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Study of Fixed Points and Chaos in Wave Propagation for the Generalized Damped Forced Korteweg-de Vries Equation using Bifurcation Analysis

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ABSTRACT In this article, we consider the Generalized Damped Forced Korteweg-de Vries (GDFKdV) equation. The forcing term considered is of the form $F(U) = U(U - v_1)(U - v_2)$, where v_1 and v_2 are free parameters. We investigate the behaviour of fixed points evaluated for the corresponding dynamical system of our model problem. With respect to these fixed points, we investigate the effects of a few significant parameters involved in the model, namely, the free parameters v_1 and v_2, the nonlinear, dispersion and damping coefficients using the tools from bifurcation analysis. We also obtain the wave plots for the critical values of the nonlinear and dispersion coefficients for which the system becomes unstable and exhibit chaotic behaviour. We confirm the chaos in our dynamical system under various conditions with the help of Lyapunov exponents.

KEYWORDS
GDFKdV equation
Nonlinear dynamics
Chaos
Wave propagation
Lyapunov exponent
Phase portraits

INTRODUCTION

In dispersive media, weakly non-linear long wave propagation is described by the universal mathematical model KdV. It is also used as a model to examine several quantum mechanics-related phenomena in theoretical physics. In many real-world applications, it is recognized that higher-order non-linearity should be included in the KdV equation in order to explain the physical phenomenon, which leads to a more generalised KdV equation. This equation accounts for the wide range of applicability; Shallow-water gravity waves, ion-acoustic waves in collisionless plasma, internal waves in the atmosphere and ocean, and waves in bubbly fluids are only a few examples of the physical uses of the generalized KdV equation (Stuhlmeier 2009; Khater 2022; Vasavi *et al.* 2021; Crighton 1995).

In this article, we study the Generalized Korteweg-de Vries (GKdV) equation with damping and external force of the following form

$$U_t + PU^n U_x + QU_{xxx} + SU = \gamma F(U, x, t, v_i), \tag{1}$$

where U denotes the excitation, t, x denote time, and space coordinates, respectively. P, Q, and S denote coefficients of nonlinearity, dispersion, damping, respectively; n is the exponent which controls the non-linearity. The coefficients P and Q, which can either be constants or functions of x and t, are determined by the characteristics of the medium. The Generalized KdV equation describes the combined effect of the basic long wave dispersion (U_{xxx}) and, ($U^n U_x, n > 0$) which has the same form as that in the KdV or 1-dimensional Navier-Stokes equations, stabilizes by transferring energy between large and small scales (Alshenawy *et al.* 2020; Zhang 2014; El 2007). In equation 1, the function F denotes an additional forcing term. The parameter γ is the force coefficient. The range of γ governs the strength of the force field. For instance, for $\gamma > 1$, $\gamma \approx 1$, and $\gamma \ll 1$ the force field can be considered strong, weak or very weak.

In this study, we consider $F(U, x, t, v_i) \equiv F(U) = U(U - v_1)(U - v_2)$. The roots of the polynomial $F(U) = 0$ are $U = 0, v_1$ and v_2. The parameters v_1 and v_2 are referred to as forcing parameters. Under certain conditions imposed on these parameters, the forcing term in its present form may act as an attenuator or amplifier for the solitary waves (Engelbrecht and Peipman 1992). Such forcing terms have been used to study the wave propagation in different media within the framework of a perturbed KdV equation (see for example (Engelbrecht and Khamidullin 1988; Engelbrecht and Peipman 1992; Engelbrecht 1991). See also (Peterson and Salupere 1997; Peterson 1997) for the numerical treatment of the KdV equation with forcing term in cubic polynomial form with periodic boundary conditions and the harmonic initial condition. It is important to

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note that due to the presence of the forcing term in the model equation 1 does not satisfy the conservation laws.

For $n = 1, 2$, the well-developed techniques are available to obtain the analytical solutions of KdV equation and its variants without forcing term in the right hand side, refer to (Wazwaz 2004; Zuo and Zhang 2011) and references therein. However, for $n \geq 3$, the generalized KdV equation becomes non-integrable. According to (Merle 2001; Bona *et al.* 1987; Hereman and Takaoka 1990; Zabusky 1967), the solutions of GKdV are stable for $n \leq 4$, unstable for $n \geq 6$ and conditionally stable for $n = 5$. Even for the numerical treatment of generalized KdV equation for $n \geq 5$, one has to take a very small mesh width in space or time step in order to obtain an acceptable computed solution. The problem becomes more challenging when $P >> Q$ (Alvarado and Omel'yanov 2012). This poses serious limitations on the conventional analytical and numerical methods. To analyse such intricate non-linear systems, it is desirable to employ tools available in the bifurcation analysis (Guckenheimer and Holmes 2013). Bifurcation analysis is a mathematical framework to study qualitative changes to investigate the unexpected appearance, disappearance, or change in the stability of equilibrium points with respect to certain parameters or perturbations. Bifurcation analysis has long been used to investigate dynamical systems emerging from varied real world problems, refer to (Hilborn *et al.* 2000) and references therein.

Many authors have studied the KdV equation and its variants using bifurcation analysis. Zao Li $et.el.$ (Li *et al.* 2021) studied the fractional generalized Hirota–Satsuma coupled KdV equations with the help of bifurcation theory. Yiren Chen and Shaoyong Li (Chen and Li 2021) investigated the generalized KdV-mKdV-like equation with the help of bifurcation analysis; see also (Saha and Chatterjee 2014; Tamang and Saha 2020) for similar studies. To the best of our knowledge in most of these studies, the authors have considered fixed values of the parameters involved in the equations. The novelty of this work presented here is that we have not put any restriction on the range of any of these parameters and investigated the nature of the dynamical system corresponding to the generalized damped forced KdV equation given by equation 1 with respect to all the equilibrium points. In view of the facts mentioned above, we analyse our model problem for $n \geq 3$, $P >> Q$ and various values of S using bifurcation tools. The authors (Chadha *et al.* 2023; Tomar *et al.* 2023; Chen and Li 2021) used bifurcation analysis to study the behaviour of the dynamical system for the equilibrium points and found chaotic behaviours in the Damped Forced KdV and Generalized KdV equations under certain conditions on the parameters involved. The interested reader may also refer to (Haidong *et al.* 2023; Sami *et al.* 2022; Xu *et al.* 2022) for some recent work on fractional order dynamical systems and their applications where the authors have used phase portraits, time series plots, the Lyapunov spectrum and other related tools from the bifurcation analysis to study the chaotic behaviour of the systems.

The organization of this study is as follows: First, we evaluate three different equilibrium points obtained from a three-dimensional dynamical system corresponding to the generalized DFKdV equation. We study the behaviour of these equilibrium points and wave propagation in the dynamical system using the bifurcation analysis. For the first equilibrium point, we investigate the system with respect to the free parameters v_1, v_2, and S. It is important to mention that the other two equilibrium points have locational dependence on v_1, v_2, and S, which further complicates the problem. For these equilibrium points, we investigate the system for $n = 5$ and P and Q ratio up to 10^4. The system exhibits chaotic behaviour. These theoretical findings are confirmed by the wave propagation plots and the Lyapunov exponents. We conclude with our major findings in this study.

BIFURCATION ANALYSIS OF GDFKDV EQUATION

In this section, we investigate the dynamical behaviour of the generalized damped forced KdV equation 1 with forcing term $F(U) = U(U - v_1)(U - v_2)$ and $\gamma = 1$ with respect to the different equilibrium points.

The Generalized DFKdV equation is

$$U_t + PU^n U_x + QU_{xxx} + SU = F(U). \tag{2}$$

Consider a wave transformation,

$$U(x,t) = U(z),\ z = (x - ct). \tag{3}$$

Using the wave transformation in equation 2, we get the ordinary differential equation:

$$-cU_z + PU^n U_z + QU_{zzz} + SU = F(U). \tag{4}$$

Equation 4 can be rewritten as follows

$$\begin{aligned} U' &= V, \\ V' &= W, \\ W' &= \frac{1}{Q}(cV - PU^n V - SU + U(U - v_1)(U - v_2)). \end{aligned} \tag{5}$$

By solving this system of equation, we obtain the equilibrium points $(0, 0, 0)$, $(h, 0, 0)$ and $(k, 0, 0)$, here $h, k = \frac{1}{2}(v_1 + v_2 \mp ((v_1 - v_2)^2 + 4S)^{1/2})$.

For the dynamical system equation 5, the Jacobian matrix is

$$J = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{-nPU^{n-1}V - S + 3U^2 - 2(v_1+v_2)U + v_1 v_2}{Qw_0^2} & \frac{c - PU^n}{Qw_0^2} & 0 \end{bmatrix}$$

Corresponding characteristic equation is

$$\begin{aligned} &\frac{-1}{Q}(QE^3 + (PU^n - c)E + \\ &(S + 2U(v_1 + v_2) - 3U^2 - v_1 v_2 + PU^{n-1}V^n) = 0. \end{aligned} \tag{6}$$

The eigenvalues for this system are

$$\begin{aligned} E_1 &= T + \frac{Uc - PUU^n}{3QUT}, \\ E_{2,3} &= -\left(\frac{T}{2} + \frac{Uc - PUU^n}{6QUT}\right) \pm \frac{\sqrt{3}i}{2}\left(T - \frac{Uc - PUU^n}{3QUT}\right). \end{aligned} \tag{7}$$

Here,

$$\begin{aligned} T &= \left(\left(\frac{A}{4Q^2U^2} - \frac{(Uc - PUU^n)^3}{27Q^3U^3}\right)^{1/2} - \frac{A}{2Q^2U^2}\right)^{1/3}, \\ A &= (SU + 2U^2 v_1 + 2U^2 v_2 + 3U^3 - Uv_1 v_2 + PU^n Vn)^2. \end{aligned}$$

To study the behaviour of the dynamical system equation 5, it is important to investigate the nature of the eigenvalues for all the equilibrium points.

For a three dimensional system: $(E_1, E_2, E_3) = (-, -, -)$ corresponds to fixed point, $(E_1, E_2, E_3) = (0, 0, -)$ corresponds to limit cycle, $(E_1, E_2, E_3) = (0, 0, -)$ corresponds to two dimensional annulus, $(E_1, E_2, E_3) = (+, +, -)$ correspond to unstable limit cycle, $(E_1, E_2, E_3) = (+, 0, -)$ correspond to strange attractor, $(E_1, E_2, E_3) = (+, 0, 0)$ corresponds to strange attractor, (Layek *et al.* 2015).

For study of these equilibrium points, this investigation is divided into two sections: at first equilibrium point $(0, 0, 0)$ and second equilibrium point $(h, 0, 0)$ and third equilibrium point $(k, 0, 0)$.

At equilibrium point $(0, 0, 0)$**:**

The first equilibrium point $(0, 0, 0)$ is independent of the parameters. In this case, we are investigating the effect of the damping parameter S and the forcing parameters v_1, and v_2. Exactly at the equilibrium point, we could not find any interesting feature worth reporting. So, we take a point $(1e-5, 1e-5, 1e-5)$ which is in the close vicinity of the equilibrium point.

In Figure 1, we have generated the phase portraits which show the quasi-periodic movement in trajectories for the constant value of the parameters $c = 0.7; P = 2; Q = 0.2; v_1 = 0.2; v_2 = 0.5; n = 3$, and $S = 0.15$ in the time interval $0 : \pi/1000 : 6\pi$. The projections in $U-V$, $V-W$, and $U-W$ planes are shown in Figure 1 (a), (b), (c), respectively.

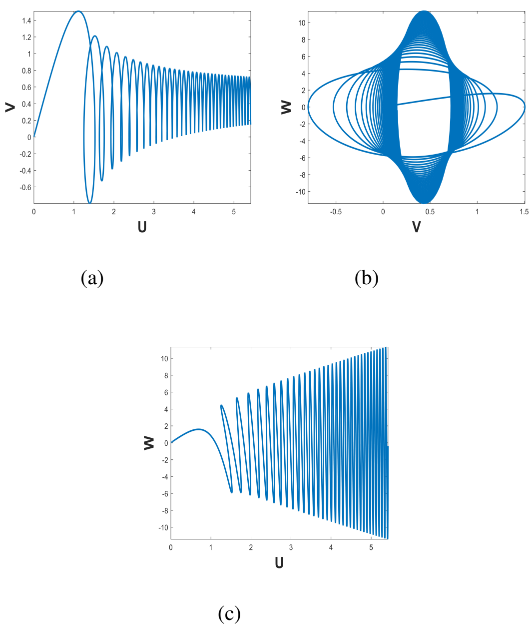


Figure 1 Phase portraits of the dynamical system equation 5 with respect to first equilibrium point $(1e-5, 1e-5, 1e-5)$. Parameters are $c = 0.7; P = 2; Q = 0.2; v_1 = 0.2; v_2 = 0.5; n = 3$, and $S = 0.15$. Time interval is $0 : \pi/1000 : 6\pi$.

Figure 2 $(a), (b)$, and (c) exhibit the quasi-periodic behaviour of two trajectories for the range of parameter $v_1 \in [0.1, 0.97)$; $v_1 = 1.1$ is the break point for the system. These figures are generated for the four representing values from this range, $v_1 = [0.1, 0.8, 0.95, 0.97]$ to show the complete behaviour for the defined range. The value of other parameters are $c = 0.7; P = 2; Q = 0.2; v_2 = 0.5; n = 3$, and $S = 0.15$; time interval considered is $0 : \pi/1000 : 6\pi$. This behaviour of the dynamical system can be justified by the eigenvalues for this equilibrium point, which are given in Table 1. We observe that we have two negative and one positive eigenvalues. Since one of the eigenvalue is always possible for this range of v_1, the equilibrium point will be unstable and system will be weak chaotic. In this forcing term, we have one more parameter v_2 for which we found almost identical behaviour corresponding to v_1. Thus, plots for v_2 are not presented here.

For the damping parameter S, we have investigated the nature of the trajectories for the four representing values from the range of $S \in [0.1, 0.48]$. From Figure 3 $(a), (b)$, and (c), it is evident that the movement of trajectories is quasi-periodic and $S = 0.48$ is the break-point which can be seen in Figure 3 (d). The corresponding behaviour of

Table 1 Eigenvalues of the Jacobian matrix with respect to equilibrium points $(1e-5, 1e-5, 1e-5)$ for $v_1 \in [0.1, 0.97]$. Other parameters are $c = 0.7; P = 2; Q = 0.2; v_2 = 0.5; n = 3$, and $S = 0.15$.

v_1	E_1	E_2	E_3
0.1	1.7948	0.1437	−1.9385
0.8	2.0288	−0.3718	−1.6570
0.95	2.0700	−0.5000	−1.5701
0.97	2.0753	−0.5183	−1.5570

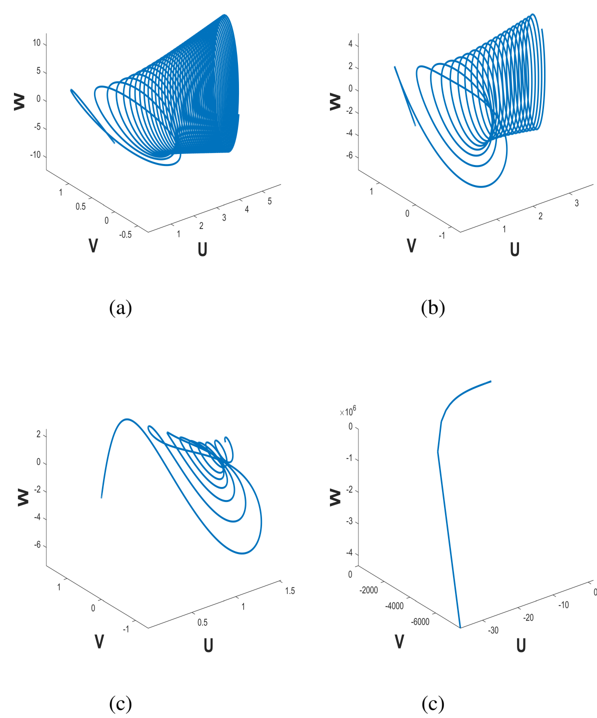


Figure 2 Three dimensional phase portraits of the dynamical system equation 5 with respect to the range value of parameter $v_1 = [0.1, 0.8, 0.95, 0.97]$ are shown for first equilibrium point $(1e-5, 1e-5, 1e-5)$. Parameters are $c = 0.7; P = 2; Q = 0.2; v_2 = 0.5; n = 3$, and $S = 0.15$. Time interval is $0 : \pi/1000 : 6\pi$.

the dynamical system is strongly chaotic, which can be justified by the nature of the eigenvalues shown in Table 2.

Table 2 Eigenvalues of the Jacobian matrix with respect to equilibrium points $(1e-5, 1e-5, 1e-5)$ for $S \in [0.1, 0.48]$. Other parameters are $c = 0.7; P = 2; Q = 0.2; v_2 = 0.5; n = 3$, and $v_1 = 0.2$.

S	E_1	E_2	E_3
0.1	1.7948	0.1437	−1.9385
0.3	1.6009	0.4556	−2.0565
0.45	1.3573	0.7768	−2.1341
0.48	1.2674	0.8813	−2.1487

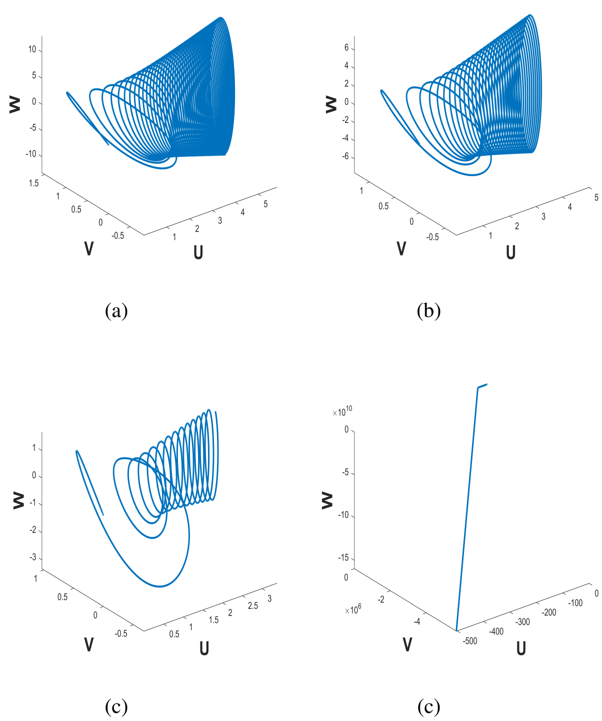


Figure 3 Three dimensional phase portraits of the dynamical system equation 5 with respect to the range value of parameter $S = [0.1, 0.3, 0.45, 0.48]$ are shown for first equilibrium point $(1e-5, 1e-5, 1e-5)$. Parameters are $c = 0.7; P = 2; Q = 0.2; v_2 = 0.5; n = 3$, and $S = 0.15$. Time interval is $0 : \pi/1000 : 6\pi$.

Study of the second equilibrium point $(h, 0, 0)$ and third equilibrium point $(k, 0, 0)$ here, $(h, k) \equiv (\frac{1}{2}(v_1 + v_2 \mp ((v_1 - v_2)^2 + 4S)^{1/2})$:

In this case, we investigate the behaviour of the dynamical system equation 5 with respect to the second $(h, 0, 0)$, and third equilibrium points $(k, 0, 0)$. These equilibrium points involve other three parameters v_1, v_2, and S. The location of these equilibrium points may vary depending on the range of these parameters.

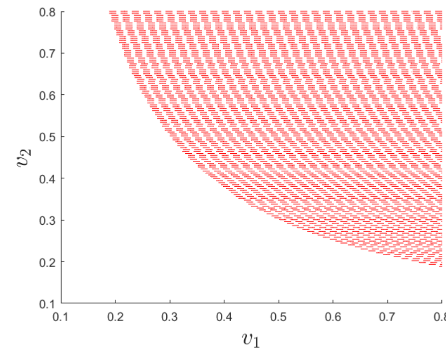


Figure 4 Behaviour of the second equilibrium point of the dynamical system equation 5 for the range the parameters v_1, and v_2. Here $v_1 \in (0.1, 0.8), v_2 \in (0.1, 0.8)$, and $S = 0.5$.

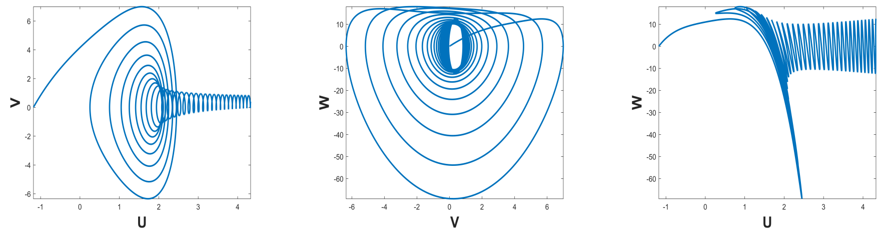
In Figure 4, the coloured portion depicts the nature of the second equilibrium point. Here the boundary of the shaded region shows the conversion of the nature of the equilibrium point from negative to positive. Below the boundary, the value of the equilibrium point is negative and it is positive in the shaded region. From this figure, we get three different ranges for the parameters v_1, and v_2 for which the nature of the second equilibrium point changes from negative to positive and tends to zero at the boundary.

To see the behaviour of the dynamical system for these equilibrium points, we have generated the phase portraits. The Figure 5(a), (b), and (c) show the movement of the trajectories in $U - V$, $V - W$, and $U - W$ planes, respectively for the equilibrium point $(h, 0, 0)$. The values of the parameters involved are as follows: $c = 0.7; P = 2; Q = 0.2; v_1 = 0.01; v_2 = 0.5; n = 3$, and $S = 2$ and taken time interval is $0 : \pi/1000 : 6\pi$.

For the comparison purpose, while studying the nature of the third equilibrium point $(k, 0, 0)$, we have considered the same parameter values as for the second equilibrium point $(h, 0, 0)$ and generated few phase portraits. For these same value of the parameters, the behaviour of both the equilibrium points is the same but the location of both the equilibrium points is different. On this basis, the movement in the trajectories is totally different. They are depicted by the phase portraits shown in Figure 6(a), (b), and (c).

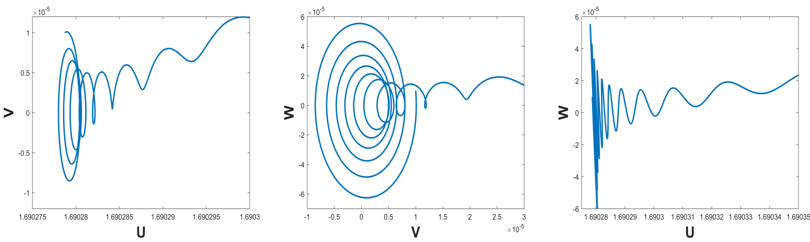
This generalized DFKdV equation is having two more important parameters: one is the non-linear parameter P and the other one is the dispersion parameter Q. The ratio of these two parameters may significantly affect the nature of the wave propagation for this dynamical system. To see the effect of the ratio of these parameters, we present some wave propagation plots shown in Figure 7 and Figure 8. For the second equilibrium point, the nature of the wave is quasi-periodic for $\frac{P}{Q} = 10^2$, refer to Figure 7(a). But when we increase the ratio $\frac{P}{Q} = 10^4$, the oscillations are significantly increased in the waves, and the system becomes chaotic; this is clearly visible in Figure 7(c). The value of the parameters considered to generate these plots are as follows: $c = 0.7, v_1 = 0.2; v_2 = 0.5; n = 5$, and $S = 0.5$. Time interval is $0 : \pi/100 : 2\pi$.

For the third equilibrium point, for the same value of the parameters considered for the second equilibrium point, the wave propagation is quasi-periodic; refer to Figure 8(a). For a higher ratio of P, and Q, the oscillations become more complex, shown in Figure 8(b), and (c). This suggests that the system may be a chaotic system. To confirm this, we use the Lyapunov exponents (Hilborn *et al.* 2000). From Figure 7(d), (e), and (f) and Figure 8(d), (e), and (f), it is clearly visible that one of the Lyapunov exponent is always positive. This confirms that the system is chaotic.



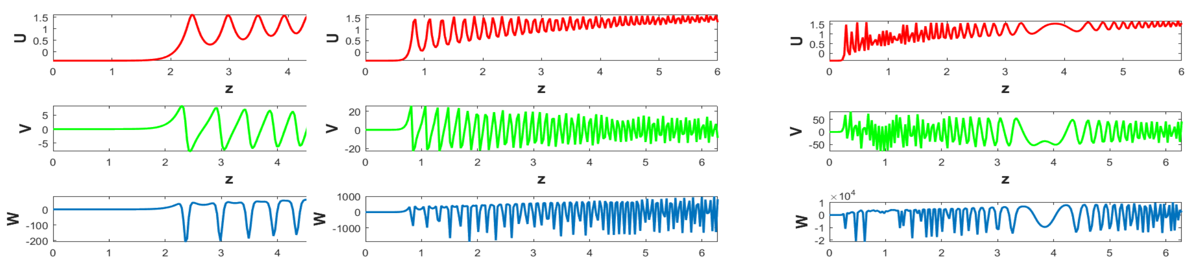
(a) (b) (c)

Figure 5 Phase portraits of the dynamical system equation 5 with respect second equilibrium point $(h, 0, 0)$. Parameters are $c = 0.7; P = 2; Q = 0.2; v_1 = 0.01; v_2 = 0.5; n = 3$, and $S = 2$. Time interval is $0 : \pi/1000 : 6\pi$.

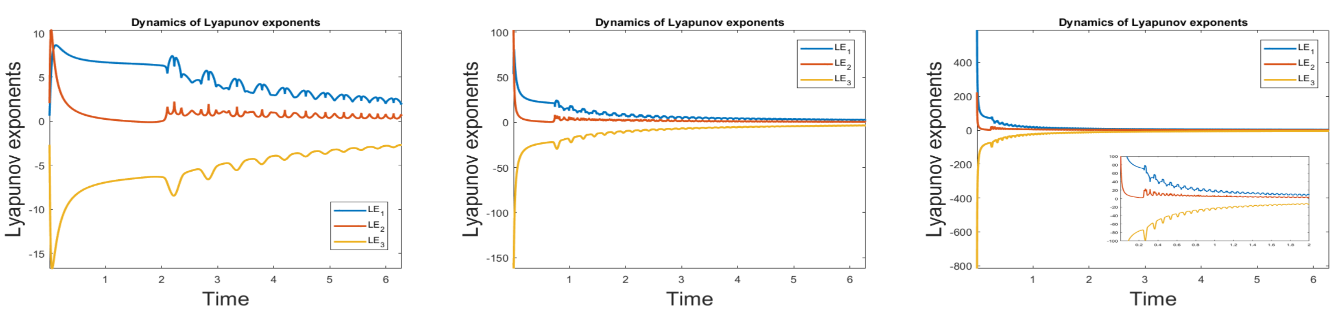


(a) (b) (c)

Figure 6 Phase portraits of the dynamical system equation 5 with the third equilibrium point $(k, 0, 0)$. Corresponding parameters are same as in Figure 5.



(a) (b) (c)



(d) (e) (f)

Figure 7 Wave propagation and the Lyapunov exponent plots of the dynamical system equation 5 with respect to second equilibrium point $(h, 1e-5, 1e-5)$ for the ratio between non-linear and dispersion parameters. Parameters are $c = 0.7; P = 2; Q = 2 * [1e-2, 1e-3, 1e-4]; v_1 = 0.2; v_2 = 0.5; n = 5$, and $S = 0.5$. Time interval is $0 : \pi/1000 : 2\pi$.

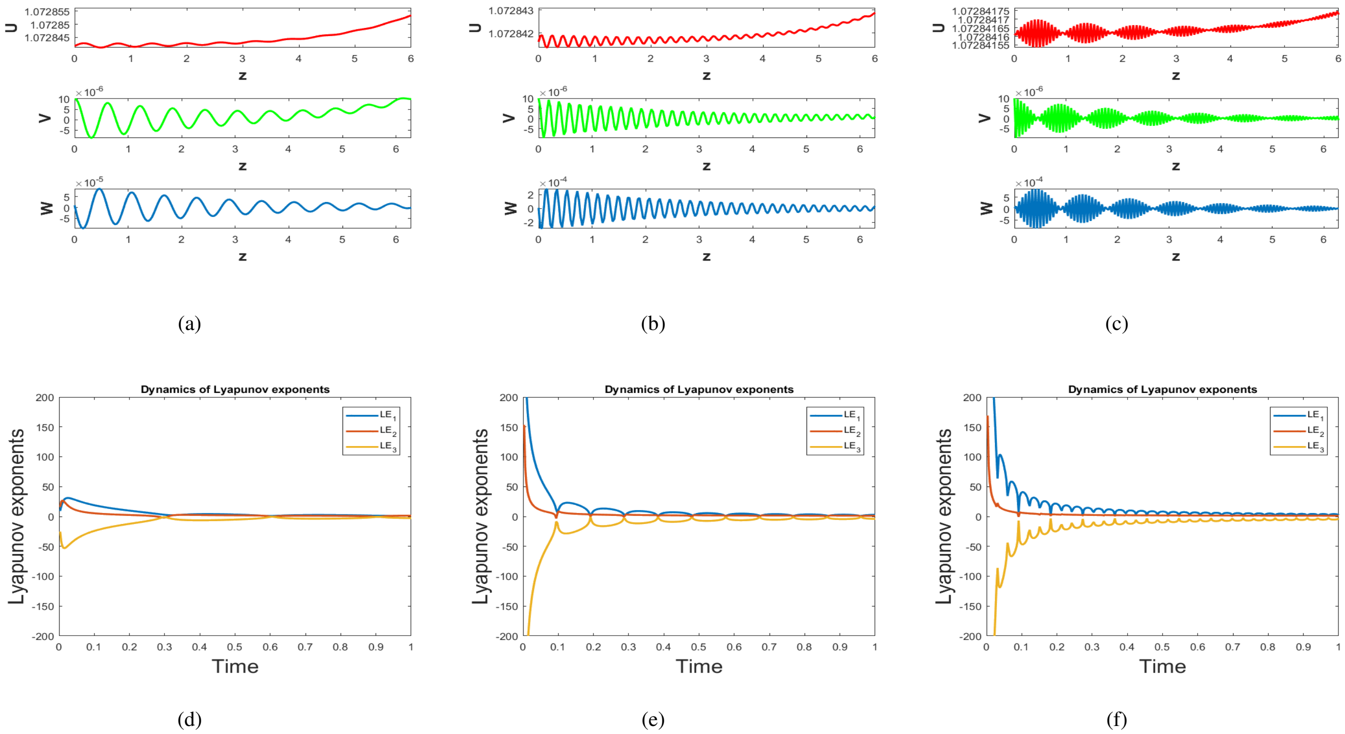


Figure 8 Wave propagation and the Lyapunov exponent plots of the dynamical system equation 5 with respect to third equilibrium point $(k, 1e-5, 1e-5)$ for the ratio between non-linear and dispersion parameters. Parameters are $c = 0.7; P = 2; Q = 2 * [1e-2, 1e-3, 1e-4]; v_1 = 0.2; v_2 = 0.5; n = 5$, and $S = 0.5$. Time interval is $0 : \pi/1000 : 2\pi$.

CONCLUSION

In this study, we studied a higher-order non-linear generalized damped forced KdV equation by employing the tools available in the bifurcation analysis such as phase portraits, time-series plots, Lyapunov exponents etc. The model equation was converted into a three dimensional dynamical system which was investigated for certain parameters involved, namely, P, Q, S which denote the coefficients of non-linearity, dispersion, and damping respectively. Furthermore, the dynamical system was investigated for two forcing parameters v_1, and v_2 which appear in the forcing term appearing in the right hand side of our model problem.

For the first equilibrium point, we can conclude that the dynamical system exhibits the unstable limit cyclic behaviour with respect to the damping parameter S and the forcing parameter v_1. The location of the second and third equilibrium points further depend on the parameters. Thus, the dynamical system exhibited different behaviour at these points. One noteworthy point here is that the behaviour of the system is significantly affected by the ratio of P, and Q. With the help of phase portraits, wave propagation plots and Lyapunov exponents, we showed that the system changes its behaviour from being quasi-periodic to become chaotic for an increased ratio. It is well known that for a highly non-linear Generalized KdV equation with a forcing term in the right hand side such as our model problem considered here, the conventional analytical and numerical methods may not produce acceptable results. In particular a higher ratio of P, and Q may pose a serious challenge for conventional numerical methods. In view of the results presented here regarding the range of the parameters and their corresponding effect on the dynamical system, the investigation may be helpful to devise advance analytical and numerical methods.

Availability of data and material

Not applicable.

Conflicts of interest

The authors declare that there are no competing interests in the publication of this research.

Ethical standard

The authors have no relevant financial or non-financial interests to disclose.

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