1}PP + \varepsilon_2^{-1}A^TA; \;\; A_2 := D_2 + \varepsilon_3^{-1}B^TB;$$

$$A_3 := D_3 + \varepsilon_4^{-1}C^TC; \;\; D_1 := A^TP + PA + A^TA + R + 3\big(1 + \sum_{i=1}^{4}\varepsilon_i\big)\lambda_1^2 I;$$

$$D_2 := B^TB - (1-\alpha_2)R + 3\big(1 + \sum_{i=1}^{4}\varepsilon_i\big)\lambda_2^2 I; \;\; \text{and} \;\; D_3 := C^TC - (1-\alpha_2)I + 3\big(1 + \sum_{i=1}^{4}\varepsilon_i\big)\lambda_3^2 I.$$

Then the solution of system (2.1) is asymptotically stable.

Proof. Let $x = x(t),\ \tau = \tau(t)$ and $x_\tau = x(t-\tau(t))$, the derivative of V with respect to the independent variable t along the solution path of (2.1) is

$$\dot{V}_{(2.1)} = x^T(PA + A^TP)x + 2x_\tau^T B^T Px + 2\dot{x}_\tau^T C^T Px + 2Q^T Px + \sum_{i=1}^{2} \dot{W}_i, \tag{2.4}$$

where

$$\begin{aligned} \dot{W}_1 = {} & x^T A^T Ax + x_\tau^T B^T Bx_\tau + \dot{x}_\tau^T C^T C\dot{x}_\tau + Q^T Q + 2x^T A^T Bx_\tau + 2x^T A^T C\dot{x}_\tau \\ & + 2x_\tau^T B^T C\dot{x}_\tau + 2Q^T Ax + 2Q^T Bx_\tau + 2Q^T C\dot{x}_\tau - (1-\dot{\tau})\dot{x}_\tau^T \dot{x}_\tau \end{aligned} \tag{2.5}$$

and

$$\dot{W}_2 = x^T Rx - (1-\dot{\tau})x_\tau^T Rx_\tau. \tag{2.6}$$

Re-arranging $2Q^TPx, 2Q^TAx, 2Q^TBx_\tau$ and $2Q^TC\dot{x}_\tau$ using Lemma 2.1, we obtain

$$\begin{aligned} 2Q^T Px &\le \varepsilon_1 Q^T Q + \varepsilon_1^{-1} x^T PPx, \\ 2Q^T Ax &\le \varepsilon_2 Q^T Q + \varepsilon_2^{-1} x^T A^T Ax, \\ 2Q^T Bx_\tau &\le \varepsilon_3 Q^T Q + \varepsilon_3^{-1} x_\tau^T B^T Bx_\tau, \\ 2Q^T C\dot{x}_\tau &\le \varepsilon_4 Q^T Q + \varepsilon_4^{-1} \dot{x}_\tau^T C^T C\dot{x}_\tau. \end{aligned} \tag{2.7}$$

In view of inequality (2.2) and the fact that $2ab\|x\|\|y\| \le a^2\|x\|^2 + b^2\|y\|^2$, it follows that

$$Q^T Q \le 3(\lambda_1^2 x^T x + \lambda_2^2 x_\tau^T x_\tau + \lambda_3^2 \dot{x}_\tau^T \dot{x}_\tau). \tag{2.8}$$

From inequalities (2.7) and (2.8) we find

$$\big(1 + \sum_{i=1}^{4} \varepsilon_i\big) Q^T Q \le 3\big(1 + \sum_{i=1}^{4} \varepsilon_i\big)(\lambda_1^2 x^T x + \lambda_2^2 x_\tau^T x_\tau + \lambda_3^2 \dot{x}_\tau^T \dot{x}_\tau). \tag{2.9}$$

Engaging (2.5), (2.6), estimates (2.7), and (2.9) in (2.4), we have

$$\begin{aligned} \dot{V}_{(2.1)} \le {} & x^T \big[A^T P + PA + \varepsilon_1^{-1} PP + (1+\varepsilon_2^{-1})A^T A + R + 3\big(1 + \sum_{i=1}^{4} \varepsilon_i\big)\lambda_1^2 I\big] x \\ & + x_\tau^T \big[(1+\varepsilon_3^{-1})B^T B + 3\big(1 + \sum_{i=1}^{4} \varepsilon_i\big)\lambda_2^2 I - (1-\alpha_2)R\big] x_\tau \\ & + \dot{x}_\tau^T \big[(1+\varepsilon_4^{-1})C^T C + 3\big(1 + \sum_{i=1}^{4} \varepsilon_i\big)\lambda_3^2 I - (1-\alpha_2)I\big] \dot{x}_\tau \\ & + 2x^T(A^T B + PB)x_\tau + 2x^T(A^T C + PC)\dot{x}_\tau + 2x_\tau^T B^T C\dot{x}_\tau. \end{aligned} \tag{2.10}$$

Inequality (2.10) can be recast as

$$\dot{V}_{(2.1)} \le \begin{bmatrix} x \\ x_\tau \\ \dot{x}_\tau \end{bmatrix}^T M_1 \begin{bmatrix} x \\ x_\tau \\ \dot{x}_\tau \end{bmatrix}$$

where

$$M_1 = \begin{bmatrix} A_1 & PB + A^T B & PC + A^T C \\ B^T P + B^T A & A_2 & B^T C \\ C^T P + C^T A & C^T B & A_3 \end{bmatrix}.$$

Therefore, $\dot{V}_{(2.1)}$ is negative definite if the matrix M_1 is negative definite. This completes the proof. □

Remark 2.1. We have the following observations:

(i) If $\tau(t) = h$ for some constant $h > 0$, then Theorem 2.1 coincide with Theorem 1 in [25]. Thus system (2.1) and the Lyapunov-Krasovskiǐ functional defined by (2.3) is an extension of the one used in [25];

(ii) If the perturbed function $Q(t) = 0$ in (2.1) then the sufficient condition for stability of the trivial solution is

$$M_2 = \begin{bmatrix} A_4 & A^T B + PB & A^T C + PC \\ B^T A + B^T P & A_5 & B^T C \\ C^T A + C^T P & C^T B & A_6 \end{bmatrix} < 0 \tag{2.11}$$

where

$$A_4 := A^T P + PA + A^T A + R, \quad A_5 := B^T B - (1-\alpha_2)R, \text{ and } A_6 := C^T C - (1-\alpha_2)I.$$

Thus if $\tau(t) = h$ inequality (2.11) coincide with inequality (17) in [25].

Next, we shall discuss the delay-dependent stability criteria for system (2.1). Let $x(t)$ be continuously differentiable on $[-2\alpha_1, -\alpha_1]$, then system (2.1) can be represented as

$$\dot{x}(t) = A_0 x(t) - B\sum_{i=1}^{3}\eta_i - BCx_\tau(t) + BCx_{2\tau}(t) + C\dot{x}_t(t) + Q(t), \tag{2.12}$$

where

$$A_0 := A + B, \ \eta_1 := \int_{t-\tau}^{t} Ax(\theta)d\theta, \ \eta_2 := \int_{t-\tau}^{t} Bx_\tau(\theta)d\theta, \text{ and } \eta_3 := \int_{t-\tau}^{t} Q(\theta)d\theta.$$

A continuously differentiable functional $W = W(x)$ used in this case is defined by

$$W = x^T(t)Px(t) + \sum_{i=1}^{4} W_i, \tag{2.13}$$

where

$$W_1 := \int_{t-\tau}^{t} \dot{x}^T(\theta)\dot{x}(\theta)d\theta, \ W_2 := \int_{t-\tau}^{t} x^T(\theta)R_1 x(\theta)d\theta, \ W_3 := \int_{t-2\tau}^{t} x^T(\theta)R_2 x(\theta)d\theta,$$
$$W_4 = \int_{-\tau}^{0}\int_{t+\mu}^{t} \left[\|Ax(\theta)\|^2 + \|Bx_\tau(\theta)\|^2 + \|Q(\theta)\|^2\right] d\theta d\mu,$$

R_1 and R_2 are positive semi-definite symmetric matrices to be determined later.

Remark 2.2. If $\tau = h$ ($h > 0$ a constant) then the functional (2.13) specialized to that of Park and Won [25].

The stability result for the delay-dependent (2.12) is as follows.

Theorem 2.2. *Suppose there exist positive constants ξ_i $(i = 1, 2, \cdots, 12)$, P, R_1 and R_2 are $n \times n$ symmetric, positive definite matrices which satisfy the following LMI:*

$$M_3 = \begin{bmatrix} A_7 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & A_8 \end{bmatrix} < 0$$

with $21B^T B = (1-\alpha_2)I$ where

$$A_7 := A_0^T P + PA_0 + \gamma_1 A_0^T A_0 + \gamma_2 PP + \alpha_1 A^T A + [(1-2\alpha_2)^{-1}\gamma_3 + (1-\alpha_2)^{-1}\gamma_4]C^T B^T BC$$
$$+ (1-\alpha_2)^{-1}\left[\alpha_1 B^T B + (\xi_7^{-1} + \xi_8^{-1})C^T C\right] + \left[\lambda_1^2 + (1-\alpha_2)\lambda_2^2\right]\gamma_3 I,$$
$$A_8 := \gamma_6 C^T C + \left[\lambda_3^2\gamma_3 - (1-\alpha_2)\right]I, \ \gamma_1 := 1 + 3\alpha_1 + \xi_3^{-1} + \xi_6 + \xi_7 + \xi_{10},$$
$$\gamma_2 := 3(\alpha_1 + \xi_1^{-1}) + \xi_5 + \xi_8 + \xi_9, \ \ \gamma_3 := 3\big(1 + 4\alpha_1 + \xi_3 + \sum_{i=1}^{4}\xi_i\big),$$
$$\gamma_4 := 1 + 3\alpha_1 + \xi_3^{-1} + \xi_5^{-1} + \xi_6^{-1} + \xi_{11} + \xi_{12}, \ \ \gamma_5 := 1 + 3\alpha_1 + \xi_3^{-1} + \xi_9^{-1} + \xi_{10}^{-1} + \xi_{11}^{-1} + \xi_{12}, \textit{and}$$
$$\gamma_6 := 1 + 3\alpha_1 + \xi_4^{-1} + 2\xi_{12}^{-1}.$$

Then the solution of (2.12) *is asymptotically stable.*

Proof. The derivative of the functional (2.13) along solution of (2.12) is given by

$$\begin{aligned}\dot{W}_{(2.12)} &= x^T(A_0^TP + PA_0)x - 2\sum_{i=1}^{3}\eta_i^TB^TPx - 2x_\tau^TC^TB^TPx + 2x_{2\tau}^TC^TB^TPx\\&\quad + 2\dot{x}_\tau^TC^TPx + 2Q^TPx + \sum_{i=1}^{4}\dot{W}_i,\end{aligned} \tag{2.14}$$

where

$$\begin{aligned}\dot{W}_1 &:= x^TA_0^TA_0x + x_\tau^TC^TB^TBCx_\tau + x_{2\tau}^TC^TB^TBCx_{2\tau} + \dot{x}_\tau^TC^TC\dot{x}_\tau - 2x^TA_0^TBCx_\tau\\&+ 2x^TA_0^TBCx_{2\tau} + 2x^TA_0^TC\dot{x}_\tau - 2x_\tau^TC^TB^TBCx_{2\tau} - 2x_\tau^TC^TB^TC\dot{x}_\tau + 2x_{2\tau}^TC^TB^TC\dot{x}_\tau\\&+ \sum_{i=1}^{3}\eta_i^TB^TB\sum_{i=1}^{3}\eta_i - 2x^TA_0^TB\sum_{i=1}^{3}\eta_i + 2\sum_{i=1}^{3}\eta_i^TB^TBCx_\tau - 2\sum_{i=1}^{3}\eta_i^TB^TBCx_{2\tau}\\&- 2\sum_{i=1}^{3}\eta_i^TB^TC\dot{x}_\tau - 2\sum_{i=1}^{3}\eta_i^TB^TQ + 2x^TA_0^TQ - 2x_\tau^TC^TB^TQ + 2x_{2\tau}^TC^TB^TQ + 2\dot{x}_\tau^TC^TQ\\&+ Q^TQ - (1-\dot{\tau})\dot{x}_\tau^T\dot{x}_\tau,\end{aligned} \tag{2.15}$$

$$\dot{W}_2 := x^TR_1x - (1-\dot{\tau})x_\tau^TR_1x_\tau, \tag{2.16}$$

$$\dot{W}_3 := x^TR_2x - (1-2\dot{\tau})x_{2\tau}^TR_2x_{2\tau}, \text{ and} \tag{2.17}$$

$$\dot{W}_4 := \tau\big[\|Ax\|^2 + \|Bx_\tau\|^2 + \|Q(t)\|^2\big] - (1-\dot{\tau})\int_{t-\tau}^{t}\big[\|Ax(\theta)\|^2 + \|Bx_\tau(\theta)\|^2 + \|Q(\theta)\|^2\big]d\theta. \tag{2.18}$$

By Lemma 2.1, the following inequalities are fulfilled:

$$\begin{aligned}-2\sum_{i=1}^{3}\eta_i^TB^TPx &\le 3\tau x^TPPx + \tau^{-1}\sum_{i=1}^{3}\eta_i^TB^TB\sum_{i=1}^{3}\eta_i;\\-2\sum_{i=1}^{3}\eta_i^TB^TA_0x &\le 3\tau x^TA_0^TA_0x + \tau^{-1}\sum_{i=1}^{3}\eta_i^TB^TB\sum_{i=1}^{3}\eta_i;\\2\sum_{i=1}^{3}\eta_i^TB^TBCx_\tau &\le 3\tau x_\tau^TC^TB^TBCx_\tau + \tau^{-1}\sum_{i=1}^{3}\eta_i^TB^TB\sum_{i=1}^{3}\eta_i;\\-2\sum_{i=1}^{3}\eta_i^TB^TBCx_{2\tau} &\le 3\tau x_{2\tau}^TC^TB^TBCx_{2\tau} + \tau^{-1}\sum_{i=1}^{3}\eta_i^TB^TB\sum_{i=1}^{3}\eta_i;\\-2\sum_{i=1}^{3}\eta_i^TB^TC\dot{x}_\tau &\le 3\tau\dot{x}_\tau^TC^TC\dot{x}_\tau + \tau^{-1}\sum_{i=1}^{3}\eta_i^TB^TB\sum_{i=1}^{3}\eta_i;\end{aligned} \tag{2.19}$$

$$\begin{aligned}-2\sum_{i=1}^{3}\eta_i^TB^TQ &\le 3\tau Q^TQ + \tau^{-1}\sum_{i=1}^{3}\eta_i^TB^TB\sum_{i=1}^{3}\eta_i;\\2Q^TPx &\le \xi_1Q^TQ + \xi_1^{-1}x^TPPx;\\2Q^TA_0x &\le \xi_2Q^TQ + \xi_2^{-1}x^TA_0^TA_0x;\\-2Q^TBCx_\tau &\le \xi_3Q^TQ + \xi_3^{-1}x_\tau^TC^TB^TBCx_\tau;\\2Q^TBCx_{2\tau} &\le \xi_3Q^TQ + \xi_3^{-1}x_{2\tau}^TC^TB^TBCx_{2\tau};\\2Q^TC\dot{x}_\tau &\le \xi_4Q^TQ + \xi_4^{-1}\dot{x}_\tau^TC^TC\dot{x}_\tau.\end{aligned} \tag{2.20}$$

First collate the like-terms in Q^TQ from (2.15) and (2.20) using estimate (2.2) we find

$$\Big(1 + 3\tau + \xi_3 + \sum_{i=1}^{3}\xi_i\Big)Q^TQ \le 3\Big(1 + 3\tau + \xi_3 + \sum_{i=1}^{3}\xi_i\Big)\big(\lambda_1^2x^Tx + \lambda_2^2x_\tau^Tx_\tau + \lambda_3^2\dot{x}_\tau^T\dot{x}_\tau\big), \tag{2.21}$$

also terms involving η_i from (2.15) and (2.19) are

$$\sum_{i=1}^{3}\eta_i^T B^T B\sum_{i=1}^{3}\eta_i + 18\tau^{-1}\sum_{i=1}^{3}\eta_i^T B^T B\sum_{i=1}^{3}\eta_i \le 21B^T B\int_{t-\tau}^{t}\big[\|Ax(\theta)\|^2 + \|Bx_\tau(\theta)\|^2 + \|Q(\theta)\|^2\big]d\theta, \tag{2.22}$$

and the first term of (2.18) is

$$\tau\big[\|Ax\|^2 + \|Bx_\tau\|^2 + \|Q(t)\|^2\big] \le \tau\big[x^T(A^TA + 3\lambda_1^2 I)x + x_\tau^T(B^TB + 3\lambda_2^2 I)x_\tau + 3\lambda_3^2\dot{x}_\tau^T\dot{x}_\tau\big]. \tag{2.23}$$

Next, engaging (2.15)-(2.17), second term of (2.18), and inequalities (2.19)-(2.23) in (2.14), noting that $\tau \le \alpha_1$, $\dot{\tau} \le \alpha_2$, and $21B^TB = (1-\alpha_2)I$, we obtain

$$\begin{aligned}\dot{W}_{(2.12)} &= x^T\big[A_0^TP + PA_0 + (1 + 3\alpha_1 + \xi_2^{-1})A_0^TA_0 + (3\alpha_1 + \xi_1^{-1})PP + \alpha_1A^TA + R_1 + R_2\\ &+ 3(1 + 4\alpha_1 + \xi_3 + \sum_{i=1}^{4}\xi_i)\lambda_1^2 I\big]x + x_\tau^T\big[(1 + 3\alpha_1 + \xi_3^{-1})C^TB^TBC + \alpha_1B^TB - (1-\alpha_2)R_1\\ &+ 3(1 + 4\alpha_1 + \xi_3 + \sum_{i=1}^{4}\xi_i)\lambda_2^2 I\big]x_\tau + x_{2\tau}^T\big[(1 + 3\alpha_1 + \xi_2^{-1})C^TB^TBC - (1-2\alpha_2)R_2\big]x_{2\tau}\\ &+ \dot{x}_\tau^T\big[(1 + 3\alpha_1 + \xi_4^{-1})C^TC + 3(1 + 4\alpha_1 + \xi_3 + \sum_{i=1}^{4}\xi_i)\lambda_3^2 I - (1-\alpha_2)\big]\dot{x}_\tau\\ &- 2x^T(PBC + A_0^TBC)x_\tau + 2x^T(A_0^TC + PC)\dot{x}_\tau + 2x^T(PBC + A_0^TBC)x_{2\tau}\\ &- 2x_\tau^TC^TB^TBCx_{2\tau} - 2x_\tau^TC^TB^TC\dot{x}_\tau + 2x_{2\tau}^TC^TB^TC\dot{x}_\tau.\end{aligned} \tag{2.24}$$

Rearranging the mixed terms in (2.24) using Lemma 2.1, the following inequalities hold:

$$\begin{aligned}-2x^TPBCx_\tau &\le \xi_5x^TPPx + \xi_5^{-1}x_\tau^TC^TB^TBCx_\tau,\\ -2x^TA_0^TBCx_\tau &\le \xi_6x^TA_0^TA_0x + \xi_6^{-1}x_\tau^TC^TB^TBCx_\tau,\\ 2x^TA_0^TC\dot{x}_\tau &\le \xi_7x^TA_0^TA_0x + \xi_7^{-1}x_\tau^TC^TCx_\tau,\\ 2x^TPC\dot{x}_\tau &\le \xi_8x^TPPx + \xi_8^{-1}x_\tau^TC^TCx_\tau,\\ -2x^TPBCx_{2\tau} &\le \xi_9x^TPPx + \xi_9^{-1}x_{2\tau}^TC^TB^TBCx_{2\tau},\\ -2x^TA_0^TBCx_{2\tau} &\le \xi_{10}x^TA_0^TA_0x + \xi_{10}^{-1}x_{2\tau}^TC^TB^TBCx_{2\tau},\\ -2x_\tau^TC^TB^TBCx_{2\tau} &\le \xi_{11}x_\tau^TC^TB^TBCx_\tau + \xi_{11}^{-1}x_{2\tau}^TC^TB^TBCx_{2\tau},\\ -2x_\tau^TC^TB^TC\dot{x}_\tau &\le \xi_{12}x_\tau^TC^TB^TBCx_\tau + \xi_{12}^{-1}\dot{x}_\tau^TC^TC\dot{x}_\tau,\\ 2x_{2\tau}^TC^TB^TC\dot{x}_\tau &\le \xi_{12}x_{2\tau}^TC^TB^TBCx_{2\tau} + \xi_{12}^{-1}\dot{x}_\tau^TC^TC\dot{x}_\tau.\end{aligned} \tag{2.25}$$

Employing inequalities (2.25) in (2.24), we obtain

$$\begin{aligned}\dot{W}_{(2.12)} &\le x^T\big[A_0^TP + PA_0 + \gamma_1A_0^TA_0 + \gamma_2PP + \alpha_1A^TA + \lambda_1^2\gamma_3I + R_1 + R_2\big]x\\ &+ x_\tau^T\big[\gamma_4C^TB^TBC + \alpha_1B^TB + (\xi_7^{-1} + \xi_8^{-1})C^TC + \gamma_3\lambda_2^2I - (1-\alpha_2)R_1\big]x_\tau\\ &+ x_{2\tau}^T\big[\gamma_5C^TB^TBC - (1-2\alpha_2)R_2\big]x_{2\tau} + \dot{x}_\tau^T\big[\gamma_6C^TC + \gamma_3\lambda_3^2I - (1-\alpha_2)\big]\dot{x}_\tau.\end{aligned} \tag{2.26}$$

Choose $R_1 := (1-\alpha_2)^{-1}\big[\gamma_4C^TB^TBC + \alpha_1B^TB + (\xi_7^{-1} + \xi_8^{-1})C^TC + \gamma_3\lambda_2^2I\big]$ and $R_2 := (1-2\alpha_2)^{-1}\gamma_5C^TB^TBC$ inequality (2.26) yields

$$\dot{W}_{(2.12)} \le \begin{bmatrix} x \\ x_\tau \\ x_{2\tau} \\ \dot{x}_\tau \end{bmatrix}^T M_3 \begin{bmatrix} x \\ x_\tau \\ x_{2\tau} \\ \dot{x}_\tau \end{bmatrix}$$

where

$$M_3 := \begin{bmatrix} A_7 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & A_8 \end{bmatrix},$$

$$A_7 := A_0^T P + PA_0 + \gamma_1 A_0^T A_0 + \gamma_2 PP + \alpha_1 A^T A + \left[(1-2\alpha_2)^{-1}\gamma_3 + (1-\alpha_2)^{-1}\gamma_4\right] C^T B^T BC$$
$$+ (1-\alpha_2)^{-1}\left[\alpha_1 B^T B + (\xi_7^{-1} + \xi_8^{-1}) C^T C\right] + \left[\lambda_1^2 + (1-\alpha_2)^{-1}\gamma_3 I\right] \text{ and}$$
$$A_8 := \gamma_6 C^T C + \left[\lambda_3^2 \gamma_3 - (1-\alpha_2)\right] I.$$

The function $\dot{W}_{(2.12)}$ is negative definite if M_3 is negative definite, thus the solution of system (2.12) is asymptotically stable. □

3. Examples

Example 3.1. Consider the following neutral delay differential system

$$\dot{x} = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} x + \begin{bmatrix} 2 & 10 \\ 10 & 3 \end{bmatrix} x_\tau + \begin{bmatrix} 6 & 7 \\ 7 & 8 \end{bmatrix} \dot{x}_\tau + Q(t). \tag{3.1}$$

Equations (2.1) and (3.1) established that

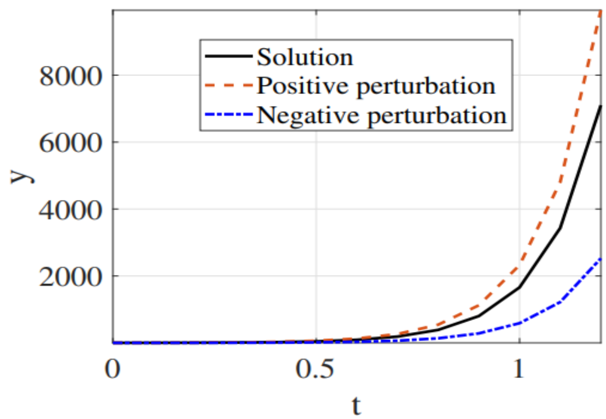

Figure 1. Path of solution of (3.1) and its perturbations in the neighbourhood $(-1, 1)$.

$$A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 10 \\ 10 & 3 \end{bmatrix}, \quad C = \begin{bmatrix} 6 & 7 \\ 7 & 8 \end{bmatrix}$$

and since β_i $(i = 1, 2, 3)$ is positive, the nonlinear perturbation Q is estimated to be

$$\|Q(t)\| \leq 0.1\|x\| + 0.01\|x_\tau\| + 0.001\|\dot{x}_\tau\|$$

so that $\beta_1 = 0.1,\ \beta_2 = 0.01$ and $\beta_3 = 0.001$. Moreover, since $0 < \alpha_2 < 1$ it follows that for any $\alpha_2 \in [0.001,\ 0.9]$ with $\varepsilon_1 = 0.1,\ \varepsilon_2 = 0.11,\ \varepsilon_3 = 0.111$ and $\varepsilon_4 = 0.1111$ the matrix $M_2 < 0$ if matrices

$$P := \begin{bmatrix} 2 & 0.1 \\ 0.1 & 3 \end{bmatrix} \text{ and } R := \begin{bmatrix} 4 & 0.01 \\ 0.01 & 2 \end{bmatrix}.$$

If $\alpha_2 = 0$, the case discussed in [25] is verified, i.e., $M_2 < 0$, thus the solution of system (3.1) is stable.
In addition, the exact solution of (3.1) using Matlab software is shown in Figure 1 in the neighbourhood $(-1, 1)$ with $0 < \alpha_2 < 1$, i.e., $\alpha_2 \in [0.001, 09]$, hence the solutions of (3.1) is not only stable but asymptotically stable.

4. Conclusion

In this paper, we have investigated the asymptotic stability of neutral differential systems with variable delay and nonlinear perturbations. We have established sufficient conditions for the asymptotic stability of the systems using Lyapunov-Krasovskiĭ functionals technique. The results obtained in this paper provide important insights into the stability properties of neutral differential systems with variable delay and nonlinear perturbations. Our findings contribute to the existing body of knowledge on stability analysis of time-delay systems and have potential applications in various engineering and scientific fields. Further research can be conducted to extend the results to more general classes of systems and to explore practical implementation of the stability conditions derived in this paper.

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